Category Theory (UMV/TK/07)

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Project 2005/NP1-051 11230100466

European Social Fund

Basic information

- Extent: 2 hrs lecture/1 hrs seminar per week.
- Assessment: Written tests during the semester, written and oral examination.
- Grading: According to results of the tests and the examination.
- Course objective: To obtain basic knowledge about categories, functors and natural transformations and the categorical approach to various mathematical objects and constructions.

Syllabus

1. Abstract and concrete categories.
2. Monomorphisms, epimorphisms, isomorphisms.
3. Subobjects, quotient objects, free objects.
4. Limits and colimits, completeness.
5. Functors.
6. Natural transformations.
7. Adjoint functors.

1) Introduction

The category theory enables

- to investigate common properties of various mathematical objects;
- to understand better the logical structure of various mathematical theories;
- to transfer ideas from one part of mathematics to another.

General approach: instead of studying the internal structure of mathematical objects, we rather investigate their behaviour and place in the class of similar objects.

A category C is given by

- the class Ob(C) of *objects* of C;
- for every $U, V \in$Ob(C) the set $C(U, V)$ of *morphisms* form U to V;
- the composition rule, i. e. mapping $\circ : \ C(U, V) \times C(V, W) \to C(U, W)$.

1) Categories

Requirements on a category C:

- the composition is associative:

$$(f \circ g) \circ h = f \circ (g \circ h);$$

- for every $U \in$Ob(C) there is a *unit morphism* 1_U such that

$$f \circ 1_U = f, \quad 1_U \circ g = g$$

  for every suitable morphisms f, g.

Concrete categories:

- the objects are structured sets;
- the morphisms are functions between the underlying sets of objects;
- the composition of morphisms is the usual set-theoretical composition of functions;
- the unit morphisms are the identity functions.

Natural examples of the concrete categories:

- sets;
- algebras (groups, lattices, vector spaces,...);
- relational structures (graphs, ordered sets...);
- continuity structures (topological spaces, metric spaces,...);
- mixed structures (ordered groups, topological fields,...).

Some denotations:

- SET ... the category of all sets and all mappings (i.e the objects are all sets and the morphisms are all mappings);
- GRP ... the category og all groups and group homomorphisms (other kinds of algebras analogously: SGR for semigroups, MON for monoids, RNG for rings, FIELD for fields, LAT for lattices, DLAT for distributive lattices, BOOL for Boolean algebras, VECT_F for vector spaces over a given field F, etc.);
- POS ... the category of all partially ordered sets and order-preserving mappings;
- GRAPH ... the category of all (directed) graphs (sets endowed with any binary relation) and all arrow-preserving mappings.

1) Categories

- TOP ... the category of all topological spaces and all continuous mappings;
- COMP ... the category of all compact Hausdorff spaces and all continuous mappings;
- MTP ... the category of all metric spaces and all continuous mappings;
- MET ... the category of all metric spaces and all contractions (maps f satisfying $\rho(f(x), f(y)) \leq \rho(x, y)$).

Every monoid M can be viewed as a category with a single object X, where M is considered as the set of all morphisms $X \to X$ and the composition of morphisms coincides with the monoid multiplication.

More generally, every category C can be viewed a large partial monoid, whose elements are the morphisms of C and the multiplication is the composition, whenever defined.

A quasi-ordered set is any set Q endowed with a binary relation $\leq$ which is:

- reflexive: $x \leq x$ for every $x \in Q$;
- transitive: $x \leq y$ and $y \leq z$ imply $x \leq z$.

We do not require the antisymmetry ($x \leq y$ and $y \leq x$ imply $x = y$). An antisymmetric quasi-ordered set is called (partially) ordered.

1) Categories

Every quasi-ordered set $(Q, \leq)$ can be viewed as a category, where

- objects are the elements of Q;
- morphisms are all pairs (x, y) with $x, y \in Q$, $x \leq y$ (every such pair is a morphism $x \to y$);
- the composition of the pairs (x, y) and (y, z) is the pair (x, z), i.e. $(y, z) \circ (x, y) = (x, z)$.

This category is *small* (the objects form a set) and *thin* (there is at most one morphism between any two objects).

Factorization of categories.
A *congruence* on a category C is an equivalence $\sim$ on the class of all morphisms of C such that:

- every two equivalent morphisms have common domain and range;
- if $f \sim f'$ and $g \sim g'$ for $f, f' \in C(X,Y)$, $g, g' \in C(Y,Z)$, then $g \circ f \sim g' \circ f'$.

Every congruence gives rise to the quotient category $C/\sim$, which has the same objects as C, but the morphisms are the equivalence classes of $\sim$. (Hence, the category is not concrete.)

Example of a non-concrete category:
Consider the category TOP. We say that two continuous maps $f, f' : X \to Y$ are *homotopically equivalent* ($f \sim f'$) if there is a continuous function $h : X \times I \to Y$ (where $I = \langle 0, 1 \rangle$ is the interval with the usual topology) such that

$$h(x, 0) = f(x) \text{ and } h(x, 1) = f'(x)$$

for every $x \in X$.
The quotient category HTOP=TOP$/\sim$ is important in algebraic topology.

Duality.
For every category C we define the dual category C^{op} such that

- $\mathsf{Ob}(C^{op}) = \mathsf{Ob}(C)$;
- $C^{op}(U, V) = C(V, U)$ for all objects U, V.

This gives rise to a dualization of many categorial concepts, like limit and colimit, retraction and coretraction, etc.

2) Morphisms

A morphism $f: U \to V$ is called

- *a monomorphism* if, for every $g, h: W \to U$,

$$f \circ g = f \circ h \quad \text{implies} \quad g = h;$$

- *an epimorphism* if, for every $g, h: V \to W$,

$$g \circ f = h \circ f \quad \text{implies} \quad g = h.$$

2) Morphisms

A morphism $f : U \to V$ is called

- *an isomorphism* if there exists $g : V \to U$ such that

$$f \circ g = 1_V \quad \text{and} \quad g \circ f = 1_U.$$

In the category of sets,

- monomorphisms are the injective functions;
- epimorphisms are the surjective functions;
- isomorphisms are the bijective functions.

2) Morphisms

An example of a concrete category with non-surjective epimorphisms.
TOP_2 ... the category of Hausdorff topological spaces and continuous maps;

- $f: X \to Y$ in TOP_2 is an epimorphism if and only if $f(X)$ is dense in Y.

A bijective morphism need not be an isomorphism.
In GRAPH, any bijective edge-preserving map is both monomorphism and epimorphism, but not necessarily an isomorphism.
On the other hand in categories of algebras (SGR, GRP, RNG, LAT, etc.), the bijective morphisms are isomorphisms. (Also in COMP !)

A morphism $f: U \to V$ is called

- *a retraction* if there exists $g: V \to U$ such that $f \circ g = 1_V$.
- *a coretraction* if there exists $g: V \to U$ such that $g \circ f = 1_U$.

Theorem

Every retraction is an epimorphism. Every coretraction is a monomorphism.

Initial objects.
An object $Z \in \mathrm{Ob}(C)$ is called *initial* if for every $X \in \mathrm{Ob}(C)$ there exists a unique morphism $Z \to X$.
Examples.

- In SET, POS, GRAPH, TOP, MET, LAT, SGR and many other categories: the object with the empty underlying set is initial.
- In GRP: the one-element group is initial.
- In RNG: the ring of integers is initial.
- In BOOL: the 2-element Boolean algebra is initial.

2) Morphisms

Terminal objects.
An object $Z \in \mathrm{Ob}(C)$ is called *terminal* if for every $X \in \mathrm{Ob}(C)$ there exists a unique morphism $X \to Z$.
Examples.

- In SET, POS, GRAPH, TOP, MET, LAT, SGR, GRP, RNG, BOOL and many other categories: the object with the one-element underlying set is terminal.
- In a quasi-ordered set Q viewed as a category, the largest element (if it exists) is terminal.

3) Subobjects, quotient objects, free objects

Let X and Y be objects of a concrete category C with $Y \subseteq X$ (the inclusion of underlying sets). We say that Y is *a subobject* of X if

- the inclusion map $v: Y \to X$ is a morphism;
- for every object Z, a mapping $h: Z \to Y$ is a morphisms if and only if $v \circ h$ is morphism.

If only the first condition is satisfied, we speak about *weak subobjects*.

Subobjects in categories of algebras.
Typically, subobjects and weak subobjects are the same and are called subalgebras (subgroups, subrings, sublattices, vector subspaces etc.). The subobjects exist on those subsets of the underlying sets, which are closed under basic operations. Examples:

- the additive group of integers is a subgroup of the additive group of reals;
- the ring of polynomials with rational coefficients is a subring of the ring of polynomials with real coefficients;
- the lattice of natural numbers is a sublattice of the lattice of integers (with respect to the natural order);

Subobjects in categories of relational structures.
Typically, subobjects exist on any subsets of the underlying sets, but weak subobjects need not be subobjects. In GRAPH, the weak subobjects are called subgraphs, while subobjects are called induced subgraphs. Hence:

- a graph (V_1, E_1) is a subgraph of (V_2, E_2) if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$;
- a graph (V_1, E_1) is an induced subgraph of (V_2, E_2) if $V_1 \subseteq V_2$ and $E_1 = E_2 \cap (V_1 \times V_1)$.

Subobjects in categories of continuity.
Typically, subobjects exist on any subsets of the underlying sets and are called subspaces (topological subspace, metric subspace, etc.).
Weak subobjects need not be subobjects. The exception is the category COMP, where

- weak subobjects are subobjects (a nontrivial topological fact);
- subobjects only exist on topologically closed subsets of the underlying sets.

Let X be an object of a concrete category C and let $\sim$ be an equivalence on (the underlying set of) X. An object Y with the underlying set $X/\sim$ is *a quotient object* of X if

- the factor map $p: X \to Y$ is a morphism;
- for every object Z, a mapping $h: Y \to Z$ is a morphism if and only if $h \circ p$ is a morphism.

If the quotient object exists, then $\sim$ is called *a congruence*.

Factorization of sets.
In the category SET,

- every equivalence relation is a congruence;
- the elements of a quotient object $X/\sim$ are the equivalence classes of $\sim$ and are denoted by

$$x/\sim = \{y \in X \mid y \sim x\}.$$

The construction of quotient objects is called a factorization.

An example of quotient objects.
Consider the ring Z of integers as an object of RNG. Then

- the congruences of Z are precisely the congruences "modulo n" (including the trivial cases $n = 0$ and $n = 1$);
- the quotient objects of Z are exactly the rings Z_n of integers modulo n (including the one-element ring Z_1 and Z itself).

Factorization of graphs.
In GRAPH, any equivalence relation on any object is a congruence.
If (V, E) is a graph and $\sim$ is an equivalence relation on V, then we define the graph structure on $V/\sim$ by

$$(x/\sim, y/\sim) \in E/\sim \quad \text{if there are} \quad x', y' \in V$$

$$\text{such that } x \sim x',\ y \sim y',\ (x', y') \in E.$$

Factorization of topological spaces.
In TOP, every equivalence relation is a congruence. If T is a topological space and $\sim$ is an equivalence on T, then the topology on the set $T/\sim$ is defined by the rule that $A \subseteq T/\sim$ is open if and only if the set

$$\bigcup \{x/{\sim} \mid x/{\sim} \in A\}$$

is open in T.

Let X be an object of a concrete category C and let M be a subset of its underlying set. We say that X is *free over M* if

- for every object Z and every mapping $f_0 : M \to Z$ there exists a unique morphism $f : X \to Z$ extending f_0.

Examples: free algebras, β-hulls of compact spaces.

Existence of free objects.

- in SET: every set is free over itself;
- in POSETS: free objects are the antichains (ordered sets where no two distinct elements are comparable);
- in GRAPH: free objects are the discrete graphs (without edges);
- in general: for categories of relational structures, the free objects usually are discrete.

Existence of free objects.

- in SGR: the free semigroup over a set X is the semigroup of all nonempty words over X with the operation of concatenation;
- in VEC$_F$: every vector space is free over any basis;
- in BOOL: the free Boolean algebra with n generators is the Boolean algebra of all subsets of a 2^n-element set;
- in FIELD: free objects do not exist;
- in general: free objects in the categories of algebras are useful and interesting, often quite complicated.

Existence of free objects.

- in TOP: the free topological spaces are discrete (every set is open);
- in MTP: the discrete spaces are free;
- in MET: free objects with more than one generator do not exist;
- in COMP: the free objects are the β-hulls of discrete spaces (the largest compactifications);
- in general: the continuity categories behave like the categories of relational structures, but the compactness condition makes them similar to the categories of algebras.

Existence of free objects in general.

Theorem

Every nontrivial, transferable concrete category with Cartesian products and equalizers and with bounded generating has free objects.

4) Limits and colimits

A diagram in a category C is a collections of objects and a collection of morphisms between these objects.
A diagram is *commuting*, if all compositions of morphisms with common domain and range coincide.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g\big\downarrow & & s\big\downarrow \\ Z & \xrightarrow{t} & T \end{array}$$

In this example: $s \circ f = t \circ g$.

4) Limits and colimits

Let $D = (\{D_i \mid i \in I\}, \{f_j \mid j \in J\})$ be a diagram in C. *The limit* of D is an object X together with morphisms $p_i : X \to D_i$ such that

- $p_k = f_j \circ p_i$ for every $j \in J$, $f_j : D_i \to D_k$;
- if an object Y and morphisms $g_i : Y \to D_i$ satisfy $g_k = f_j \circ g_i$ for every $j \in J$, then there exists a unique morphism $h : Y \to X$ with $g_i = p_i \circ h$ for every i.

4) Limits and colimits

Let $D = (\{D_i \mid i \in I\}, \{f_j \mid j \in J\})$ be a diagram in C. *The colimit* of D is an object X together with morphisms $e_i : D_i \to X$ such that

- $e_i = e_k \circ f_j$ for every $j \in J$, $f_j : D_i \to D_k$;
- if an object Y and morphisms $g_i : D_i \to Y$ satisfy $g_i = g_k \circ f_j$ for every $j \in J$, then there exists a unique morphism $h : X \to Y$ with $g_i = h \circ e_i$ for every i.

Example of the limit construction:
The product of the objects A_i $(i \in I)$ is an object A together with a collection of morphisms $p_i: A \to A_i$ (called *projections*) having the following property:

- for every object B and every collection of morphisms $q_i: B \to A_i$ there is a unique morphisms $f: B \to A$ such that $q_i = p_i \circ f$ for every $i \in I$.

Equivalently: The product is the limit of a diagram which contains no morphisms.
Examples: Cartesian products of algebras, structures, topological spaces.

A category with non-Cartesian products.
MET$^\bullet$ ... (pointed metric spaces) is the category where:

- objects are all metric spaces with one distinguished point;
- morphisms are all contractions which preserve the distinguished points;

If M_i $(i \in I)$ are metric spaces (M_i having the metrics ρ_i and the distinguished point a_i) then the product is a *subset* of the Cartesian product ΠM_i consisting of all $h = (h_i)_{i \in I} \in \Pi M_i$ such that $\sup\{\rho_i(h_i, a_i) \mid i \in I\}$ is finite.

Example of the colimit construction:
The coproduct (or *sum*) of the objects A_i $(i \in I)$ is an object A together with a collection of morphisms $e_i: \ A_i \to A$ (called *injections*) having the following property:

- for every object B and every collection of morphisms $q_i: \ A_i \to B$ there is a unique morphisms $f: \ A \to B$ such that $q_i = f \circ p_i$ for every $i \in I$.

Equivalently: The coproduct is the colimit of a diagram with no morphisms.
Examples: disjoint union of structures, free product of algebras.

Construction of coproducts.

- in POS, GRAPH, TOP: the coproduct is the disjoint union;
- in SGR, GRP, LAT, and other categories of algebras: the coproducts are the algebras "freely" generated by the disjoint union;
- in MET, FIELD: the coproducts of more that one objects do not exist;
- in COMP: the coproduct of a finite family of objects is the disjoint union; for an infinite family it is the largest compactification of the disjoint union.

Products and coproducts are defined also for the empty family of objects. It is easy to see that

- the product of the empty family is the terminal object;
- the coproduct of the empty family is the initial object;

Products and coproducts in VEC_F. Let V and W be vector spaces over a field F. Then

- the product $V \times W$ is the Cartesian product together with projections $p_1 : V \times W \to V$ and $p_2 : V \times W \to W$ given by $p_1(u, v) = u$, $p_2(u, v) = v$.
- the coproduct $V + W$ is the Cartesian product $V \times W$ together with injections $e_1 : V \to V \times W$, $e_2 : W \to V \times W$ given by $e_1(u) = (u, 0)$, $e_2(v) = (0, v)$;

Some other algebraic categories have the same property.

Directed union as a colimit.
Let $A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ be an increasing sequence of sets. Then $\bigcup_{i=1}^{\infty} A_i$ is called the *directed union*. (More generally, the directed union is the union of an increasing collection of sets indexed by any directed index set.)
It is easy to see that this directed union is the colimit of the diagram

$$A_0 \xrightarrow{\varepsilon_0} A_1 \xrightarrow{\varepsilon_1} A_2 \xrightarrow{\varepsilon_2} \dots$$

where ε_i are the set inclusions, in the category SET. (Similarly for many other concrete categories.)

Products and coproducts in thin categories.

Theorem

Let Q be a quasi-ordered set viewed as a category. Let $M \subseteq Q$ be a family of objects. Then

- *the product ΠM is equal to $\inf M$, provided the infimum exists;*
- *the coproduct ΣM is equal to $\sup M$, provided the supremum exists;*

Equalizers and coequalizers.
Consider the following diagram in a category C.

$$X \underset{g}{\overset{h}{\rightrightarrows}} Y$$

The limit of this diagram (if it exists) is called the *equalizer* of the morphisms g and h. The colimit of this diagram (if it exists) is called the *coequalizer* of g and h.

Example: equalizers and coequalizers in SET.
In the category SET the equalizers and coequalizers exists for every sets X, Y and every pair of mappings $g, h : X \to Y$. More precisely,

- the equalizer of g and h is the set

  $$E = \{x \in X \mid g(x) = h(x)\}$$

  together with the set inclusion $e : E \to X$;
- the coequalizer of g and h is the quotient set $Y/{\sim}$ together with the natural projection $p : Y \to Y/{\sim}$, where $\sim$ is the smallest equivalence on Y containing the set

  $$\{(g(x), h(x)) \mid x \in X\}.$$

Pullback of the morphisms $g: A_1 \to A_0$ and $h: A_2 \to A_0$ is the limit of the diagram

$$A_1 \xrightarrow{\;g\;} A_0 \xleftarrow{\;h\;} A_2$$

Hence, it is an object A together with morphisms $\pi_i: A \to A_i$, having the universal property.
Example: in SET, the pullback is the set

$$A = \{(x,y) \in A_1 \times A_2 \mid g(x) = h(y)\}$$

together with natural projections $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$.

Pushout of the morphisms $g: A_0 \to A_1$ and $h: A_0 \to A_2$ is the colimit of the diagram

$$A_1 \xleftarrow{\ g\ } A_0 \xrightarrow{\ h\ } A_2$$

Hence, it is an object A together with morphisms $\pi_i: A_i \to A$, having the universal property.
Example: in SET, the pullback is the set

$$A = A_1 + A_2/\sim,$$

where $A_1 + A_2$ denotes the disjoint union and $\sim$ is the least equivalence on $A_1 + A_2$ such that $g(x) \sim h(x)$ for every $x \in A_0$.

Completeness of categories.
A category C is called *complete*, if every diagram in C has a limit.

Theorem

A category C is complete if and only if every family of objects has a product and every pair of morphism (with a common domain and a range) has an equalizer.

Sketch of the proof.
Let D be a diagram in C containing objects D_i ($i \in I$) and morphisms p_t ($t \in T$). For every $t \in T$ there are $i(t), j(t) \in I$ with $p_t : D_{i(t)} \to D_{j(t)}$. Consider the products
$A = \Pi_{i \in I} D_i$ (together with morphisms $\pi_i :\ A \to D_i$) and
$A^* = \Pi_{t \in T} D_{j(t)}$ (together with morphisms $\pi_t^* :\ A^* \to D_{j(t)}$.
By the definition of the product, there are morphisms
$g, h :\ A \to A^*$ such that $\pi_t^* \circ g = \pi_{j(t)}$ and $\pi_t^* \circ h = p_t \circ \pi_{i(t)}$ for every $t \in T$. And it is possible to show that the equalizer of g and h is the limit of D.

Co-completeness of categories.
A category C is called *co-complete*, if every diagram in C has a colimit.

Theorem

A category C is co-complete if and only if every family of objects has a coproduct and every pair of morphism (with a common domain and a range) has an coequalizer.

Example.

Theorem

Let Q be an ordered set viewed as a category. The following conditions are equivalent.

- *Q is a complete category;*
- *Q is a co-complete category;*
- *Q is a complete lattice (every set has a supremum and an infimum).*

5) Functors

A functor $F: C \to D$ between the categories C and D is an assignment, which

- to each $X \in \mathrm{Ob}(C)$ assigns $FX \in \mathrm{Ob}(D)$;
- to each $f \in C(X, Y)$ assigns $Ff \in D(FX, FY)$;
- preserves the composition

$$F(g \circ f) = Fg \circ Ff$$

  and units

$$F(1_X) = 1_{FX}.$$

5) Functors

The identity functor $1_C : C \to C$ on a category C is defined by

- $FX = X$ for each $X \in \mathrm{Ob}(C)$;
- $Ff = f$ for each $f \in C(X,Y)$.

The constant functor.
Let C and D be categories and $B \in \mathrm{Ob}(D)$. The constant functor $F_B: C \to D$ is defined by

- $FX = B$ for each $X \in \mathrm{Ob}(C)$;
- $Ff = 1_B$ for each $f \in C(X, Y)$.

The Cartesian product functor $Q : \mathrm{SET} \to \mathrm{SET}$ is defined by

- $QX = X \times X$ for each set X;
- if $f : X \to Y$ is a mapping, then $Qf : X \times X \to Y \times Y$ is defined by $f(u, v) = (f(u), f(v))$.

5) Functors

The power set functor $\mathcal{P}$: SET $\to$ SET is defined by

- $\mathcal{P}X$ is the set of all subsets of X;
- $\mathcal{P}f$ for $f: X \to Y$ is the mapping determined by the rule

$$\mathcal{P}(f)(A) = \{f(a) \mid a \in A\}.$$

5) Functors

A forgetful functor $U : C \to \mathrm{SET}$ for a concrete category C is an assigmnent, which

- to each $X \in \mathsf{Ob}(C)$ assigns its underlying set;
- to each $f \in C(X, Y)$ assigns f itself (viewed as a plain mapping.

5) Functors

We say that D is a subcategory of C if $\mathsf{Ob}(D) \subseteq \mathsf{Ob}(C)$ and $D(X,Y) \subseteq C(X,Y)$ for every $X, Y \in \mathsf{Ob}(D)$.
Every such subcategory gives rise to the *inclusion functor*

$$V:\ D \to C$$

defined by $DX = X$ on objects and $Df = f$ on morphisms.

Hom-functor.
Let C be a category and $A \in \mathsf{Ob}(C)$. We define

$$\mathrm{Hom}(A, -) : \ C \to \mathrm{SET}$$

as follows:

- $\mathsf{Hom}(A, -)(X) = C(A, X)$ for each $X \in \mathsf{Ob}(C)$;
- $\mathsf{Hom}(A, -)(f)$ for each $f \in C(X, Y)$ is a mapping $C(A, X) \to C(A, Y)$ defined by $g \mapsto f \circ g$.

The free objects functor.
Let C be a category which has free objects. We define the free object functor $F:\ \mathrm{SET} \to C$ as follows:

- for each set X, FX is the free object in C over X;
- for each mapping $f:\ X \to Y$ assigns Ff is the unique morphism $FX \to FY$ extending f.

A composition of functors $F:\ C \to D$ and $G:\ D \to E$ is the functor $GF:\ C \to E$ defined by $GFX = G(FX)$ and $GFf = G(Ff)$.

5) Functors

A functor $F: C \to D$ is called

- *faithful* if $Ff \neq Fg$ whenever $f, g \in C(X, Y)$ are different;
- *full* if for every $g \in D(FX, FY)$ there exists $f \in C(X, Y)$ with $Ff = g$;
- *an isofunctor* if there exists a functor $G: D \to C$ with $FG = 1_D$ and $GF = 1_C$.(Then C and D are *isomorphic*.)

The structure of non-concrete categories.

Theorem

For every category C there exists a concrete category K and a congruence $\sim$ on K such that C is isomorphic to the factor category $K/\sim$.

5) Functors

A contravariant functor $F: C \to D$ between the categories C and D is an assignment, which

- to each $X \in \mathrm{Ob}(C)$ assigns $FX \in \mathrm{Ob}(D)$;
- to each $f \in C(X,Y)$ assigns $Ff \in D(FY,FX)$;
- preserves the composition

$$F(g \circ f) = Ff \circ Fg$$

and units

$$F(1_X) = 1_{FX}.$$

(The usual functors are sometimes called *covariant*.)

An example of a contravariant functor O : TOP$\to$ LAT.

- For each topological space X let $O(X)$ be the lattice of all open subsets of X;
- to each continuous mapping $f: \ X \to Y$ define $O(f): \ O(Y) \to O(X)$ by

$$O(f)(A) = f^{-1}(A).$$

Exercise: check that O is a contravariant functor.

Let $F, G: C \to D$ be two functors with a common domain and a range. *A transformation* $\tau: F \to G$ is an assingment, which assigns to each $X \in \mathrm{Ob}(C)$ a morphism $\tau_X: FX \to GX$ such that the diagram

$$\begin{array}{ccc} FX & \xrightarrow{\tau_X} & GX \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle Gh} \\ FY & \xrightarrow{\tau_Y} & GY \end{array}$$

commutes for every morphism $h: X \to Y$.

6) Natural transformations

Let $F, G, H : C \to D$ be three functors with a common domain and a range. Let $\tau : F \to G$ and $\sigma : G \to H$ be transformations. The *composition* of τ and σ is a transformation $F \to H$ defined by

$$(\sigma \cdot \tau)_X = \sigma_X \circ \tau_X .$$

The *unit* transformation $1_F : F \to F$ is defined by $(1_F)_X = 1_{FX}$ for every $X \in \mathrm{Ob}(C)$.

A transformation $\tau : F \to G$ is called *natural equivalence* if there exists a transformation τ^{-1} such that $\tau^{-1} \cdot \tau = 1_F$ and $\tau \cdot \tau^{-1} = 1_G$.

Theorem

A transformation $\tau : F \to G$ is a natural equivalence if and only if τ_X is an isomorphism for every X.

An example of a natural equivalence.

Theorem

If a concrete category C has the free object Z with one generator, then the forgetful functor $U : C \to$ SET is naturally equivalent to the hom-functor $\mathrm{Hom}(Z, -)$.

Another example.

Theorem

Two hom-functors $\mathrm{Hom}(X, -)$ *and* $\mathrm{Hom}(Y, -)$ *are naturally equivalent if and only if the objects* X *and* Y *are isomorphic.*

Categories C and D are called *equivalent* if there are functors $E: C \to D$ and $F: D \to C$ such that the compositions FE and EF are naturally equivalent to the identity functors 1_C and 1_D, respectively.

Theorem

The categories C and D are equivalent if and only if there exists a full and faithful functor $E: C \to D$ such that every $Y \in Ob(D)$ is isomorphic to EX for some $X \in Ob(C)$.

A pair of *adjoint functors* between two categories C and D consists of functors $F: C \to D$ and $G: D \to C$ and a natural equivalence

$$\Phi: D(F-,-) \to C(-,G-).$$

The natural equivalence Φ consists of bijections

$$\Phi_{X,Y} : D(FX,Y) \to C(X,GY)$$

for every $X \in \mathrm{Ob}(C)$, $Y \in \mathrm{Ob}(D)$.
If (F,G) is a pair of adjoint functors, then F is also called *the left adjoint* and G *the right adjoint*.

7) Adjoint functors

Theorem

(F, G) is a pair of adjoint functors if and only if there are transformations

$$\eta : \ 1_D \to F \cdot G, \quad \varepsilon : \ G \cdot F \to 1_C,$$

such that $F\varepsilon \cdot \eta F = 1_F$ and $\varepsilon G \cdot G\eta = 1_G$.

7) Adjoint functors

In diagrams:

$$\begin{array}{ccc} FA & \xrightarrow{\eta_{FA}} & FGFA \\ \big\downarrow{\scriptstyle 1_{FA}} & & \big\downarrow{\scriptstyle F\varepsilon_A} \\ FA & \xrightarrow{1_{FA}} & FA \end{array} \qquad \begin{array}{ccc} GM & \xrightarrow{1_{GM}} & GM \\ \big\downarrow{\scriptstyle G\eta_M} & & \big\downarrow{\scriptstyle 1_{GM}} \\ GFGM & \xrightarrow{\varepsilon_{GM}} & GM \end{array}$$

Example of an adjoint pair.
Let C be a concrete category which has free objects. Let $F:$ SET $\to C$ be the free objects functor.
Let $U:$ $C \to$ SET be the forgetful functor.

Theorem

(F, U) is apair of adjoint functors. The functor F is the left adjoint and G is the right adjoint.

Example of an adjoint pair.
Let C be a category with products. Let $D: C \to C \times C$ be the diagonal functor defined by $DX = (X, X)$. Let $P: C \times C \to C$ be the product functor defined by $P(Y, Z) = Y \times Z$.

Theorem

(D, P) is a pair of adjoint functors.

Example of an adjoint pair.
Let Q, R be ordered sets viewed as categories. Let $f: Q \to R$ and $g: R \to Q$ be functors, i.e.order-preserving mappings.

Theorem

(f, g) is a pair of adjoint functors if and only if the following condition holds for every $x \in Q$, $y \in R$:

$$f(x) \leq y \iff x \leq g(y).$$

Reflective subcategories.
Let D be a subcategory of a category C. Then D is called *reflective*, if the inclusion functor $V: D \to C$ is a right adjoint, i.e. if there exist a functor $W: C \to D$ such that (W, V) is a pair of adjoint functors.
Example: The category AB of abelian groups is a reflective subcategory of the category GRP. The functor W assigns to each group its maximal abelian quotient.

Reflective subcategories - another example.
The category POS of ordered sets is a reflective subcategory of the category QOS of quasiordered sets. The functor W : QOS $\to$ POS assigns to each quasiordered set Q its quotient

$$W(Q) = Q/\sim,$$

where

$$x \sim y \quad \Longrightarrow \quad x \leq y \leq x.$$

Preservation of limits.
We say that a functor $F: C \to D$ *preserves limits*, if $F(\lim H) = \lim F(H)$ for every diagram H in C.

Theorem

A functor preserves limits if and only if it preserves the products and the equalizers.

Theorem

Every right adjoint functor preserves limits.

Appendix) Definitions from algebra and topology

A *semigroup* is a set S endowed with a binary operation $\cdot$ satisfying the associative law:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

A mapping $f: S \to T$ between semigroups S and T is a *homomorphism* if it preserves the operation $\cdot$:

$$f(x \cdot y) = f(x) \cdot f(y).$$

A *monoid* is a semigroup with a distinguished neutral element e satisfying

$$x \cdot e = e \cdot x = x.$$

A *group* is a monoid, additionally endowed with a unary operation $^{-1}$ satisfying

$$x \cdot x^{-1} = x^{-1} \cdot x = e.$$

A mapping $f : S \to T$ between groups S and T is a *homomorphism* if it preserves the operations $\cdot$, $^{-1}$ and the neutral element.
The group is *abelian* (or *commutative*) if it satisfies the identity

$$x \cdot y = y \cdot x.$$

A *ring* is a set R endowed with binary operations $+$ and $\cdot$, unary operation $-$ and a constant 0 such that

- $(R, +, 0, -)$ is an abelian group;
- $(R, \cdot)$ is a semigroup;
- the *distributive laws* hold:

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z);$$

$$(y + z) \cdot x = (y \cdot x) + (z \cdot x).$$

A *field* is a ring $(R, +, -, 0, \cdot)$ satisfying the following additional requirements:

- $x \cdot y = y \cdot x$ for every $x, y \in R$;
- there exists $1 \in R$ such that $x \cdot 1 = x$ for every $x \in R$;
- for every $x \neq 0$ there exists $x^{-1} \in R$ such that $x \cdot x^{-1} = 1$;

A *vector space* over a field F is a set V endowed with a binary operation $+$, a unary operation $-$, a constant 0 and a set $\{u_\alpha \mid \alpha \in F\}$ of unary operations, satisfying the following conditions. (As usual, we write αx instead of $u_\alpha(x)$.)

- $(V, +, 0, -)$ is an abelian group;
- $\alpha(y + z) = (\alpha y) + (\alpha z)$;
- $(\alpha + \beta)x = \alpha x + \beta x$;
- $(\alpha \cdot \beta)x = \alpha(\beta x)$;
- $1x = x$.

Appendix) Definitions from algebra and topology

A *(partially) ordered set* is a set P endowed with a binary relation $\leq$, which is

- reflexive: $x \leq x$;
- antisymmetric: $x \leq y$ and $y \leq x$ imply $x = y$;
- transitive: $x \leq y$ and $y \leq z$ imply $x \leq z$.

A map $f: P \to Q$ between posets is *order-preserving* (or *isotone*), if $x \leq y$ implies $f(x) \leq f(y)$.

Appendix) Definitions from algebra and topology

A *lattice* is a partially ordered set L in which every two elements have the supremum (the least upper bound) and the infimum(the largest lower bound). The supremum and the infimum are usually denoted by $\vee$ and $\wedge$, respectively, and regarded as binary operations on L.
A map $f: L \to M$ between lattices is a homomorphism, if $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for every $x, y \in L$.

A lattice L is called *distributive* if it satisfies the following identities:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z);$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

A lattice L is called *bounded* if it contains a smallest element 0 and a largest element 1. (That is, $0 \leq x \leq 1$ for every $x \in L$.)

Appendix) Definitions from algebra and topology

A *Boolean algebra* is a set B endowed with binary operations $\vee$ and $\wedge$, a unary operation $'$ and constants 0, 1 such that

- $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice;
- $'$ is *the complementation*:

$$x \vee x' = 1;$$

$$x \wedge x' = 0.$$

Appendix) Definitions from algebra and topology

A *graph* is a set G (the set of *vertices*) endowed with a binary relation E (the set of oriented edges).
An *unoriented graph* is a set G endowed with a symmetric and irreflexive binary relation E.
A map $f:\ G \to H$ between (unoriented) graphs is called a homomorphism, if it is edge-preserving:

$$(x, y) \in E \quad \text{implies} \quad (f(x), f(y)) \in E.$$

A *topological space* is a set T endowed with a collection τ of its subsets satisfying

- $\emptyset, T \in \tau$;
- $X, Y \in \tau$ implies $X \cap Y \in \tau$;
- $\{X_i \mid i \in I\} \subseteq \tau$ implies $\bigcup_{i \in I} X_i \in \tau$.

The members of the collection τ are called *open*.
A map $f : T \to V$ between topological spaces is called *continuous* if the set $f^{-1}(A)$ is open for every open set A.

Appendix) Definitions from algebra and topology

A topological space T is called *Hausdorff* if for every $x, y \in T$, $x \neq y$ there are open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.
A topological space T is called *compact* if for every collection $\{A_i \mid i \in I\}$ of open sets which cover T (that is, $\bigcup_{i \in I} A_i = T$) there exists a finite subcover (that is, a finite set $J \subseteq I$ with $\bigcup_{i \in J} A_i = T$).

A *metric space* is a set M endowed with a function $d: \ M^2 \to \mathbb{R}$ such that

- $d(x, y) \geq 0$;
- $d(x, y) = 0$ if and only if $x = y$;
- $d(x, y) = d(y, x)$;
- $d(x, z) \leq d(x, y) + d(y, z)$;

for all $x, y, z \in M$.

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