The strong endomorphism kernel property for modular p-algebras and for distributive lattices

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Abstract. We study strong endomorphism kernel property (SEKP) for some classes of universal algebras. Using Katriňák-Mederly triple construction we prove a universal equivalent condition under which a modular p-algebra has SEKP. As a consequence, we characterize distributive lattices with top element which enjoy SEKP. Using Priestley duality we also characterize unbounded distributive lattices which have SEKP.

1. Introduction

The concept of the (strong) endomorphism kernel property for an universal algebra has been introduced by Blyth, Fang and Silva as follows. (See [2] and [3].)

Definition 1.1. *An algebra A has the* endomorphism kernel property *(EKP) if every congruence relation on A different from the universal congruence $\iota_A = A \times A$ is the kernel of an endomorphism on A.*

Let $\theta \in \mathrm{Con}(A)$ be a congruence on A. We say that a mapping $f: A \to A$ is *compatible* with θ if $a \equiv b(\theta)$ implies $f(a) \equiv f(b)(\theta)$. An endomorphism of A is called *strong*), if it is compatible with every congruence $\theta \in \mathrm{Con}(A)$.

The compatibility of functions (of any arity) with congruences has been widely studied in various contexts. We refer to the monograph [12] for an overview. Compatible functions are sometimes called "congruence preserving functions" or "functions with substitution property".

Definition 1.2. *An algebra A has the* strong endomorphism kernel property *(SEKP) if every congruence relation on A different from the universal congruence ι_A is the kernel of a strong endomorphism of A.*

The exception for the universal congruence ι_A appears in the above definitions with the purpose that algebras with two or more nullary operations have a chance to satisfy the conditions. It is not necessary for algebras with

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one-element subagebras, such as distributive lattices in the second half of this paper.

Blyth and Silva considered the case of Ockham algebras and in particular of MS-algebras and provided a full characterization of MS-algebras having SEKP. For instance, a Boolean algebra has SEKP if and only if it has exactly two elements. Further Blyth, J. Fang and Wang in [4] proved a full characterization of finite distributive double p-algebras and finite double Stone algebras having SEKP. SEKP for distributive p-algebras and Stone algebras has been studied and fully characterized by G. Fang and J. Fang in [7]. J. Fang and Sun fully described semilattices with SEKP in [8]. The main approach in papers [3], [4] and [7] is to regard algebras in question as Ockham algebras and use the Priestley duality.

The original paper [3] of Blyth and Silva contains one additional assumption, namely that all considered algebras contain two nullary operations (denoted by 0 and 1, $0 \neq 1$). We do not keep this assumption in our paper and we would like to remark that in our more general context some of the results from [3] are no longer true. For instance, Corollary 1 in [3] says that a finite algebra with SEKP is directly indecomposable, while it is easy to check that the 4-element lattice $\{0,1\}^2$ has SEKP. Also, all one-element algebras trivially have SEKP.

2. SEKP and modular p-algebras

We shall use Katriňák-Mederly triple construction for Stone algebras and modular p-algebras (see [13]) in this section. This approach enables us to prove some general characterization of SEKP for modular p-algebras and using this, to translate results of the paper [3] or [7] to the case of distributive $\{1\}$-lattices ($\{0\}$-lattices).

A (modular, distributive) *p-algebra* is an algebra $L = (L; \vee, \wedge, ^*, 0, 1)$ of type (2,2,1,0,0), where $(L; \vee, \wedge, 0, 1)$ is a bounded (modular, distributive) lattice and, for every $a \in L$, the element a^* is a *pseudocomplement* of a, i.e. $x \leq a^*$ if and only if $x \wedge a = 0$. The standard results on p-algebras may be found in [10].

An $S-$*algebra* is a p-algebra satisfying the Stone identity $x^* \vee x^{**} = 1$. The $S-$algebra L is a *Stone algebra*, if it is distributive.

Every modular p-algebra possesses two important parts: the Boolean algebra of *closed* elements $(S(L); +, \wedge, ^*, 0, 1)$, where $x \in S(L)$ if and only if $x = x^{**}$ ($S(L)$ stands for a skeleton of L) and

$$x^{**} + y^{**} = (x \vee y)^{**}$$

for every $x, y \in L$; the second key subset of L is the filter $D(L)$ of *dense* elements, that means, x is dense if and only if $x^* = 0$. We regard $D(L)$ as a lattice with 1, that is, an algebra of type $(2, 2, 0)$. A modular p-algebra L is uniquely determined (up to isomorphism) by its *associated triple* $(S(L), D(L), \varphi(L))$, where $\varphi(L)$ is a mapping of $S(L)$ into the lattice of all filters of $D(L)$ defined by

$$\varphi(L) : a \mapsto D(L) \cap [a^*)$$

for every $a \in S(L)$ (see [13, Theorems 2 and 4]).

The triple construction of a modular p-algebra L enables to decompose any congruence relation $\theta \in \mathrm{Con}(L)$ and any endomorphism $f \in \mathrm{End}(L)$ of a modular p-algebra L as follows.

Theorem 2.1. (See [13].) *Let L be a modular p-algebra. For every $\theta \in \mathrm{Con}(L)$ the restrictions $\theta_S = \theta \restriction S(L)$ and $\theta_D = \theta \restriction D(L)$ are congruences on $S(L)$ and $D(L)$, respectively. A pair $(\theta_1, \theta_2) \in \mathrm{Con}(S(L)) \times \mathrm{Con}(D(L))$ is equal to (θ_S, θ_D) for some congruence $\theta \in \mathrm{Con}(L)$ iff*

$$a \equiv 0(\theta_1) \text{ implies } x \equiv 1(\theta_2) \text{ for all } x \in \varphi(L)(a).$$

The pair (θ_S, θ_D) determines θ uniquely.

A pair (θ_1, θ_2) satisfying the condition from Theorem 2.1 is called *a congruence pair* of L.

Theorem 2.2. (See [13].) *Let L be a modular p-algebra. For every $f \in \mathrm{End}(L)$ the restrictions $f_S = f \restriction S(L)$ and $f_D = f \restriction D(L)$ are endomorphisms of $S(L)$ and $D(L)$, respectively. A pair $(h, g) \in \mathrm{End}(S(L)) \times \mathrm{End}(D(L))$ is equal to (f_S, f_D) for some $f \in \mathrm{End}(L)$ iff the conditions*

(i) $g(a \vee a^*) = h(a) \vee h(a)^*$;

(ii) $\{g(x); x \in \varphi(L)(a)\} \subseteq \varphi(L)(h(a))$;

are satisfied for all $a \in S(L)$.

The pair (f_S, f_D) determines f uniquely.

A pair (h, g) satisfying the conditions from Theorem 2.2 is called *fair*.

Now we can begin our investigation of SEKP for modular p-algebras. By [11, Theorem 2.6(i)] we know that if a modular p-algebra L has SEKP, then it is an S-algebra (compare also with Corollary 3.2 of [7]).

In fact, we can say something more precise. Given a p-algebra L, we may form the *Glivenko congruence* Γ on L as follows:

$$x \equiv y(\Gamma) \quad \text{iff} \quad x^* = y^* \quad \text{iff} \quad x^{**} = y^{**}.$$

Each Glivenko congruence class $[c]\Gamma$ contains a unique closed element $a = c^{**}$, which is the greatest element of $[c]\Gamma$. Thus every Glivenko congruence class can be expressed as $[a]\Gamma$ for some $a \in S(L)$.

Lemma 2.3. *Let Γ be the Glivenko congruence on a modular p-algebra L. Then Γ is a kernel of a strong endomorphism on L if and only if L is an $S-$algebra.*

Proof. The proof is the same as for [11, Lemma 2.3], we need only to add that for an S-algebra L, the mapping $h : x \mapsto x^{**}$ on L is a strong endomorphism on L. $\square$

Now we can prove our first result

Theorem 2.4. *Let L be a modular p-algebra. An endomorphism $f : L \to L$ is strong if and only if f_S and f_D are strong.*

Proof. Let $\theta_1 \in \mathrm{Con}(S(L))$ and let $\iota_D = D(L) \times D(L)$ be the universal congruence on $D(L)$. Then (θ_1, ι_D) is a congruence pair of L (because now we have $x \equiv 1(\iota_D)$ for any $x \in D(L)$). It means that there is a congruence $\theta \in \mathrm{Con}(L)$ such that $\theta_S = \theta_1$ (and $\theta_D = \iota_D$). Let $a, b \in S(L)$, $(a, b) \in \theta_1$. Then $(a, b) \in \theta$ and, as f is a strong, we see that

$$f_S(a) = f(a) \equiv_{(\theta)} f(b) = f_S(b),$$

which means that f_S is compatible with every congruence $\theta_1 \in \mathrm{Con}(S(L))$ and therefore it is a strong endomorphism on $S(L)$.

Let $\theta_2 \in \mathrm{Con}(D(L))$ and let $\omega_S = \{(a, a);\, a \in S(L)\}$ be a trivial congruence on $S(L)$. Then (ω_S, θ_2) is a congruence pair on the triple $(S(L), D(L), \varphi(L))$. Indeed, $a \equiv 0\ (\omega_S)$ if and only if $a = 0$, it means that $x \in \varphi(L)(a) = \varphi(L)(0) = [1)$ holds only for $x = 1$ and therefore $x \equiv 1(\theta_2)$.

Now, let $a, b \in D(L)$, $(a, b) \in \theta_2$ and a congruence $\theta \in \mathrm{Con}(L)$ be such that $\theta_D = \theta_2$ (and $\theta_S = \omega_S$). Then $(a, b) \in \theta$ and as f is a strong, we have

$$f_D(a) = f(a) \equiv_{(\theta)} f(b) = f_D(b),$$

which means that f_D is compatible with every congruence $\theta_2 \in \mathrm{Con}(D(L))$ and therefore it is a strong endomorphism on $D(L)$.

For the converse, let f be an endomorphism of L, such that f_S and f_D are strong. Let $\theta \in \mathrm{Con}(L)$ and $a, b \in L$ with $(a, b) \in \theta$.

The modularity of L implies that $a = a^{**} \wedge (a \vee a^*)$, $b = b^{**} \wedge (b \vee b^*)$, with a^{**}, $b^{**} \in S(L)$, $a \vee a^*$, $b \vee b^* \in D(L)$. We have $f(a) = f(a^{**}) \wedge f(a \vee a^*)$ and $f(b) = f(b^{**}) \wedge f(b \vee b^*)$.

Now, $(a, b) \in \theta$ means that $a^{**} \equiv b^{**}(\theta)$, it means $a^{**} \equiv b^{**}(\theta_S)$ and as f_S is strong, we have also $f(a^{**}) = f_S(a^{**}) \equiv_{(\theta_S)} f_S(b^{**}) = f(b^{**})$ and therefore also $f(a^{**}) \equiv f(b^{**})(\theta)$.

Next, as $(a, b) \in \theta$, we see that $a^* \equiv b^*(\theta)$, therefore also $a \vee a^* \equiv b \vee b^*(\theta)$, in other words $a \vee a^* \equiv b \vee b^*(\theta_D)$ and as f_D is strong, we have also $f(a \vee a^*) = f_D(a \vee a^*) \equiv_{(\theta_D)} f_D(b \vee b^*) = f(b \vee b^*)$ and therefore $f(a \vee a^*) \equiv f(b \vee b^*)(\theta)$.

And as $f(a^{**}) \equiv f(b^{**})(\theta)$ and $f(a \vee a^*) \equiv f(b \vee b^*)(\theta)$, we have also

$$f(a^{**}) \wedge f(a \vee a^*) \equiv f(b^{**}) \wedge f(b \vee b^*)(\theta),$$

it means that $f(a) \equiv f(b)(\theta)$ and therefore f is strong on L. □

It is clear that the trivial modular p-algebra has SEKP. We can now prove

Theorem 2.5. *Let L be a non-trivial modular p-algebra. Then L satisfies SEKP if and only if*

(i) $S(L) \cong \mathbf{2}$ *(two element Boolean algebra)*
(ii) $D(L)$ *has SEKP as* $\{1\}$*-lattice.*

Proof. Assume that L satisfies SEKP. To prove (i), we shall prove that $S(L)$ has SEKP and, as the only Boolean algebra which has SEKP is $\mathbf{2}$, the result follows. Let $\theta_1 \in \mathrm{Con}(S(L))$. The same argument as in the proof of 2.4 shows that θ_1 is a restriction of some $\theta \in \mathrm{Con}(L)$. As $\theta_1 \neq \iota_{S(L)}$, we also have $\theta \neq \iota_L$, so θ is the kernel of some strong endomorphism h. Then θ_1 is the kernel of h_S, which is a strong endomorphism by 2.4.

Therefore, $S(L)$ has SEKP and (i) is done. Similarly we can establish (ii). Any congruence $\theta_2 \in \mathrm{Con}(D(L))$, $\theta \neq \iota_{D(L))}$ is a restriction of some $\theta' \in \mathrm{Con}(L)$, $\theta' \neq \iota_L$, so θ' is the kernel of a strong endomorphism h' and consequently, θ_2 is the kernel of a strong endomorphism h'_D.

Conversely, let L satisfy (i) and (ii). Take an arbitrary non-universal congruence $\theta \in \mathrm{Con}(L)$. Then θ_S is clearly the trivial congruence on $S(L) = \{0, 1\}$, which is the kernel of the identity mapping $f : S(L) \to S(L)$. As for θ_D, there are two possible cases. If $\theta_D = \iota_{D(L)}$ then we define $g : D(L) \to D(L)$ as the constant function $g(x) = 1$ for every x. Clearly, g is a strong homomorphism with θ_D as the kernel. If $\theta_D \neq \iota_{D(L)}$, then SEKP for $D(L)$ implies the existence of a strong $g \in \mathrm{End}(D(L))$ with θ_D as the kernel.

In both cases, the verification that (f, g) is fair is routine. By Theorem 2.2, $(f, g) = (h_S, h_D)$ for some $h \in \mathrm{End}(L)$. As h_S and h_D are strong, h is strong by Theorem 2.4. The kernel of h is a congruence θ' on L such that $\theta'_S = \theta_S$, $\theta'_D = \theta_D$. By Theorem 2.1, $\theta' = \theta$. The proof is complete. □

So, modular p-algebras with SEKP are just the modular $\{1\}$-lattices with SEKP, with a new bottom element 0 added. We admit that this description is not quite satisfactory, as we do not have a good description of modular $\{1\}$-lattices with SEKP. On the other hand, our result is in full accordance with the program of reducing the problems on p-algebras into corresponding problems on Boolean algebras and $\{1\}$-lattices.

A satisfactory description for the special case of distributive p-algebras is given by T. Blyth and H. Silva in [3] and also by G. Fang and J. Fang in [7]. Characterizations [3, Theorem 14] (or [7, Theorem 3.8]) can be reformulated as follows:

Theorem 2.6. *Let L be a distributive p-algebra with $0 \neq 1$. Then it has SEKP if and only if*

(i) $S(L) \cong \mathbf{2}$ *(two element Boolean algebra)*
(ii) $D(L)$ *is isomorphic to the lattice of all cofinite subsets of some set Z.*

Combining this and Theorem 2.5 we can characterize distributive $\{1\}$-lattices which have SEKP:

Theorem 2.7. *Let L be a distributive $\{1\}$-lattice. Then L has SEKP if and only if it is isomorphic to the lattice of all cofinite subsets of some set Z.*

Let us remark that Theorem 2.6 was proved using Priestley duality, without any decomposition result like 2.5. Hence, 2.7 does not follow from 2.6 alone.

By an order duality we can use this theorem to describe distributive $\{0\}$-lattices which possess SEKP as $\{0\}$-lattices (it means that only bottom element is a part of the signature and must be preserved by homomorphisms and endomorphisms).

Corollary 2.8. *Let L be a distributive $\{0\}$-lattice. Then L has SEKP if and only if it is isomorphic to the lattice of all finite subsets of some set Z.*

If a bounded distributive lattice (with 0, 1 as nullary operations) has SEKP (it means as $\{0, 1\}$-lattice), then it has SEKP also as $\{1\}$-lattice. Infinite $\{1\}$-lattices which have SEKP do not have bottom element by Theorem 2.7 and therefore by [3, Theorem 2] we have

Corollary 2.9. *Let L be a bounded distributive lattice. Then L has SEKP if and only if it is a 1- or 2- element chain.*

Obviously, $\{1\}$-lattices and $\{0\}$-lattices with SEKP also have this property when considered as lattices (without nullary operations). In the next section we describe the class of all distributive lattices with SEKP.

3. Unbounded distributive lattices

Now we shall deal with distributive lattices considered as unbounded lattices (i.e. the top and/or bottom elements - if they exists - are not a part of the signature and therefore need not be preserved by homomorphisms). Let L be an unbounded distributive lattice in all what follows.

We shall use Priestley duality for unbounded distributive lattices as a main tool. We shall follow [5, Section 1.2] to introduce its basic elements. To every distributive lattice L we assign its Priestley space

$$\mathbf{D}(L) = (Spec(L); 0, 1, \subseteq, \tau),$$

where $Spec(L)$ be the set of all prime ideals of L, including $\emptyset$ and L, $0 = \emptyset$, $1 = L$, $\subseteq$ is the set inclusion and τ is the topology on $Spec(L)$, which has as subbasis all sets $A_x = \{P \in Spec(L); x \notin P\}$ and their complements $B_y = \{P \in Spec(L); y \in P\}$ $(x, y \in L)$. Thus, $\mathbf{D}(L)$ is an ordered topological space. This space is bounded (as an ordered set), compact (as a topological space) and totally order-disconnected.

Let $\mathcal{O}(\mathbf{D}(L))$ be a set all nonempty proper clopen down sets of $\mathbf{D}(L)$, ordered by the set inclusion (a set $U \subseteq Spec(L)$ is a down set if $x \in U$, $y \in Spec(L)$ and $y \leq x$ implies $y \in U$, up sets are defined dually). The representation theorem says

Theorem 3.1. *Every distributive lattice L is isomorphic to $\mathcal{O}(\mathbf{D}(L))$. The isomorphism $e_L \colon L \to \mathcal{O}(\mathbf{D}(L))$ can be defined as $e_L(x) = A_x$.*

Congruences on $\mathcal{O}(\mathbf{D}(L))$ can be described as follows. Let X be a closed subset of $(Spec(L);\tau)$. Denote

$$\theta_X = \{(A,B) \in \mathcal{O}(\mathbf{D}(L)) \times \mathcal{O}(\mathbf{D}(L)); A \cap X = B \cap X\}$$

Then θ_X is a congruence on $\mathcal{O}(\mathbf{D}(L))$ and therefore it corresponds to a congruence on L and every congruence on L can be obtained by this construction.

Besides Priestley duality, we use the fact that the compatible functions on distributive lattices are a well investigated topic. We need two such results, see [15].

Theorem 3.2. *Let L be a distributive lattice, $f\colon L \to L$ a compatible isotone function. Then*

(1) *f is an idempotent homomorphism (a retraction);*
(2) *$\mathrm{Im}(f)$ is a convex sublattice of L.*

If $f\colon L \to L$ is an idempotent homomorphism and $[a]_f$ is a congruence block of a congruence $\ker(f) = \{(a,b) \in L^2; f(a) = f(b)\}$, then $f(a) \in [a]_f$, because $a \equiv_{\ker(f)} f(a)$ by the idempotency.

We shall describe the Priestley spaces of distributive lattices with SEKP. Let us start with some properties of the order relation $\subseteq$ of $Spec(L)$.

Lemma 3.3. *Let L have SEKP. Then L does not have four element chain C_4 as a homomorphic image.*

Proof. Let $C_4 = \{0,a,b,1\}$ be a 4-element chain $0 < a < b < 1$. For a contradiction, let $r\colon L \to C_4$ be a surjective homomorphism. Consider the equivalence θ on L with the equivalence classes $r^{-1}(\{0\})$, $r^{-1}(\{a,b\})$ and $r^{-1}(\{1\})$. Clearly, θ is a congruence, so $\theta = \ker(f)$ for some strong endomorphism $f:\ L \to L$. Thus, f satisfies 3.2 (i), (ii). We have $\mathrm{Im}(f) = \{x,y,z\}$ for some $x < y < z$. Obviously, $r(x) = 0$, $r(y) \in \{a,b\}$, $r(z) = 1$. Let us assume $r(y) = a$. (The case $r(y) = b$ is similar.) Choose $t \in L$ with $r(t) = b$ and let $u = (t \vee y) \wedge z$. Then $y \leq u \leq z$ and $r(u) = (b \vee a) \wedge 1 = b$, which shows that $y \neq u \neq z$, so $\mathrm{Im}(f)$ is not convex, a contradiction with 3.2(ii). □

Lemma 3.4. *Let L be a distributive lattice. The following conditions are equivalent:*

(1) *There are no proper prime ideals $P_0, P_1, P_2 \in Spec(L)$ such that $P_0 \subsetneq P_1 \subsetneq P_2$.*
(2) *L does not have four element chain C_4 as a homomorphic image.*

Proof. (1) ⇒ (2): Let $f\colon L \to C_4$ be a surjective homomorphism. Let $P_0 = f^{-1}(\{0\})$, $P_1 = f^{-1}(\{0,a\})$ and $P_2 = f^{-1}(\{0,a,b\})$. As $\{0\} \subsetneq \{0,a\} \subsetneq \{0,a,b\}$ are proper prime ideals of C_4, $P_0 \subsetneq P_1 \subsetneq P_2$ are proper prime ideals of L.

(2) $\Rightarrow$ (1): Let $P_0, P_1, P_2 \in \mathit{Spec}(L) \setminus \{\emptyset, L\}$ with $P_0 \subsetneq P_1 \subsetneq P_2$. Then $f\colon L \to C_4$ defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in L \setminus P_2 \\ b \text{ if } x \in P_2 \setminus P_1 \\ a \text{ if } x \in P_1 \setminus P_0 \\ 0 \text{ if } x \in P_0 \end{cases}$$

is a surjective homomorphism. □

The previous assertions yield the following description of the ordered set $(\mathit{Spec}(L), \subseteq)$.

Lemma 3.5. *If L has SEKP, then $X = \mathit{Spec}(L) \setminus \{\emptyset, L\}$ is a disjoint union of three antichains $A_0 \cup A_1 \cup A_2$, where $A_1 = \{a \in X; (\exists b \in X)(a < b)\}$ ("bottom" elements), $A_2 = \{b \in X; (\exists a \in X)(a < b)\}$ ("top" elements) and $A_0 = X \setminus (A_1 \cup A_2)$ ("incomparable" elements).*

Now we are going to describe the topology of $\mathbf{D}(L)$.

Lemma 3.6. *Let L be any distributive lattice.*

(1) *Let $P \in \mathit{Spec}(L)$, $P \neq \emptyset, P \neq L$. Then P is a discrete point in the topology τ if and only if there are $a, b \in L$ such that $a \prec b$ and $a \in P$, $b \notin P$.*
(2) *Let L have SEKP, $P \in \mathit{Spec}(L)$, $P \neq \emptyset, P \neq L$. Then P is a discrete point in the topology τ.*

Proof. (1) Let $a, b \in L$ be such that $a \prec b$ and $a \in P$, $b \notin P$. We have to prove that $\{P\}$ is open in τ.

We know that $\{P\} \subseteq A_b \cap B_a$, $A_b \cap B_a$ is the intersection of clopen sets and therefore it is open. It remains to prove that $A_b \cap B_a$ contains only P. Let $J \in A_b \cap B_a$. Thus, J is a prime ideal, $a \in J$, $b \notin J$.

Let $x \in P$. Then $a \vee x \in P$. It is clear that $a \leq b \wedge (a \vee x) \leq b$. The equality $b \wedge (a \vee x) = b$ leads to a contradiction, as it implies $b \leq a \vee x \in P$, while $b \notin P$. Since (a, b) is a covering pair, we obtain that $b \wedge (a \vee x) = a \in J$. Since $b \notin J$, the primality of J implies that $a \vee x \in J$. It means that $x \in J$, because $x \leq a \vee x \in J$. This shows that $P \subseteq J$. By the symmetry we have $P = J$ and therefore $A_b \cap B_a = \{P\}$.

Conversely, let $\{P\}$ be open. It is one element set and therefore it must be an element of a basis of the topology. Every element of the basis is an intersection of finitely many elements of a subbasis, it means that there are $x_1, \dots, x_k, y_1, \dots, y_l \in L$ such that

$$\{P\} = A_{x_1} \cap \dots \cap A_{x_k} \cap B_{y_1} \cap \dots \cap B_{y_l}.$$

Since $P \notin \{\emptyset, L\}$, we can assume that $k \geq 1$, $l \geq 1$. But $A_{x_1} \cap \dots \cap A_{x_k} = A_{x_1 \wedge \dots \wedge x_k}$ and $B_{y_1} \cap \dots \cap B_{y_l} = B_{y_1 \vee \dots \vee y_l}$, so there are $a, b \in L$ such that $\{P\} = A_b \cap B_a$. Let Q be an ideal in L with $a \in Q$. Then it is clear that

$b \notin Q \Leftrightarrow a \vee b \notin Q$. Therefore $\{P\} = A_{a\vee b} \cap B_a$. If there is c such that $a < c < a \vee b$, then there are prime ideals P_0 and P_1 such that $a \in P_0$, $c \notin P_0$ and $c \in P_1$, $a \vee b \notin P_1$, what means that $P_0, P_1 \in A_{a\vee b} \cap B_a$ and this is not possible. Therefore $a \prec a \vee b$, $a \in P$, $a \vee b \notin P$.

(2) Take a congruence θ_P with blocks $P, L \setminus P$, let $f \colon L \to L$ be a strong endomorphism with $\ker(f) = \theta_P$.

We know that $\mathrm{Im}(f)$ has 2 elements and it is a convex sublattice of L. So that there $a, b \in L$ such that $\mathrm{Im}(f) = \{a, b\}$, $a \prec b$ and $a \in P$, $b \in L \setminus P$ because f is idempotent. The result follows from (1). □

Lemma 3.7. *Let P be a discrete point for every $P \in Spec(L) \setminus \{\emptyset, L\}$. Then, for every $x, y \in L$, the set $Y = A_x \cap B_y$ is finite.*

Proof. It is clear that $Y = A_x \cap B_y \subseteq Spec(L) \setminus \{\emptyset, L\}$ is a closed set, hence it is compact. By the assumption, $Y = \bigcup\{\{P\}; P \in Y\}$ is a cover of Y by open sets and therefore this open cover has a finite subcover, which means that Y is finite. □

A lattice is called locally finite if every interval is finite.

Lemma 3.8. *A distributive lattice L is locally finite if and only if every $P \in Spec(L) \setminus \{\emptyset, L\}$ is a discrete point in the topology τ.*

Proof. Let every interval be finite and $P \in Spec(L) \setminus \{\emptyset, L\}$. There are c, d such that $c \in P$, $d \notin P$. Therefore $c \vee d \notin P$, as well. As the interval $[c, c \vee d]$ is finite, there are $a, b \in [c, c \vee d]$ such that $a \prec b$ and $a \in P, b \notin P$. We see that P is a discrete point by Lemma 3.6(1).

Conversely, let $a < b$. Then $C = \{P; a \in P, b \notin P\} = A_b \cap B_a$ is finite by Lemma 3.7. In a distributive lattice, every two elements can be separated by a prime ideal. Hence, every two elements of the interval $[a, b]$ can be separated by some $P \in C$. Hence, the cardinality of $[a, b]$ is at most 2^n, where $n = card(C)$. □

Lemma 3.9. *Let L have SEKP. Then for every $A \in \mathcal{O}(\mathbf{D}(L))$ the sets $A \cap A_2$ and $A_1 \setminus A$ are finite.*

Proof. Suppose that the set $C \cap A_2$ is infinite for some $C \in \mathcal{O}(\mathbf{D}(L))$. Consider the relation θ on $\mathcal{O}(\mathbf{D}(L))$ given by

$$(A, B) \in \theta \text{ if and only if } A \cap A_1 = B \cap A_1.$$

Clearly, θ is a congruence. (Notice that $A \cap A_1 = B \cap A_1$ if and only if $A \cap X_1 = B \cap X_1$, where X_1 is the topological closure of A_1.) Since $\mathcal{O}(\mathbf{D}(L))$ is isomorphic to L, it has SEKP. Let $f \colon \mathcal{O}(\mathbf{D}(L)) \to \mathcal{O}(\mathbf{D}(L))$ be a strong homomorphism with $\ker(f) = \theta$. By 3.2, f is idempotent and $\mathrm{Im}(f)$ is a convex subset of $\mathcal{O}(\mathbf{D}(L))$.

Denote $A = f(C)$. Then we have $f(A) = A$, because f is idempotent. The set $C \setminus A$ is clopen and discrete (by 3.6), so it is finite, which implies that

$A \cap A_2$ is infinite. Let $Q \in A \cap A_2$. There is $P \in A_1$ such that $P \subseteq Q$, $P \in A \cap A_1$, because A is down set.

The set $\uparrow P = \{I \in \mathit{Spec}(L); P \subseteq I\}$ is closed (a property of Priestley spaces), so $A \cap \uparrow P$ is a closed subset of the discrete set $A_1 \cup A_2$, which means it must be finite, and hence clopen. That is why

$$B = A \setminus \uparrow P = A \setminus (A \cap \uparrow P)$$

is a clopen down set.

Clearly, $f(B) \subseteq f(A) = A$. We have $(f(B), B) \in \ker(f) = \theta$, hence

$$f(B) \cap A_1 = B \cap A_1 = A \cap A_1 \setminus \{P\}.$$

The set $M = f(B) \cup \{P\}$ is also a clopen down set, $f(B) \subsetneq M \subsetneq A = f(A)$. (The inequality $M \neq A$ holds because $Q \in A \setminus M$.) Since $M \cap A_1 = A \cap A_1$, we have $(A, M) \in \theta$, so $f(M) = f(A) = A \neq M$, which means that $M \notin \operatorname{Im}(f)$. This is a contradiction with the requirement that $\operatorname{Im}(f)$ is a convex subset of $\mathcal{O}(\mathbf{D}(L))$.

The second statement can be proved using the order duality. The proper clopen up sets in $\mathbf{D}(L)$ form a distributive lattice which is dual to $\mathcal{O}(\mathbf{D}(L))$, so it has SEKP, too. For every $A \in \mathcal{O}(\mathbf{D}(L))$ its complement $B = \mathit{Spec}(L) \setminus A$ is a clopen up set, and similarly as above, we can prove that $B \cap A_1 = A_1 \setminus A$ is finite. □

Lemma 3.10. *Let L have SEKP. Then there exists clopen down set $C \in \mathcal{O}(\mathbf{D}(L))$ such that $A_1 \subseteq C$ and $C \cap A_2 = \emptyset$. Moreover, for any such C and for $A \in \mathcal{O}(\mathbf{D}(L))$ such that $A \subseteq C$ the interval $[A, C]$ of $\mathcal{O}(\mathbf{D}(L))$ is (finite) Boolean and also for $A \in \mathcal{O}(\mathbf{D}(L))$ such that $C \subseteq A$ the interval $[C, A]$ of $\mathcal{O}(\mathbf{D}(L))$ is (finite) Boolean.*

Proof. Take any $A \in \mathcal{O}(\mathbf{D}(L))$. By 3.9, the sets $A \cap A_2$ and $(A_1 \setminus A)$ are finite, consisting of discrete points, and hence clopen. Therefore,

$$C = A \cup (A_1 \setminus A) \setminus (A \cap A_2)$$

is a clopen down set, $A_1 \subseteq C$, $A_2 \cap C = \emptyset$.

Let $A \in \mathcal{O}(\mathbf{D}(L))$, $C \subseteq A$. By 3.9, $C = A \cup M$ for some finite $M \subseteq A_2$. Every set X with $C \subseteq X \subseteq A$ is a clopen down set, so interval $[C, A]$ is isomorphic to the power set of M.

Let $A \in \mathcal{O}(\mathbf{D}(L))$, $A \subseteq C$. By 3.9, $C = A \setminus M$ for some finite $M \subseteq A_1$. Every set X with $A \subseteq X \subseteq C$ is a clopen down set, so interval $[A, C]$ is isomorphic to the power set of M. □

Now we can prove the following characterization theorem.

Theorem 3.11. *Let L be an unbounded distributive lattice. Then the following are equivalent:*

(1) *L has SEKP.*
(2) *L is locally finite and there exists $c \in L$ such that for every $x < c$ or $x < c$ intervals $[x, c]$ (if $x < c$) and $[c, x]$ (if $x > c$) are (finite) Boolean.*

Proof. If L has SEKP, it is locally finite by Lemmas 3.6(2) and 3.8. Second part of the condition (2) follows from Lemma 3.10.

Conversely let (2) be satisfied. We prove that $\mathcal{O}(\mathbf{D}(L))$ has SEKP. Let c be a special element of L from the condition (2). Denote $C = A_c = \{P \in Spec(L); c \notin P\}$. Clearly, $C \in \mathcal{O}(\mathbf{D}(L))$.

First we claim that $C \setminus \{\emptyset\}$ is an antichain. For contradiction, let $P_1, P_2 \in C$ be such that $P_1 \subsetneq P_2$. Let $a \in P_1$, $b \in P_2 \setminus P_1$. Put $b' = c \wedge b$ and $a' = c \wedge b \wedge a$. We have that $b' \in P_2$, but $b' \notin P_1$, $a' \in P_1$. Let $Q_1 = P_1 \cap [a', c]$, $Q_2 = P_2 \cap [a', c]$. As $b' \in [a', c]$, Q_1, Q_2 are prime ideals in a boolean algebra $[a', c]$ such that $Q_1 \subsetneq Q_2$, because $b' \in Q_2 \setminus Q_1$, and this is impossible.

The set $B_c = \{P \in Spec(L); c \in P\}$ is the complement of C. Similarly as above we can prove that $B_c \setminus \{L\}$ is an antichain.

The local finiteness of L implies that all points $P \in \mathcal{O}(\mathbf{D}(L)) \setminus \{\emptyset, L\}$ are discrete.

Now let θ be a congruence on $\mathcal{O}(\mathbf{D}(L))$. Hence, there exists a (closed) set $U \subseteq Spec(L)$ such that

$$(A, B) \in \theta \text{ if and only if } A \cap U = B \cap U.$$

We define a function $f_U : \ \mathcal{O}(\mathbf{D}(L)) \to \mathcal{O}(\mathbf{D}(L))$ by

$$f_U(A) = (A \cap U) \cup (C \setminus U).$$

and we claim that (1) $f_U(A)$ is always a nonempty proper clopen down set, (2) f_U is a strong endomorphism, and (3) $\ker(f_U) = \theta$.

To prove (1), let us firstly show that $f_U(A)$ is clopen. It is easy to see that $A \cap C \subseteq f_U(A) \subseteq A \cup C$. Both $A \cup C$ and $A \cap C$ are clopen sets, so their difference $(A \cup C) \setminus (A \cap C)$ is clopen (and hence compact), consisting of discrete points. It follows that $(A \cup C) \setminus (A \cap C)$ is finite, and every set between $A \cap C$ and $A \cup C$ is clopen. So, $f_U(A)$ is a clopen set.

Now we show that $f_U(A)$ is a down set. Let $Q \in f_U(A)$, $P \in Spec(L)$ and $P \subsetneq Q$. If $P = \emptyset$, then clearly $P \in f_U(A)$. Let $P \neq \emptyset$. Clearly, $Q \neq L$, so the only possibility is $P \in C$, $Q \in B_c$. Now $Q \in f_U(A)$ implies $Q \in A \cap U$. Then $P \in A$, as A is a down set. We distinguish two cases. If $P \in U$, then $P \in A \cap U \subseteq f_U(A)$. If $P \notin U$, then $P \in C \setminus U$. In both cases $P \in f_U(A)$. Clearly, $L \notin f_U(A)$, so $f_U(A) \in \mathcal{O}(\mathbf{D}(L))$.

To prove (2), let $Z \subseteq Spec(L)$. We can see that

$$f_U(A) \cap Z = [(A \cap Z) \cap U] \cup [(C \setminus U) \cap Z].$$

Let $(A, B) \in \theta_Z$, which means that $A \cap Z = B \cap Z$. Then

$$\begin{aligned} f_U(A) \cap Z =& [(A \cap Z) \cap U] \cup [(C \setminus U) \cap Z] \\ =& [(B \cap Z) \cap U] \cup [(C \setminus U) \cap Z] \\ =& f_U(B) \cap Z, \end{aligned}$$

which means that $(f_U(A), f_U(A)) \in \theta_Z$ and therefore f_U is a compatible function. It is clear, that f_U is an isotone function and by Theorem 3.2(1) every compatible isotone function on $\mathcal{O}(\mathbf{D}(L))$ (on L) is a homomorphism (it can be also checked by a routine calculation). That means, that f_U is a strong endomorphism.

To prove (3), we shall use an evident equality $f_U(A) \cap U = A \cap U$. Now, let $f_U(A) = f_U(B)$. Then $f_U(A) \cap U = f_U(B) \cap U$, which means that $A \cap U = B \cap U$, or that $(A, B) \in \theta$.

For the opposite inclusion, let $(A, B) \in \theta$, which means that $A \cap U = B \cap U$. Then also

$$f_U(A) = (A \cap U) \cup (C \setminus U) = (B \cap U) \cup (C \setminus U) = f_U(B)$$

and we see that $\ker(f_U) = \theta$. □

If L is finite, it is locally finite, so that we can omit this fact from the condition (2) in Theorem 3.11.

Theorem 3.12. *Let L be a finite distributive lattice. Then the following are equivalent:*

(1) *L has SEKP;*
(2) *L does not have C_4 as a homomorphic image;*
(3) *the poset $P(L)$ of all proper prime ideals (and/or $Ji(L)$ of all join irreducible elements) has length (height) at most 1;*
(4) *there exists $c \in L$ such that for every $x < c$ or $x < c$ intervals $[x, c]$ (if $x < c$) and $[c, x]$ (if $x > c$) are Boolean;*

Proof. (1) ⇒ (2) is Lemma 3.3, (2) ⇔ (3) holds by Lemma 3.4, (1) ⇔ (4) by Theorem 3.11.

So that it is enough to prove for example the implication (3) ⇒ (4). But in the finite case it is enough to take the down set

$$C = \{P \in P(L); (\exists Q \in P(L)(P \subsetneq Q))\}$$

and the corresponding element $c \in L$ has all necessary properties, because every subset and every superset of C is a down set. □

Finally, let us present one example. Let C_3 be a 3-element chain $0 < a < 1$. Let I be any set and let L be the sublattice of C_3^I consisting of all $(x_i)_{i \in I}$ with $\{i \in I : x_i \neq a\}$ finite. Then L has SEKP. Indeed, it is easy to see that L is locally finite, and $c = (c_i)_{i \in I}$ with $c_i = a$ for every i satisfies 3.11(2).

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