Uncountable critical points for congruence lattices

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Abstract. The critical point between two classes $\mathcal{K}$ and $\mathcal{L}$ of algebras is the cardinality of the smallest semilattice isomorphic to the semilattice of compact congruences of some algebra in $\mathcal{K}$, but not in $\mathcal{L}$. Our paper is devoted to the problem of determining the critical point between two finitely generated congruence-distributive varieties. For a homomorphism $\varphi: S \to T$ of $(0, \vee)$-semilattices and an automorphism τ of T, we introduce the concept of a τ-symmetric lifting of φ. We use it to prove a criterion, which ensures that the critical point between two finitely generated congruence-distributive varieties is less or equal to $\aleph_1$. We illustrate the criterion by constructing two new examples with the critical point exactly $\aleph_1$.

1. Introduction

For a class $\mathcal{K}$ of algebras we denote $\operatorname{Con}\mathcal{K}$ the class of all lattices isomorphic to $\operatorname{Con}(A)$ (the congruence lattice of an algebra A) for some $A \in \mathcal{K}$. Despite many partial results, a good description of $\operatorname{Con}\mathcal{K}$ has proved to be a very difficult (and probably intractable) problem, even for the most common classes of algebras. A more promising approach seems to be to compare the classes $\operatorname{Con}\mathcal{K}$ and $\operatorname{Con}\mathcal{L}$ for different $\mathcal{K}$ and $\mathcal{L}$. The following definition of a critical point has been introduced by P. Gillibert in his thesis. ([3], see also [4] or [7].)

Let L_c denote the set of all compact elements of an algebraic lattice L.

Definition 1.1. *Let $\mathcal{K}$ and $\mathcal{L}$ be classes of algebras. The* critical point of $\mathcal{K}$ under $\mathcal{L}$, *denoted* $\operatorname{crit}(\mathcal{K}, \mathcal{L})$, *is the smallest cardinality of L_c for $L \in \operatorname{Con}\mathcal{K} \setminus \operatorname{Con}\mathcal{L}$ (if $\operatorname{Con}\mathcal{K} \nsubseteq \operatorname{Con}\mathcal{L}$) or ∞ (if $\operatorname{Con}\mathcal{K} \subseteq \operatorname{Con}\mathcal{L}$).*

We are mainly interested in the case when $\mathcal{K}$ and $\mathcal{L}$ are varieties of algebras. For most of pairs of varieties, the critical point is either finite or $\aleph_0$. Examples with an uncountable critical point are quite rare and the proofs are difficult. First examples with the critical point $\aleph_2$ have been exhibited in [9] and [10]. More $\aleph_2$ examples are contained in [5]. The first example with the critical point $\aleph_1$ has been presented in [4]. The result from [11] shows that the critical point between the variety of all majority algebras and the variety of all lattices is $\aleph_2$.

The following recent result says that, under some reasonable restrictions, the critical point cannot be greater than $\aleph_2$.

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Theorem 1.2. (See [6].) *Let $\mathcal{K}$ and $\mathcal{L}$ be locally finite varieties of algebras. Assume that for each finite $A \in \mathcal{K}$ there are, up to isomorphism, only finitely many $B \in \mathcal{L}$ such that $\operatorname{Con} A \cong \operatorname{Con} B$, and every such B is finite. Then either $\operatorname{crit}(\mathcal{K}, \mathcal{L}) \leq \aleph_2$ or $\operatorname{Con} \mathcal{K} \subseteq \operatorname{Con} \mathcal{L}$.*

So, under the above restrictions, the critical point can be finite, $\aleph_0$, $\aleph_1$, $\aleph_2$, or ∞. There are several methods how to distinguish these cases, but no algorithm is known, not even in the most tractable situation when $\mathcal{K}$ and $\mathcal{L}$ are both finitely generated, congruence-distributive varieties. In this paper we use the method of lifting of semilattice diagrams. Other known methods are based on a topological representation of congruence lattices (see e.g. [9]) and on various refinement properties (e.g. [13], [14], [15], [16], [11]).

The present paper is devoted to the least investigated case: the critical point $\aleph_1$. Based on Gillibert's result (see Theorem 2.1), we prove a new upper bound criterion, using a concept of a τ-symmetric lifting. This allows to investigate diagrams indexed by a chain, instead of diagrams indexed by products of two chains. We also present two examples, demonstrating the use of our result.

Now we recall basic denotations and facts. We assume familiarity with the fundamentals of lattice theory and universal algebra. For all undefined concepts and unreferenced facts we refer to [8] and [2].

The smallest and the largest congruence on A will be denoted by 0_A and 1_A, respectively. The congruence lattice of $\operatorname{Con} A$ of an algebra A is always algebraic and its compact elements form a $(0, \vee)$-subsemilattice of $\operatorname{Con} A$, denoted $\operatorname{Con}_c A$. Recall that the semilattice $\operatorname{Con}_c A$ determines the lattice $\operatorname{Con} A$ uniquely. For $x, y \in A$ let $\theta(x, y)$ denote the smallest congruence containing the pair (x, y). The semilattice $\operatorname{Con}_c(A)$ consists precisely of all finitely generated congruences, i.e. congruences of the form $\theta(x_1, y_1) \vee \cdots \vee \theta(x_n, y_n)$. The smallest congruence (the equality relation) is considered as compact, so $\operatorname{Con}_c A$ always has the smallest element. If $f : A \to B$ is a homomorphism of algebras, then we define $\operatorname{Con} f : \operatorname{Con} A \to \operatorname{Con} B$ by the rule that $(\operatorname{Con} f)(\alpha)$ is the congruence on B generated by all pairs $(f(x), f(y))$ with $(x, y) \in \alpha$. The restriction of $\operatorname{Con} f$ to $\operatorname{Con}_c A$ is a mapping $\operatorname{Con}_c f : \operatorname{Con}_c A \to \operatorname{Con}_c B$. It is easy to see that $\operatorname{Con}_c f$ is a homomorphism of $(0, \vee)$-semilattices.

Now, every $(0, \vee)$-homomorphism $\varphi : K \to L$ between complete lattices K and L determines the mapping $\varphi^{\leftarrow} : L \to K$ by the rule

$$\varphi^{\leftarrow}(\beta) = \bigvee \{\alpha \mid \varphi(\alpha) \leq \beta\}.$$

If $\varphi = \operatorname{Con}_c f$ for a homomorphism $f : A \to B$ of finite algebras, then

$$(\operatorname{Con}_c f)^{\leftarrow}(\beta) = \{(x, y) \in A^2 \mid (f(x), f(y)) \in \beta\}.$$

The pair $(\varphi, \varphi^{\leftarrow})$ is sometimes referred to as residuated mappings. The following facts are rather well known. (See [1], Section 1.3.)

Lemma 1.3. *Let $\varphi : K \to L$ be a $(0, \vee)$-homomorphism of finite lattices.*

(1) *$\varphi^{\leftarrow}$ preserves $\wedge$ and the largest element.*

(2) $\varphi(\alpha) = \bigwedge\{\beta \mid \alpha \leq \varphi^{\leftarrow}(\beta)\}$ *for every* $\alpha \in L$.
(3) *If* $\psi : L \to M$ *is another* $(0, \vee)$*-homomorphism of finite lattices, then* $(\psi\varphi)^{\leftarrow} = \varphi^{\leftarrow}\psi^{\leftarrow}$.

We need some basic facts about congruences on direct products. If algebras B_1, B_2 belong to a congruence-distributive variety, then the lattice $\mathrm{Con}(B_1 \times B_2)$ is isomorphic to the direct product of lattices $\mathrm{Con}\, B_1 \times \mathrm{Con}\, B_2$. (See [2].) The isomorphism $\iota :\ \mathrm{Con}\, B_1 \times \mathrm{Con}\, B_2 \to \mathrm{Con}(B_1 \times B_2)$ maps every pair (β_1, β_2) into the product congruence $\beta_1 \times \beta_2 = \{((x_1, x_2), (y_1, y_2)) \mid (x_i, y_i) \in \beta_i\}$. Taking into account this isomorphism we also have the following assertion. Recall that any two mappings $f_1 :\ A \to B_1$ and $f_2 :\ A \to B_2$ induce the product mapping $f_1 \times f_2 :\ A \to B_1 \times B_2$.

Lemma 1.4. *Let* $f_i :\ A \to B_i$ $(i = 1, 2)$ *be homomorphisms of algebras belonging to a congruence-distributive variety. Then* $\mathrm{Con}(f_1 \times f_2) = \mathrm{Con}\, f_1 \times \mathrm{Con}\, f_2$.

Proof. Let $\alpha \in \mathrm{Con}\, A$. We know that $\mathrm{Con}(f_1 \times f_2)(\alpha)$ is a congruence on $B_1 \times B_2$, hence it is the product $\beta_1 \times \beta_2$ for some $\beta_i \in \mathrm{Con}\, B_i$ $(i = 1, 2)$. We just need to show that $\beta_i = (\mathrm{Con}\, f_i)(\alpha)$.

The congruence $\mathrm{Con}(f_1 \times f_2)(\alpha)$ is generated by all pairs of the form $((f_1 \times f_2)(x), (f_1 \times f_2)(y))$ with $(x, y) \in \alpha$. Every such pair clearly belongs to $(\mathrm{Con}\, f_1)(\alpha) \times (\mathrm{Con}\, f_2)(\alpha)$, so $\beta_i \subseteq (\mathrm{Con}\, f_i)(\alpha)$. On the other hand, for every $(f_1(x), f_1(y)) \in (\mathrm{Con}\, f_1)(\alpha)$ we have $((f_1 \times f_2)(x), (f_1 \times f_2)(y)) \in \beta_1 \times \beta_2$, so $(\mathrm{Con}\, f_1)(\alpha) \subseteq \beta_1$, and similarly $(\mathrm{Con}\, f_2)(\alpha) \subseteq \beta_2$. □

Recall that if $\alpha \in \mathrm{Con}\, A$, then the congruence lattice of the quotient algebra A/α is isomorphic to the interval $[\alpha, 1_A]$ of $\mathrm{Con}\, A$. In this isomorphism, every $\delta \in [\alpha, 1_A]$ corresponds to $\delta/\alpha = \{(x/\alpha, y/\alpha) \mid x, y \in \delta\} \in \mathrm{Con}(A/\alpha)$. (By x/α we denote the α-equivalence class containing x.)

Lemma 1.5. *Let* D *be any algebra with* $\mathrm{Con}\, D$ *distributive, let* L_1 *and* L_2 *be distributive lattices. Further, let* $\Phi :\ \mathrm{Con}\, D \to L_1 \times L_2$ *be an isomorphism of distributive lattices. Then there are* $\alpha_1, \alpha_2 \in \mathrm{Con}\, D$ *with* $\alpha_1 \wedge \alpha_2 = 0_A$ *and isomorphisms* $\Phi_i :\ \mathrm{Con}(D/\alpha_i) \to L_i$ *such that*

$$\Phi(\delta) = (\Phi_1(\delta \vee \alpha_1/\alpha_1), \Phi_2(\delta \vee \alpha_2/\alpha_2))$$

for every $\delta \in \mathrm{Con}\, D$.

Proof. Take α_1, α_2 with $\Phi(\alpha_1) = (0, 1)$, $\Phi(\alpha_2) = (1, 0)$. (Notice that both L_i must be bounded, because $\mathrm{Con}\, D$ is bounded.) The isomorphism Φ maps the interval $[\alpha_1, 1_D]$ onto the interval $[(0, 1), (1, 1)]$, so there is an isomorphism $\Psi_1 :\ [\alpha_1, 1_D] \to L_1$ with $\Phi(\delta) = (\Psi_1(\delta), 1)$ for every $\delta \in [\alpha_1, 1_D]$. Similarly, we have an isomorphism $\Psi_2 :\ [\alpha_2, 1_D] \to L_1$ with $\Phi(\delta) = (1, \Psi_2(\delta))$ for every $\delta \in [\alpha_2, 1_D]$. Now we define Φ_i by $\Phi_i(\delta/\alpha_i) = \Psi_i(\delta)$. By the distributivity we have $\Phi(\delta) = \Phi(\delta \vee \alpha_1) \wedge \Phi(\delta \vee \alpha_2) = (\Psi_1(\delta \vee \alpha_1), 1) \wedge (1, \Psi_2(\delta \vee \alpha_2)) = (\Phi_1(\delta \vee \alpha_1/\alpha_1)), \Phi_2(\delta \vee \alpha_2/\alpha_2))$ for every $\delta \in \mathrm{Con}\, D$. □

The above Lemma also means that D is isomorphic to a subdirect product of D/α_1 and D/α_2 and $\operatorname{Con} D$ is isomorphic to $\operatorname{Con}(D/\alpha_1)\times\operatorname{Con}(D/\alpha_2)$. If ι is the natural embedding of D into $D/\alpha_1 \times D/\alpha_2$, that is $\iota(x) = (x/\alpha_1, x/\alpha_2)$, then $\operatorname{Con}\iota$ is an isomorphism which maps $\delta \in \operatorname{Con} D$ to $(\delta\vee\alpha_1/\alpha_1, \delta\vee\alpha_2/\alpha_2)$.

If $f\colon\ X \to Y$ is a mapping then $\operatorname{rng} f$ denotes the range of this mapping, and $\operatorname{Ker} f$ (the kernel of f) is the binary relation on X defined by $(x, y) \in \operatorname{Ker} f$ iff $f(x) = f(y)$.

2. Lifting of semilattice diagrams

Let P be an ordered set. Let $\mathcal{K}$ be a class of algebras. A P-indexed diagram $\vec{A}$ in $\mathcal{K}$ consists of a family $(A_p, p \in P)$ of algebras in $\mathcal{K}$ and a family $(f_{p,q}, p \leq q)$ of homomorphisms $f_{p,q}:\ A_p \to A_q$ such that $f_{p,p}$ is the identity on A_p and $f_{p,r} = f_{q,r}f_{p,q}$ for all $p \leq q \leq r$.

For any such diagram $\vec{A}$ we consider the diagram $\operatorname{Con}_c\vec{A}$ of semilattices with 0, which consists of the family $(\operatorname{Con}_c A_p, p \in P)$ and the mappings $\operatorname{Con}_c f_{p,q}$: $\operatorname{Con}_c A_p \to \operatorname{Con}_c A_q$. It is easy to see that $\operatorname{Con}_c\vec{A}$ is a P-indexed diagram of $(0,\vee)$-semilattices.

Now let $\vec{A} = (A_p, f_{p,q} \mid p \leq q \text{ in } P)$ be a P-indexed diagram of nonempty algebras in $\mathcal{K}$ and let $\vec{S} = (S_p, g_{p,q} \mid p \leq q \text{ in } P)$ be a P-indexed diagram of $(0,\vee)$-semilattices. We say that $\vec{A}$ is *a lifting of* $\vec{S}$ (or that $\vec{A}$ *lifts* $\vec{S}$) if the diagrams $\vec{S}$ and $\operatorname{Con}_c\vec{A}$ are isomorphic, which means that there are isomorphisms $\varphi_p : \operatorname{Con}_c A_p \to S_p$, $p \in P$ such that the diagram

$$\begin{array}{ccc}
\operatorname{Con}_c A_p & \xrightarrow{\operatorname{Con}_c f_{p,q}} & \operatorname{Con}_c A_q \\
{\scriptstyle\varphi_p}\big\downarrow & & {\scriptstyle\varphi_q}\big\downarrow \\
S_p & \xrightarrow{g_{p,q}} & S_q
\end{array}$$

commutes for every $p \leq q$.

We use the following result. (See [4], Corollaries 7.6 and 7.12, or [7], Theorem 4.9.2 and Corollary 4.9.7.)

Theorem 2.1. *Let $\mathcal{K}$ and $\mathcal{L}$ be finitely generated congruence-distributive varieties of algebras and n a nonnegative integer. Then (ii) implies (i), where*

(i) $\operatorname{crit}(\mathcal{K},\mathcal{L}) \leq \aleph_n$*;*

(ii) *there exists a diagram of finite $(\vee, 0)$-semilattices indexed by a product of $n + 1$ finite chains liftable in $\mathcal{K}$ but not in $\mathcal{L}$*

If $n = 0$ then also (i)$\Longrightarrow$ (ii).

We need the above theorem for $n = 0$ and $n = 1$. In the case $n = 1$ we use the following, slightly modified form.

Theorem 2.2. *Let $\mathcal{K}$ and $\mathcal{L}$ be finitely generated congruence-distributive varieties of algebras with* $\operatorname{crit}(\mathcal{K},\mathcal{L}) > \aleph_1$*. Let f_i : $A_i \to B$ $(i \in \{1, 2\})$ be* *homomorphisms of finite algebras in* $\mathcal{K}$ *such that* $\operatorname{rng} f_1 \cap \operatorname{rng} f_2 \neq \emptyset$. *Then the diagram*

$$\operatorname{Con}_c A_1 \xrightarrow{\operatorname{Con}_c f_1} \operatorname{Con}_c B \xleftarrow{\operatorname{Con}_c f_2} \operatorname{Con}_c A_2 \tag{*}$$

has a lifting

$$C_1 \xrightarrow{g_1} D \xleftarrow{g_2} C_2.$$

in $\mathcal{L}$ *such that* $\operatorname{rng} g_1 \cap \operatorname{rng} g_2 \neq \emptyset$.

Proof. The pullback $A_0 = \{(x, y) \in A_1 \times A_2 \mid f_1(x) = f_2(y)\}$, of A_1 and A_2 with respect to f_1 and f_2, is a subalgebra of $A_1 \times A_2$. According to our assumption, it is nonempty. Further, let $p_i : A_0 \to A_i$ ($i \in \{1, 2\}$) be the projections. Then

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B \\ {\scriptstyle p_1}\uparrow & & \uparrow{\scriptstyle f_2} \\ A_0 & \xrightarrow{p_2} & A_2 \end{array}$$

is a commutative diagram in $\mathcal{K}$ indexed by the product of two 2-element chains. By Theorem 2.1, the diagram

$$\begin{array}{ccc} \operatorname{Con}_c A_1 & \xrightarrow{\operatorname{Con}_c f_1} & \operatorname{Con}_c B \\ {\scriptstyle \operatorname{Con}_c p_1}\uparrow & & \uparrow{\scriptstyle \operatorname{Con}_c f_2} \\ \operatorname{Con}_c A_0 & \xrightarrow{\operatorname{Con}_c p_2} & \operatorname{Con}_c A_2 \end{array}$$

has a lifting

$$\begin{array}{ccc} C_1 & \xrightarrow{g_1} & D \\ {\scriptstyle q_1}\uparrow & & \uparrow{\scriptstyle g_2} \\ C_0 & \xrightarrow{q_2} & C_2 \end{array}$$

in $\mathcal{L}$. It is easy to see that g_1 and g_2 form a lifting of (*). The condition $\operatorname{rng} g_1 \cap \operatorname{rng} g_2 \neq \emptyset$ follows from $C_0 \neq \emptyset$. $\square$

3. Symmetries of liftings

Let $\varphi : S \to T$ be a homomorphism of $(\vee, 0)$-semilattices and let τ be an automorphism of T. A τ*-symmetric lifting* of φ in a variety $\mathcal{K}$ consists of algebras $A_1, A_2, B_1, B_2 \in \mathcal{K}$, homomorphisms $f_{ij} : A_i \to B_j$, isomorphisms $\psi_i : \operatorname{Con}_c A_i \to S$ and $\tau_{ij} : \operatorname{Con}_c B_j \to T$ such that

$$\operatorname{rng}(f_{11} \times f_{12}) \cap \operatorname{rng}(f_{21} \times f_{22}) \neq \emptyset,$$

the diagram

$$\begin{array}{ccc} \operatorname{Con}_c A_i & \xrightarrow{\operatorname{Con}_c f_{ij}} & \operatorname{Con}_c B_j \\ {\scriptstyle \psi_i}\downarrow & & \downarrow{\scriptstyle \tau_{ij}} \\ S & \xrightarrow{\varphi} & T \end{array}$$

commutes for every $i, j \in \{1, 2\}$, and

$$\tau = \tau_{11}\tau_{21}^{-1}\tau_{22}\tau_{12}^{-1}.$$

Hence, a τ-symmetric lifting consists of four liftings bound together by the above equality and the condition that ψ_i serves both τ_{i1} and τ_{i2}. Let us remark, that a "simplified definition" using f_{11} and f_{21} only would not work. Indeed, if φ has liftings $f_{i1} : A_i \to B_1$, then τ_{i1} can be any isomorphism $\mathrm{Con}_c B_1 \to T$ with a suitable choice of ψ_i, so $\tau_{11}\tau_{21}^{-1}$ can be any automorphism on T. That's why four liftings are needed in order to define a nontrivial concept.

Now we can prove our main result.

Theorem 3.1. *Let $\mathcal{K}$ and $\mathcal{L}$ be finitely generated congruence-distributive varieties with $\operatorname{crit}(\mathcal{K}, \mathcal{L}) > \aleph_1$. Let $\varphi : S \to T$ be a homomorphism of finite $(\vee, 0)$-semilattices and let τ be an automorphism of T. Let φ have a τ-symmetric lifting in $\mathcal{K}$. Then φ also has a τ-symmetric lifting in $\mathcal{L}$.*

Proof. Let A_i, B_i, f_{ij}, ψ_i, τ_{ij} form a τ-symmetric lifting of φ in $\mathcal{K}$. In a finitely generated congruence-distributive variety, every algebra with a finite congruence lattice is finite. Hence, we can write Con instead of Con_c.

Consider the diagram

$$A_1 \xrightarrow{f_{11} \times f_{12}} B_1 \times B_2 \xleftarrow{f_{21} \times f_{22}} A_2.$$

By Theorem 2.2 the diagram

$$\mathrm{Con}\, A_1 \xrightarrow{\mathrm{Con}(f_{11} \times f_{12})} \mathrm{Con}(B_1 \times B_2) \xleftarrow{\mathrm{Con}(f_{21} \times f_{22})} \mathrm{Con}\, A_2.$$

has a lifting

$$C_1 \xrightarrow{g_1} D \xleftarrow{g_2} C_2$$

in $\mathcal{L}$ such that $\operatorname{rng} g_1 \cap \operatorname{rng} g_2 \neq \emptyset$. By Lemma 1.4 we can identify $\mathrm{Con}(B_1 \times B_2)$ with $\mathrm{Con}\, B_1 \times \mathrm{Con}\, B_2$ and $\mathrm{Con}(f_{i1} \times f_{i2})$ with $\mathrm{Con}\, f_{i1} \times \mathrm{Con}\, f_{i2}$. Hence, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Con}\, A_1 \arrow[r, "\mathrm{Con}\, f_{11} \times \mathrm{Con}\, f_{12}"] & \mathrm{Con}\, B_1 \times \mathrm{Con}\, B_2 & \mathrm{Con}\, A_2 \arrow[l, "\mathrm{Con}\, f_{21} \times \mathrm{Con}\, f_{22}"'] \\
\mathrm{Con}\, C_1 \arrow[u, "\xi_1"] \arrow[r, "\mathrm{Con}\, g_1"] & \mathrm{Con}\, D \arrow[u, "\Phi"] & \mathrm{Con}\, C_2 \arrow[u, "\xi_2"'] \arrow[l, "\mathrm{Con}\, g_2"']
\end{tikzcd}$$

with isomorphisms ξ_1, ξ_2, Φ and homomorphisms $g_1 : C_1 \to D$, $g_2 : C_2 \to D$ in $\mathcal{L}$. By Lemma 1.5, we can assume that D is a subdirect product of D_1 and D_2, $\mathrm{Con}\, D = \mathrm{Con}\, D_1 \times \mathrm{Con}\, D_2$ and

$$\Phi(\delta_1, \delta_2) = (\Phi_1(\delta_1), \Phi_2(\delta_2))$$

for some isomorphisms $\Phi_i : \mathrm{Con}\, D_i \to \mathrm{Con}\, B_i$ and for every $\delta_i \in \mathrm{Con}\, D_i$. Further, $g_i : C_i \to D \subseteq D_1 \times D_2$ can be decomposed as $g_i = g_{i1} \times g_{i2}$ for some homomorphisms $g_{ij} : C_i \to D_j$.

Now we define isomorphisms $\chi_i : \mathrm{Con}\, C_i \to S$ and $\sigma_{ij} : \mathrm{Con}\, D_j \to T$ (with $i, j \in \{1, 2\}$) by

$$\chi_i = \psi_i \xi_i,$$

$$\sigma_{ij} = \tau_{ij}\Phi_j.$$

We claim that C_i, D_i, g_{ij}, χ_i, σ_{ij} form a τ-symmetric lifting of φ in $\mathcal{L}$. First, we have

$$\sigma_{11}\sigma_{21}^{-1}\sigma_{22}\sigma_{12}^{-1} = \tau_{11}\Phi_1\Phi_1^{-1}\tau_{21}^{-1}\tau_{22}\Phi_2\Phi_2^{-1}\tau_{12}^{-1} = \tau_{11}\tau_{21}^{-1}\tau_{22}\tau_{12}^{-1} = \tau.$$

Next, we need to check the commutativity of the diagram

$$\begin{tikzcd} \operatorname{Con} C_i \arrow[r, "\operatorname{Con} g_{ij}"] \arrow[d, "\chi_i"'] & \operatorname{Con} D_j \arrow[d, "\sigma_{ij}"'] \\ S \arrow[r, "\varphi"] & T \end{tikzcd}$$

Let $\gamma \in \operatorname{Con} C_i$. Using the commutativity of the previous diagrams,

$$(\operatorname{Con} f_{i1}(\xi_i(\gamma)), \operatorname{Con} f_{i2}(\xi_i(\gamma))) = \Phi(\operatorname{Con} g_i(\gamma)) = \Phi(\operatorname{Con} g_{i1}(\gamma), \operatorname{Con} g_{i2}(\gamma)),$$

so

$$(\operatorname{Con} f_{i1}(\xi_i(\gamma)), \operatorname{Con} f_{i2}(\xi_i(\gamma))) = (\Phi_1(\operatorname{Con} g_{i1}(\gamma)), \Phi_2(\operatorname{Con} g_{i2}(\gamma))),$$

and hence

$$\operatorname{Con} f_{ij}(\xi_i(\gamma)) = \Phi_j(\operatorname{Con} g_{ij}(\gamma)).$$

Then

$$\sigma_{ij} \operatorname{Con} g_{ij}(\gamma) = \tau_{ij}\Phi_j \operatorname{Con} g_{ij}(\gamma) = \tau_{ij} \operatorname{Con} f_{ij}\xi_i(\gamma) = \varphi\psi_i\xi_i(\gamma) = \varphi\chi_i(\gamma).$$

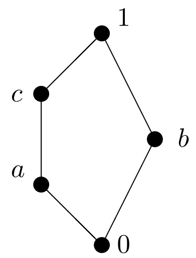
Finally,

$$\operatorname{rng}(g_{11} \times g_{12}) \cap \operatorname{rng}(g_{21} \times g_{22}) \neq \emptyset,$$

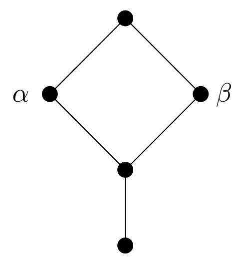
because $\operatorname{rng} g_1 \cap \operatorname{rng} g_2 \neq \emptyset$. □

4. Two examples

In this section we present two examples illustrating the use of Theorem 3.1. Let $\mathcal{N}_5$ be the variety of bounded lattices generated by the 5-element non-modular lattice N_5. (See below.) The smallest and the largest elements 0 and 1 are considered as nullary operations.

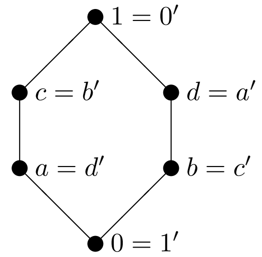


N_5

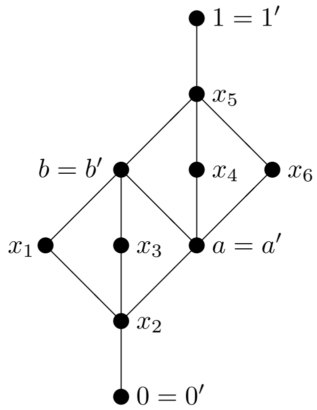


$T \cong \operatorname{Con} N_5 \cong \operatorname{Con} N_6 \cong \operatorname{Con} M$

Further, let $\mathcal{N}_6$ and $\mathcal{M}$ be varieties of bounded lattices with an additional unary operation $'$ generated by the algebras N_6 and M depicted below. In the algebra M, the unary operation on the elements x_i is defined by $x_i' = x_{i+1}$ for $1 \leq i \leq 5$ and $x_6' = x_1$.



N_6



M

All three varieties are clearly finitely generated and congruence-distributive. The congruence lattices of N_5, N_6 and M are isomorphic to the lattice T depicted above. The subdirectly irreducible members of $\mathcal{N}_5$, $\mathcal{N}_6$ and $\mathcal{M}$ can be easily determined by Jónsson's Lemma. Up to isomorphism,

$$\mathrm{SI}(\mathcal{N}_5) = \{N_5, C_2\},$$

$$\mathrm{SI}(\mathcal{N}_6) = \{N_6, N_2\},$$

$$\mathrm{SI}(\mathcal{M}) = \{M, D_2\},$$

where C_2, N_2 and D_2 are subalgebras of N_5, N_6 and M respectively, each with the underlying set $\{0, 1\}$.

Lemma 4.1. *Let $\varphi : T \to T$ be the identity homomorphism and let τ be the unique automorphism on T interchanging α and β. Then φ has a τ-symmetric lifting in $\mathcal{N}_6$, but not in $\mathcal{N}_5$.*

Proof. We set $A_1 = A_2 = B_1 = B_2 = N_6$, $f_{11} = f_{12} = f_{21} = \mathrm{id}_{N_6}$ and let f_{22} be the vertical symmetry on N_6, hence $f_{22}(0) = 0$, $f_{22}(a) = b$, $f_{22}(b) = a$, $f_{22}(c) = d$, $f_{22}(d) = c$, $f_{22}(1) = 1$. It is easy to check that f_{22} is a homomorphism. The lattice $\mathrm{Con}\, N_6$ is isomorphic to T, the two coatoms (in the block description) are $\gamma = (0ac)(bd1)$ and $\delta = (0bd)(ac1)$. We define $\psi_1, \psi_2, \tau_{11}, \tau_{12}$ and τ_{21} all equal to the unique isomorphism $\rho : \mathrm{Con}\, N_6 \to T$ with $\rho(\gamma) = \alpha$, $\rho(\delta) = \beta$. Finally, let $\tau_{22} : \mathrm{Con}\, N_6 \to T$ be defined by $\tau_{22}(\gamma) = \beta$, $\tau_{22}(\delta) = \alpha$. It is easy to check that we have a τ-symmetric lifting of φ in $\mathcal{N}_6$.

Now we show that φ has no τ-symmetric lifting in $\mathcal{N}_5$. For contradiction, let A_i, B_j, f_{ij}, ψ_i, τ_{ij} form such a lifting. Since N_5 is the only algebra in $\mathcal{N}_5$ whose congruence lattice is isomorphic to T, we can assume that $A_i = B_j = N_5$ for every i, j. There is only one homomorphism $f\colon N_5 \to N_5$ such that $\operatorname{Con} f$ is bijective, namely the identity mapping. Hence, $f_{ij} = \mathrm{id}_{N_5}$ and $\operatorname{Con} f_{ij} = \mathrm{id}_{\operatorname{Con} N_5}$ for every i, j. Now the commutativity of the diagram

$$\begin{array}{ccc} \operatorname{Con} A_i & \xrightarrow{\operatorname{Con} f_{ij}} & \operatorname{Con} B_j \\ {\scriptstyle\psi_i}\big\downarrow & & {\scriptstyle\tau_{ij}}\big\downarrow \\ S & \xrightarrow{\varphi} & T \end{array}$$

means that $\tau_{ij} = \psi_i$ for every i, j, hence

$$\tau_{11}\tau_{21}^{-1}\tau_{22}\tau_{12}^{-1} = \psi_1\psi_2^{-1}\psi_2\psi_1^{-1} = \mathrm{id}_T \neq \tau,$$

a contradiction. □

Consequence 4.2. $\operatorname{crit}(\mathcal{N}_6, \mathcal{N}_5) \leq \aleph_1$.

Our second example is slightly more complicated. Let $S = \{0, \kappa, \lambda, 1\}$ be a 4-element Boolean lattice, regarded as a $(\vee, 0)$-semilattice. Let $\varphi\colon S \to T$ be defined by $\varphi(0) = 0$, $\varphi(\kappa) = \alpha$, $\varphi(\lambda) = \beta$ and $\varphi(1) = 1$. It is a homomorphism of $(\vee, 0)$-semilattices, although the meet operation is not preserved. Let $\tau\colon T \to T$ be the same automorphism as in our first example (interchanging α and β).

Lemma 4.3. *φ has a τ-symmetric lifting in $\mathcal{N}_5$, but not in $\mathcal{M}$.*

Proof. Let $A_1 = A_2$ be a 3-element chain $0 < d < 1$. The congruence lattice of this algebra is isomorphic to S, the two nontrivial congruences are $\alpha_0 = (0d)(1)$ and $\alpha_1 = (0)(d1)$. We define isomorphisms $\psi_i\colon \operatorname{Con} A_i \to S$ by $\psi_i(\alpha_0) = \kappa$, $\psi_i(\alpha_1) = \lambda$ $(i = 1, 2)$.

Further, let $B_1 = B_2 = N_5$. We define homomorphisms $f_{ij}\colon A_i \to B_j$ by $f_{11}(d) = f_{12}(d) = f_{21}(d) = a$ and $f_{22}(d) = b$. The lattice $\operatorname{Con} N_5$ is isomorphic to T, the two coatoms are $\gamma = (0ac)(b1)$ and $\delta = (0b)(ac1)$. We define τ_{11}, τ_{12} and τ_{21} all equal to the unique isomorphism $\rho\colon \operatorname{Con} N_5 \to T$ with $\rho(\gamma) = \alpha$, $\rho(\delta) := \beta$. Finally, let $\tau_{22}\colon \operatorname{Con} N_5 \to T$ be defined by $\tau_{22}(\gamma) = \beta$, $\tau_{22}(\delta) = \alpha$. It is easy to check that we have a τ-symmetric lifting of φ in $\mathcal{N}_5$.

Now we show that φ has no τ-symmetric lifting in $\mathcal{M}$. For contradiction, let A_i, B_j, f_{ij}, ψ_i, τ_{ij} form such a lifting. Since M is the only algebra in $\mathcal{M}$ whose congruence lattice is isomorphic to T, we can assume that $B_1 = B_2 = M$. The two coatoms of $\operatorname{Con} M$ are γ with equivalence classes $\{1\}$ and $M \setminus \{1\}$, and δ whose classes are $\{0\}$ and $M \setminus \{0\}$. The monolith (smallest nonzero congruence) of M collapses the whole middle part of M.

The algebras A_i must have congruence lattices isomorphic to S, which means that they must be subdirect products of two simple algebras. The only simple algebra in $\mathcal{M}$ is D_2, so A_i are (isomorphic to) subalgebras of

$D_2 \times D_2$. There are two such algebras: the product $D_2 \times D_2$ and a 3 element chain $D_3 = \{0, e, 1\}$ with the identity unary operation $'$. It is easy to see that any homomorphism lifting φ must be injective (as $\varphi^{-1}(0) = \{0\}$). Since there are no injective homomorphisms $D_2 \times D_2 \to M$, the case $A_i = D_2 \times D_2$ is excluded, and the only possibility is $A_1 = A_2 = D_3$.

The homomorphisms $f_{ij} : D_3 \to M$ can map the element e into a or b. In both cases we have $\operatorname{Con} f_{ij}(\theta(0, e)) = \gamma$, $\operatorname{Con} f_{ij}(\theta(1, e)) = \delta$. From the commutativity of the lifting diagram we obtain that

$$\tau_{ij}(\gamma) = \varphi\psi_i(\theta(0, e)), \quad \tau_{ij}(\delta) = \varphi\psi_i(\theta(1, e)),$$

which implies that $\tau_{11} = \tau_{12}$ and $\tau_{21} = \tau_{22}$. Then obviously

$$\tau_{11}\tau_{21}^{-1}\tau_{22}\tau_{12}^{-1} = \mathrm{id}_T \neq \tau,$$

a contradiction. □

Consequence 4.4. $\operatorname{crit}(\mathcal{N}_5, \mathcal{M}) \leq \aleph_1$.

It is worth noticing that the variety $\mathcal{N}_5$ appears in both our examples, but plays two opposite roles.

5. Lower bound

This section is devoted to the proof that $\operatorname{crit}(\mathcal{N}_6, \mathcal{N}_5) > \aleph_0$, $\operatorname{crit}(\mathcal{N}_5, \mathcal{M}) > \aleph_0$, so that the upper bounds in the previous section are tight. The two proofs are very similar, we shall work out the details only in the first case.

First we need to look closer at the congruence lattices of finite algebras in $\mathcal{N}_6$, $\mathcal{N}_5$ and $\mathcal{M}$.

For an algebraic lattice L let $\mathrm{M}(L)$ denote the set of all completely $\wedge$-irreducible elements of L. It is well known that $\alpha \in \mathrm{M}(\operatorname{Con} A)$ if and only if the quotient algebra A/α is subdirectly irreducible, for any congruence α of an algebra A.

Lemma 5.1. *For every $A \in \mathcal{N}_6 \cup \mathcal{N}_5 \cup \mathcal{M}$, the ordered set* $\mathrm{M}(\operatorname{Con} A)$ *is a disjoint union of antichains P_0 and P_1 such that for every $p \in P_1$ there are exactly two $q \in P_0$ with $p < q$.*

Proof. For every $\alpha \in \mathrm{M}(\operatorname{Con} A)$, the algebra A/α is subdirectly irreducible. The set P_0 consists of those $\alpha \in \operatorname{Con} A$ with A/α isomorphic to N_2 or C_2 or D_2. The set P_1 contains those $\alpha \in \operatorname{Con} A$ with A/α isomorphic to N_6 or N_5 or M. Our claim follows from the fact that for every $\alpha \in P_1$ the lattice $\{\beta \in \operatorname{Con} A \mid \beta \geq \alpha\}$ is isomorphic to T. □

Now we state the converse to the previous Lemma for finite algebras in $\mathcal{N}_5$. A similar assertion is true for $\mathcal{N}_6$ and $\mathcal{M}$. Let h_0 and h_1 be the two surjective homomorphisms $N_5 \to C_2$, with $\operatorname{Ker} h_0 = (0ac)(b1)$, $\operatorname{Ker} h_1 = (0b)(ac1)$.

Theorem 5.2. *Let L be a finite distributive lattice such that $\mathrm{M}(L)$ is a disjoint union of antichains P_0 and P_1 such that for every $p \in P_1$ there are exactly two $q \in P_0$ with $p < q$. For $p \in \mathrm{M}(L)$ we define*

$$T_p = \begin{cases} C_2 & \text{if } p \in P_0 \\ N_5 & \text{if } p \in P_1. \end{cases}$$

Further, let $\sqsubseteq$ be any linear order on P_0. For every $p \in P_1$ we have $q_1, q_2 \in P_0$ with $p < q_1$, $p < q_2$, $q_1 \sqsubset q_2$ and we denote $f_{p,q_1} = h_0$, $f_{p,q_2} = h_1$. Let B be the limit of the diagram $(T_p, f_{p,q}; p < q \text{ in } \mathrm{M}(L))$, that is the subalgebra of $\prod_{p \in \mathrm{M}(L)} T_p$ consisting of all tuples $x = (x_p)_{p \in \mathrm{M}(L)}$ satisfying $x_q = f_{p,q}(x_p)$ whenever $p < q$.

Then $\operatorname{Con} B$ is isomorphic to L and the isomorphism $\varphi : L \to \operatorname{Con} B$ assigns to every $p \in \mathrm{M}(L)$ the kernel of the natural projection $\pi_p : B \to T_p$.

For the proof of Theorem 5.2 see [12], Theorems 3.1 and 2.4. (Actually, slightly more is proved in [12]: the homomorphisms $f_{p,q_1} \in \{h_0, h_1\}$ and $f_{p,q_2} \in \{h_0, h_1\}$ can be chosen arbitrarily (but they must be different), without taking into account any order on P_0.)

In the sequel we use the $(1, \wedge)$-homomorphisms $\varphi^{\leftarrow}$ defined in the introduction.

Lemma 5.3. *Let $f : A_0 \to A_1$ be a homomorphism of finite algebras in $\mathcal{N}_6$. Let $\alpha \in \operatorname{Con} A_1$ with $A_1/\alpha \cong N_2$. Then $A_0/\operatorname{Con} f^{\leftarrow}(\alpha) \cong N_2$.*

Proof. The algebra $A_0/\operatorname{Con} f^{\leftarrow}(\alpha)$ is isomorphic to a subalgebra of A_1/α via the natural embedding $x/\operatorname{Con} f^{\leftarrow}(\alpha) \mapsto f(x)/\alpha$. Since N_2 has no proper subalgebras (recall that 0 and 1 are nullary operations), $A_0/\operatorname{Con} f^{\leftarrow}(\alpha)$ is isomorphic to N_2. ☐

Lemma 5.4. *Let $f : A_0 \to A_1$ be a homomorphism of finite algebras in $\mathcal{N}_6$. Let $\alpha \in \operatorname{Con} A_1$ with $A_1/\alpha \cong N_6$. Let β_1, β_2 be the two congruences of A_1 above α with $A_1/\beta_1 \cong A_1/\beta_2 \cong N_2$. Then one of the following cases occurs:*

(i) *$A_0/\operatorname{Con} f^{\leftarrow}(\alpha) \cong N_6$ and $\operatorname{Con} f^{\leftarrow}(\beta_1) \neq \operatorname{Con} f^{\leftarrow}(\beta_2)$;*
(ii) *$\operatorname{Con} f^{\leftarrow}(\alpha) = \operatorname{Con} f^{\leftarrow}(\beta_1) \wedge \operatorname{Con} f^{\leftarrow}(\beta_2)$.*

Proof. Similarly as in 5.3, the algebra $A_0/\operatorname{Con} f^{\leftarrow}(\alpha)$ is isomorphic to a subalgebra of A_1/α. The subalgebras of N_6 are N_6, N_2, $\{0, 1, a, d\}$ and $\{0, 1, b, c\}$. (The last two are isomorphic to the 4-element Boolean algebra $N_2 \times N_2$.) If $A_0/\operatorname{Con} f^{\leftarrow}(\alpha)$ is isomorphic to N_2, then $\operatorname{Con} f^{\leftarrow}(\alpha)$ is a coatom of $\operatorname{Con} A_0$. By Lemma 5.3, $\operatorname{Con} f^{\leftarrow}(\beta_1) = \operatorname{Con} f^{\leftarrow}(\beta_2) = \operatorname{Con} f^{\leftarrow}(\alpha)$ and (ii) holds.

Suppose now that $A_0/\operatorname{Con} f^{\leftarrow}(\alpha)$ is isomorphic to $N_2 \times N_2$. We claim that the case (ii) occurs. Since the $\operatorname{Con}(N_2 \times N_2)$ is a 4-element Boolean lattice, we have $\operatorname{Con} f^{\leftarrow}(\alpha) = \gamma_1 \wedge \gamma_2$ for some coatoms $\gamma_1, \gamma_2 \in \operatorname{Con} A_0$. Since $\operatorname{Con} f^{\leftarrow}$ is order-preserving and γ_1, γ_2 are the only coatoms of $\operatorname{Con} A_0$ above $\operatorname{Con} f^{\leftarrow}(\alpha)$, we have $\operatorname{Con} f^{\leftarrow}(\beta_i) \in \{\gamma_1, \gamma_2\}$ for $i = 0, 1$. It remains to prove that $\operatorname{Con} f^{\leftarrow}(\beta_1) \neq \operatorname{Con} f^{\leftarrow}(\beta_2)$. Let h denote the natural projection

$A_1 \to A_1/\alpha$. Let h_1, h_2 be the two surjective homomorphisms $A_1/\alpha \to N_2$. Then $\operatorname{Con} f^{\leftarrow}(\alpha) = \operatorname{Ker} hf$, $\operatorname{Con} f^{\leftarrow}(\beta_i) = \operatorname{Ker} h_i hf$. It is easy to see that $\operatorname{Ker} h_1 hf = \operatorname{Ker} h_2 hf$ implies $\operatorname{rng} hf = \{0,1\}$, but then $A_0/\operatorname{Ker} hf$ is a 2-element algebra, a contradiction. So, $\operatorname{Ker} h_1 hf \neq \operatorname{Ker} h_2 hf$.

Finally, if $A_0/\operatorname{Con} f^{\leftarrow}(\alpha)$ is isomorphic to N_6, then we use the same argument as above to prove that $\operatorname{Con} f^{\leftarrow}(\beta_1) \neq \operatorname{Con} f^{\leftarrow}(\beta_2)$, so (i) holds. □

A similar statement is true for homomorphisms in $\mathcal{N}_5$ and $\mathcal{M}$. The main difference is that both N_5 and M contain a 4-element chain as a subalgebra, and this subalgebra is a subdirect product of three subdirectly irreducible algebras. That leads to an additional case in 5.4. We skip the details here.

Theorem 5.5. $\operatorname{crit}(\mathcal{N}_6, \mathcal{N}_5) = \aleph_1$.

Proof. The inequality $\operatorname{crit}(\mathcal{N}_6, \mathcal{N}_5) \leq \aleph_1$ has been established as Consequence 4.2. To prove $\operatorname{crit}(\mathcal{N}_6, \mathcal{N}_5) > \aleph_0$ we use Theorem 2.1. Consider the diagram

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \dots \xrightarrow{f_{n-1}} A_n$$

of finite algebras in $\mathcal{N}_6$. We need to show that the corresponding semilattice diagram

$$\operatorname{Con}_c A_0 \xrightarrow{\operatorname{Con}_c f_0} \operatorname{Con}_c A_1 \xrightarrow{\operatorname{Con}_c f_1} \operatorname{Con}_c A_2 \dots \xrightarrow{\operatorname{Con}_c f_{n-1}} \operatorname{Con}_c A_n$$

has a lifting in $\mathcal{N}_5$. So, we shall construct a diagram

$$B_0 \xrightarrow{g_0} B_1 \xrightarrow{g_1} B_2 \dots \xrightarrow{g_{n-1}} B_n$$

in $\mathcal{N}_5$ together with isomorphisms $\varphi_i : \operatorname{Con}_c B_i \to \operatorname{Con}_c A_i$ such that the diagram

$$\begin{array}{ccc}
\operatorname{Con}_c B_i & \xrightarrow{\operatorname{Con}_c g_i} & \operatorname{Con}_c B_{i+1} \\
{\scriptstyle \varphi_i}\big\downarrow & & {\scriptstyle \varphi_{i+1}}\big\downarrow \\
\operatorname{Con}_c A_i & \xrightarrow{\operatorname{Con}_c f_i} & \operatorname{Con}_c A_{i+1}
\end{array}$$

commutes for every $i = 0, \dots, n-1$.

Let us denote $L_i = \operatorname{Con}_c A_i$, $k_i = \operatorname{Con}_c f_i$. By Lemma 5.1, the ordered set $\mathrm{M}(L_i)$ is a union of antichains P_0^i and P_1^i such that for every $p \in P_1^i$ there are exactly two $q \in P_0^i$ with $p < q$. By Lemma 5.3 we have $k_i^{\leftarrow}(q) \in P_0^i$ whenever $q \in P_0^{i+1}$.

Before we can define algebras B_i and homomorphisms g_i, we need to construct linear orders $\sqsubseteq_i$ on P_0^i such that $k_i^{\leftarrow}(q_1) \sqsubseteq_i k_i^{\leftarrow}(q_2)$ whenever $q_1 \sqsubseteq_{i+1} q_2$.

We proceed by induction. Let $\sqsubseteq_0$ be any linear order on P_0^0. Now let $i \geq 0$ and suppose we have defined $\sqsubseteq_i$. Consider the following binary relation ρ_i on P_0^{i+1}:

$$(x, y) \in \rho_i \text{ if } x = y \text{ or } k_i^{\leftarrow}(x) \sqsubset_i k_i^{\leftarrow}(y).$$

It is easy to see that ρ_i is a partial order. Let $\sqsubseteq_{i+1}$ be any extension of ρ_i into a linear order. It is clear that $\sqsubseteq_{i+1}$ satisfies the requirements.

Let B_i be the algebras constructed in Theorem 5.2 together with the isomorphisms $\varphi_i : \operatorname{Con} B_i \to L_i$.

Now we define homomorphisms $g_i : B_i \to B_{i+1}$. We consider the elements of B_i in the form $\overline{x} = (x_p)_{p\in \mathrm{M}(L_i)}$.

According to Lemmas 5.3 and 5.4 the set $\mathrm{M}(L_{i+1})$ is a disjoint union of the following two sets:

$$K_1 = \{p \in \mathrm{M}(L_{i+1}) \mid p \in P_0^{i+1}, k_i^{\leftarrow}(p) \in P_0^i \text{ or } p \in P_1^{i+1}, k_i^{\leftarrow}(p) \in P_1^i\},$$

$$K_2 = \{p \in P_1^{i+1} \mid k_i^{\leftarrow}(p) = k_i^{\leftarrow}(p_1) \wedge k_i^{\leftarrow}(p_2) \text{ for } p_1, p_2 \in P_0^{i+1},\ p < p_1, p_2\}.$$

We set

$$g_i(\overline{x}) = \overline{y} = (y_p)_{p\in \mathrm{M}(L_{i+1})}$$

where the elements $y_p \in T_p$ are defined as follows. If $p \in K_1$ then

$$y_p = x_{k_i^{\leftarrow}(p)}.$$

If $p \in K_2$ then we denote $q_1 = k_i^{\leftarrow}(p_1)$, $q_2 = k_i^{\leftarrow}(p_2)$. We can assume $p_1 \sqsubset_{i+1} p_2$, which implies $q_1 \sqsubseteq_i q_2$ and define

$$y_p = \begin{cases} 0 & \text{if } x_{q_1} = x_{q_2} = 0, \\ a & \text{if } x_{q_1} = 0,\ x_{q_2} = 1, \\ b & \text{if } x_{q_1} = 1,\ x_{q_2} = 0, \\ 1 & \text{if } x_{q_1} = x_{q_2} = 1. \end{cases}$$

For the correctness of our definition we need to prove that $\overline{y} \in B_{i+1}$. Clearly $y_p \in T_p$. Now let $p \in P_1^{i+1}$, $p_1, p_2 \in P_0^{i+1}$ with $p < p_1, p_2$, $p_1 \sqsubset_{i+1} p_2$. Denote $k_i^{\leftarrow}(p) = q$, $k_i^{\leftarrow}(p_1) = q_1$, $k_i^{\leftarrow}(p_2) = q_2$.

Let $p \in K_1$. Since $\overline{x} \in B_i$, we obtain that $f_{p,p_1}(y_p) = h_0(x_q) = f_{q,q_1}(x_q) = x_{q_1} = y_{p_1}$, and similarly, $f_{p,p_2}(y_p) = y_{p_2}$.

The second possibility is $p \in K_2$. We have the following subcases.

(a) If $x_{q_1} = x_{q_2} = 0$, then $y_p = y_{p_1} = y_{p_2} = 0$.
(b) If $x_{q_1} = 0$, $x_{q_2} = 1$, then $y_p = a$, $y_{p_1} = 0$, $y_{p_2} = 1$.
(c) If $x_{q_1} = 1$, $x_{q_2} = 0$, then $y_p = b$, $y_{p_1} = 1$, $y_{p_2} = 0$.
(d) If $x_{q_1} = x_{q_2} = 1$, then $y_p = y_{p_1} = y_{p_2} = 1$.

In each case, $y_{p_1} = h_0(y_p) = f_{p,p_1}(y_p)$, $y_{p_2} = h_1(y_p) = f_{p,p_2}(y_p)$.

So, g_i is well defined. To show that g_i is a homomorphism it suffices to show that the composition $\pi_p g_i$ (where π_p is the natural projection $B_{i+1} \to T_p$) is a homomorphism for every $p \in \mathrm{M}(L_{i+1})$. However, if $p \in K_1$ then $\pi_p g_i$ is equal to the projection π_q ($q = k^{\leftarrow}(p)$). If $p \in K_2$ then $\pi_p g_i$ is the composition of $\pi_{q_1} \times \pi_{q_2}$ and an embedding $C_2 \times C_2 \to N_5$. In all cases, $\pi_p g_i$ is a homomorphism.

It remains to show that $\varphi_{i+1} \operatorname{Con} g_i = k_i \varphi_i$. Equivalently (by Lemma 1.3), we show that

$$(\operatorname{Con} g_i)^{\leftarrow} \varphi_{i+1}^{\leftarrow} = \varphi_i^{\leftarrow} k_i^{\leftarrow}.$$

Since all maps in this equality are $\wedge$-preserving, it suffices to prove the equality for all $p \in \mathrm{M}(\operatorname{Con} A_{i+1}) = \mathrm{M}(L_{i+1})$. Since φ_{i+1} is an isomorphism, we have

$\varphi_{i+1}^{\leftarrow} = \varphi_{i+1}^{-1}$, so that according to Lemma 5.2, $\varphi_{i+1}^{\leftarrow}(p) = \ker \pi_p$, where π_p is the natural projection $B_{i+1} \to T_p$. Further, let $\overline{u}, \overline{v} \in B_i$. Then

$$(\overline{u}, \overline{v}) \in (\operatorname{Con} g_i)^{\leftarrow}(\ker \pi_p) \text{ iff } (g_i(\overline{u}), g_i(\overline{v})) \in \ker \pi_p \text{ iff } g_i(\overline{u})_p = g_i(\overline{v})_p.$$

Now we distinguish the same two cases as before. Let $q = k_i^{\leftarrow}(p)$.

First, let $p \in K_1$. Then

$$g_i(\overline{u})_p = g_i(\overline{v})_p \text{ iff } \overline{u}_q = \overline{v}_q \text{ iff } (\overline{u}, \overline{v}) \in \ker \pi_q = \varphi_i^{\leftarrow}(q).$$

So, $(\operatorname{Con} g_i)^{\leftarrow}\varphi_{i+1}^{\leftarrow}(p) = \varphi_i^{\leftarrow} k_i^{\leftarrow}(p)$

Second, let $p \in K_2$, $k_i^{\leftarrow}(p_1) = q_1$, $k_i^{\leftarrow}(p_2) = q_2$. By Lemma 5.4, $q = q_1 \wedge q_2$. It is easy to see from the definition of g_i that $g_i(\overline{u})_p$ is determined uniquely by $\overline{u}_{q_1}$ and $\overline{u}_{q_2}$. Hence,

$$g_i(\overline{u})_p = g_i(\overline{v})_p \text{ iff } \overline{u}_{q_1} = \overline{v}_{q_1} \text{ and } \overline{u}_{q_2} = \overline{v}_{q_2}.$$

This is equivalent with

$$(\overline{u}, \overline{v}) \in \ker \pi_{q_1} \cap \ker \pi_{q_2} = \varphi_i^{\leftarrow}(q_1) \wedge \varphi_i^{\leftarrow}(q_2) = \varphi_i^{\leftarrow}(q),$$

and we have the same conclusion as in the first case. This completes the proof. $\square$

6. Possible variations and generalizations

Our paper is only a first step in a systematic investigation of the critical point $\aleph_1$. Our ideas can probably be modified and generalized in several ways.

Instead of considering automorphisms $\tau : T \to T$ one can consider automorphisms $\psi : S \to S$. To define an alternative concept of symmetry of liftings we consider isomorphisms ψ_{ij}, τ_i, commutativity of diagrams

$$\begin{array}{ccc} \operatorname{Con} A_i & \xrightarrow{\operatorname{Con} f_{ij}} & \operatorname{Con} B_j \\ {\scriptstyle \psi_{ij}}\Big\downarrow & & {\scriptstyle \tau_j}\Big\downarrow \\ S & \xrightarrow{\varphi} & T \end{array}$$

and the equality

$$\psi = \psi_{22}\psi_{21}^{-1}\psi_{11}\psi_{12}^{-1}.$$

Formally, the two concepts of symmetry seem nonequivalent. However, we do not have an example distinguishing them. The inequalities $\operatorname{crit}(\mathcal{N}_6, \mathcal{N}_5) \leq \aleph_1$ and $\operatorname{crit}(\mathcal{N}_5, \mathcal{M}) \leq \aleph_1$ could also be proved using this alternative concept.

As another possible generalization, one can consider a whole subgroup $G \subseteq \operatorname{Aut} T$ instead of a single automorphism τ. It seems possible that there is a τ-symmetric lifting of φ for every $\tau \in G$, but not simultaneously for all such τ. Again, we do not go into details here, as we do not have a suitable illustrating example.

Next, the concept of τ-symmetric lifting should be generalized in a way that considers liftings of any finite diagrams indexed by a chain, not just liftings of a single homomorphisms φ. The precise definition is not clear, and again, no suitable example is known. This is connected with the fact that in all known cases with $\mathrm{crit}(\mathcal{K}, \mathcal{L}) = \aleph_1$ there exists a diagram indexed by a square, which is liftable in $\mathcal{K}$ and not in $\mathcal{L}$. (Notice that Theorem 2.1 for $\aleph_1$ only assumes the existence of a diagram indexed by a product of two finite chains.)

Finally, let us remark that the first published example of $\mathrm{crit}(\mathcal{K}, \mathcal{L}) = \aleph_1$ in [4] is based on a rather different principle, which deserves a further investigation, too.

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