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COMPACT INTERSECTION PROPERTY AND DESCRIPTION OF CONGRUENCE LATTICES

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Dedicated to Professor Ján Jakubík on the occasion of his 90th birthday

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. We say that a variety $\mathcal{V}$ of algebras has the Compact Intersection Property (CIP), if the family of compact congruences of every $A \in \mathcal{V}$ is closed under intersection. We investigate the congruence lattices of algebras in locally finite congruence-distributive CIP varieties. We prove some general results and obtain a complete characterization for some types of such varieties. We provide two kinds of description of congruence lattices: via direct limits and via Priestley duality.

1. Introduction

Let $\mathcal{K}$ be a class of algebras and denote by $\operatorname{Con}\mathcal{K}$ the class of all lattices isomorphic to $\operatorname{Con} A$ (the congruence lattice of an algebra A) for some $A \in \mathcal{K}$. There are many papers investigating $\operatorname{Con}\mathcal{K}$ for various classes $\mathcal{K}$. However, the full description of $\operatorname{Con}\mathcal{K}$ has proved to be a very difficult (and probably intractable) problem, even for the most common classes of algebras, like groups or lattices. One of the sources of this difficulty is the fact that compact congruences of an infinite algebra form a join-semilattice, which is not necessarily a lattice. When trying to describe such semilattices one has to deal with various refinement properties. (See, for instance, [16], [15], or [17].)

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It is therefore not surprising that in most cases when $\operatorname{Con}\mathcal{K}$ is well understood, the algebras in $\mathcal{K}$ have a special property: the intersection of any two compact congruences of $A \in \mathcal{K}$ is compact. This is called the Compact Intersection Property (CIP). Varieties with CIP has been considered before (for instance, [4], [5], [3]), but with the main focus not on a characterization of $\operatorname{Con}\mathcal{K}$. (Although the final example in [3] describes $\operatorname{Con}\mathcal{K}$ for the variety generated by the 2-element algebra $\{0,1\}$ with the operation $p(x,y,z) = x \vee (y \wedge z)$.)

In the present paper we initiate a systematic investigation of the class $\operatorname{Con}\mathcal{K}$, where $\mathcal{K}$ is a locally finite congruence-distributive variety with CIP. Even under such restrictions, the problem of describing $\operatorname{Con}\mathcal{K}$ is still difficult. In our previous paper [10] we were able to solve several simple cases. In the present paper we try to obtain general results. First we describe the lattices in $\operatorname{Con}\mathcal{K}$ as directed limits of suitable limit system. We do not consider this characterization quite satisfactory, so we try to obtain another characterization using the Priestley duality. Our results correspond to the two main approaches to the problem of describing $\operatorname{Con}\mathcal{K}$. The approach based on lifting of diagrams has been recently greatly developed by P. Gillibert. (See [6] or [7].) The description based on topological representation has been investigated by M. Ploščica ([12], [13], [14]).

We illustrate our results by applying them to several special cases.

2. Basic facts and denotations

Let L be a lattice. An element $a \in L$ is called strictly meet-irreducible iff $a = \bigwedge X$ implies that $a \in X$, for every subset X of L. Let $\mathrm{M}(L)$ denote the set of all strictly meet-irreducible elements. The greatest element of L is not strictly meet-irreducible. By adding it to $\mathrm{M}(L)$ we obtain the set denoted by $\mathrm{M}^*(L)$.

If f is a mapping, then $\operatorname{dom}(f)$ stand for its domain. By $\ker f$ we denote the binary relation on $\operatorname{dom}(f)$ given by $(x,y) \in \ker f$ iff $f(x) = f(y)$. By $f{\restriction}X$ we mean the restriction of f to X.

Let A be an algebra. For every $a, b \in A$ by $\Theta(a,b)$ we denote the congruence generated by the pair (a,b). The congruence lattice of A will be denoted by $\operatorname{Con} A$. For $\alpha \in \operatorname{Con} A$, the α-class in A/α containing a will be denoted by $[a]_\alpha$.

The set $\operatorname{Con_c} A$ of all compact (finitely generated) congruences of A is a $(0,\vee)$-subsemilattice of $\operatorname{Con} A$. The lattice $\operatorname{Con} A$ is uniquely determined by the semilattice $\operatorname{Con_c} A$ (it is isomorphic to the ideal lattice of $\operatorname{Con_c} A$). It is often easier to describe $\operatorname{Con_c} A$ instead of $\operatorname{Con} A$.

Let P be a partially ordered set. For every $x \in P$ we set $\uparrow\! x = \{y \in P \mid y \geq x\}$, $\downarrow\! x = \{y \in P \mid y \leq x\}$. A subset $U \subseteq P$ is called an up-set (a down-set) if $\uparrow\! x \subseteq U$ for every $x \in U$ ($\downarrow\! x \subseteq U$ for every $x \in U$).

It is a well known fact that for every $\theta \in \operatorname{Con} A$ the lattice $\operatorname{Con} A/\theta$ is isomorphic to $\uparrow\theta$. Hence, $\theta \in \mathrm{M}(\operatorname{Con} A)$ if and only if the quotient algebra A/θ is subdirectly irreducible. Equivalently, $\theta \in \mathrm{M}(\operatorname{Con} A)$ if and only if $\theta = \ker f$ for some surjective homomorphism $f\colon A \to S$, with S subdirectly irreducible. This is also true if one considers one-element algebras as subdirectly irreducible and replace $\mathrm{M}(\operatorname{Con} A)$ by $\mathrm{M}^*(\operatorname{Con} A)$.

For algebras A and B, $B \leq A$ denotes that B is a subalgebra of A. If $B \leq A$ and $\theta \in \operatorname{Con} A$, then $\theta\restriction B = \theta \cap B^2$ is the restriction of θ to B. For every homomorphism $f\colon A \to B$ we define the mapping

$$\operatorname{Con_c} f\colon \operatorname{Con_c} A \to \operatorname{Con_c} B$$

by the rule that, for every $\alpha \in \operatorname{Con_c} A$, $\operatorname{Con_c} f(\alpha)$ is the congruence generated by the set $\{(f(x), f(y)) \mid (x, y) \in \alpha\}$. This mapping is a homomorphism of $(0,\vee)$-semilattices. Notice that finite $(0,\vee)$-semilattices are, in fact, lattices.

Now let $\varphi\colon K \to L$ be a $(0,\vee)$-homomorphism of finite $(0,\vee)$-semilattices. We define the map $\varphi^{\leftarrow}\colon L \to K$ by

$$\varphi^{\leftarrow}(\beta) = \bigvee\{\alpha \mid \varphi(\alpha) \leq \beta\}.$$

If $K = \operatorname{Con_c} A$, $L = \operatorname{Con_c} B$ and $\varphi = \operatorname{Con_c} f$, for some algebras A, B and a homomorphism $f\colon A \to B$, then $\varphi^{\leftarrow}(\beta) = \{(x, y) \in A \mid (f(x), f(y)) \in \beta\}$. If A is a subalgebra of B and $f\colon A \to B$ is the inclusion, then $\varphi^{\leftarrow}(\beta)$ is the restriction of $\beta \in \operatorname{Con} B$ to A.

The pair $(\varphi, \varphi^{\leftarrow})$ is sometimes referred to as residuated mappings. The following facts are rather well known. (For (1)–(4) see [1: Section 1.3], while (5) follows from Birkhoff's duality for finite distributive lattices.)

Lemma 2.1. *Let $\varphi\colon K \to L$ be a $(0,\vee)$-homomorphism of finite lattices.*

(1) *$\varphi^{\leftarrow}$ preserves $\wedge$ and the largest element.*
(2) $\varphi(\alpha) = \bigwedge\{\beta \mid \alpha \leq \varphi^{\leftarrow}(\beta)\}$.
(3) $\varphi(\alpha) \leq \beta \iff \alpha \leq \varphi^{\leftarrow}(\beta)$.
(4) *If $\psi\colon L \to M$ is another $(0,\vee)$-homomorphism of finite lattices, then $(\psi\varphi)^{\leftarrow} = \varphi^{\leftarrow}\psi^{\leftarrow}$.*
(5) *If $\varphi\colon K \to L$ is a 0-preserving homomorphism of finite distributive lattices, then $\varphi^{\leftarrow}(c) \in \mathrm{M}^*(K)$ for every $c \in \mathrm{M}^*(L)$.*

Next we recall the algebraic constructions of direct and inverse limit. Let P be an ordered set. Let $\mathcal{K}$ be a class of algebras. A P-indexed diagram $\vec{A}$ in $\mathcal{K}$ consists of a family $(A_p, p \in P)$ of algebras in $\mathcal{K}$ and a family $(f_{p,q}, p \leq q)$ of homomorphisms $f_{p,q}\colon A_p \to A_q$ such that $f_{p,p}$ is the identity on A_p and $f_{p,r} = f_{q,r} f_{p,q}$ for all $p \leq q \leq r$.

If the index set P is directed (for every $p,q \in P$ there exists $r \in P$ with $p,q \leq r$), then we define the *direct limit* of $\vec{A}$ as

$$\varinjlim \vec{A} := \varinjlim A_p := \Big(\bigsqcup_{p\in P} A_p\Big)/\sim,$$

where $\bigsqcup_{p\in P} A_p$ is the disjoint union of the family $(A_p,\ p \in P)$ and the equivalence relation $\sim$ is defined (for $x \in A_p$ and $y \in A_q$) by

$$x \sim y \iff \exists r \in P\colon f_{p,r}(x) = f_{q,r}(y).$$

A special case of the direct limit is the directed union, when all the homomorphisms are set inclusions. Note that in the category theory this construction corresponds to the (directed) colimit.

The *inverse limit* of $\vec{A}$ is defined for any partially ordered set P as a subalgebra of the direct product of $\prod_{p\in P} A_p$, namely

$$\varprojlim \vec{A} := \varprojlim A_p := \Big\{a \in \prod_{p\in P} A_p \mid a_q = f_{p,q}(a_p) \ \text{ for every } \ p,q \in P,\ \ p \leq q\Big\}.$$

(The elements of $\prod_{p\in P} A_p$ are written in the form $a = (a_p)_{p\in P}$.) A special case of this construction is the direct product, which arises when P is an antichain. In the category theory language, this construction is the limit of $\vec{A}$.

It is well known that any variety $\mathcal{V}$ is closed under the formation of direct and inverse limits.

The direct limit construction will be used to obtain the description of $\operatorname{Con}_c A$ for infinite $A \in \mathcal{V}$ from the description of $\operatorname{Con}_c A$ for finite A. This is possible due to the following two facts. First, Con_c is a functor preserving the direct limits, which means that for every directed P-indexed diagram $\vec{A}$ in $\mathcal{V}$ we have the P-indexed diagram $\operatorname{Con}_c \vec{A} = (\operatorname{Con}_c A_p, \operatorname{Con}_c \varphi_{p,q})$ in the category of $(0,\vee)$-semilattices and $(0,\vee)$-homomorphisms, and

$$\operatorname{Con}_c \varinjlim \vec{A} \simeq \varinjlim \operatorname{Con}_c \vec{A}.$$

Second, let $\vec{A} = (A_p, \varphi_{p,q})$ and $\vec{B} = (B_p, \psi_{p,q})$ be directed P-indexed diagrams and let $h_p\colon A_p \to B_p$ be isomorphisms for every $p \in P$ such that the following diagram commutes for every $p,q \in P$, $p \leq q$:

$$\begin{array}{ccc} A_p & \xrightarrow{\varphi_{p,q}} & A_q \\ {\scriptstyle h_p}\big\downarrow & & {\scriptstyle h_q}\big\downarrow \\ B_p & \xrightarrow{\psi_{p,q}} & B_q \end{array}$$

Then

$$\lim_{\rightarrow} \vec{A} \simeq \lim_{\rightarrow} \vec{B}.$$

The inverse limits will be used to construct algebras with prescribed finite (distributive) congruence lattice. For this we need special diagrams called admissible valuations.

Let $\mathrm{SI}(\mathcal{V})$ denote the class of all subdirectly irreducible members of a variety $\mathcal{V}$. In this paper we find it convenient to include one-element algebras into $\mathrm{SI}(\mathcal{V})$.

Definition 2.2. Let $\mathcal{V}$ be a variety and let M be a partially ordered set, we say that M-indexed diagram $\overrightarrow{v} = (v(\alpha), f_{\alpha,\beta})$ is a $\mathrm{SI}(\mathcal{V})$-*valuation on* M, if $v\colon M \to \mathrm{SI}(\mathcal{V})$ such that $f_{\alpha,\beta}\colon v(\alpha) \to v(\beta)$ is surjective for every $\alpha \leq \beta$ and the assignment $\beta \mapsto \ker f_{\alpha,\beta}$ is a bijection $\uparrow\alpha \to \mathrm{M}^*(\mathrm{Con}\, v(\alpha))$.

Lemma 2.3. *Let $\mathcal{V}$ be a variety, let M be a partially ordered set and let $\overrightarrow{v} = (v(\alpha), f_{\alpha,\beta})$ be a* $\mathrm{SI}(\mathcal{V})$-valuation on M. *For every $\alpha \in M$ the bijection $\uparrow\alpha \to \mathrm{M}^*(\mathrm{Con}\, v(\alpha))$ defined by $\beta \mapsto \ker f_{\alpha,\beta}$, is an isomorphism of ordered sets.*

Proof. Let $\beta, \gamma \in \uparrow\alpha$ such that $\beta \leq \gamma$. Thus $f_{\alpha,\gamma} = f_{\beta,\gamma} f_{\alpha,\beta}$ and hence $\ker f_{\alpha,\beta} \leq \ker f_{\alpha,\gamma}$.

Conversely, if $\ker f_{\alpha,\beta} \leq \ker f_{\alpha,\gamma}$, then there exists a surjective homomorphism

$$g\colon v(\beta) \to v(\gamma)$$

such that $g f_{\alpha,\beta} = f_{\alpha,\gamma}$. Now, there is a $\delta \in M$, $\delta \geq \beta$ such that $\ker f_{\beta,\delta} = \ker g$. Thus

$$\ker f_{\alpha,\delta} = \ker f_{\beta,\delta} f_{\alpha,\beta} = \ker g f_{\alpha,\beta} = \ker f_{\alpha,\gamma},$$

so $\delta = \gamma$, hence $\beta \leq \gamma$. $\square$

Definition 2.4. A P-indexed diagram $\vec{A} = (A_p, \varphi_{p,q})$ in $\mathcal{V}$ is called *admissible* if the following two conditions are satisfied:

(1) for every $p \in P$ and every $u \in A_p$ there exists

$$a \in \lim_{\leftarrow} A_p$$

such that $a_p = u$;

(2) for every $p, q \in P$, $p \nleq q$ there exist

$$a, b \in \lim_{\leftarrow} A_p$$

such that $a_p = b_p$ and $a_q \neq b_q$.

Notice that the admissibility is a purely set-theoretical property, depending only on the sets A_p and maps $\varphi_{p,q}$, and not on the algebraic structure of A_p.

The next theorem follows from [11: Theorem 2.4].

Theorem 2.5. *Let $\mathcal{V}$ be a locally finite congruence distributive variety. Let L be a finite distributive lattice and let $M = \mathrm{M}^*(L)$. Let $\vec{A} = (v(\alpha), f_{\alpha,\beta})$ be an admissible $\mathrm{SI}(\mathcal{V})$-valuation on M. Then $A := \varprojlim \vec{A}$ is an algebra whose congruence lattice is isomorphic to L. The isomorphism $h\colon \mathrm{M}^*(L) \to \mathrm{M}^*(\mathrm{Con}\, A)$ can be defined by $h(\alpha) = \ker \pi_\alpha$, where π_α is the projection $A \to v(\alpha)$.*

By the Birkhoff duality for finite distributive lattices, the isomorphism $h\colon \mathrm{M}^*(L) \to \mathrm{M}^*(\mathrm{Con}\, A)$ induces an isomorphism

$$k\colon \mathrm{Con}\, A \to L$$

by $k(x) = \bigwedge\{y \in \mathrm{M}^*(L) \mid h(y) \geq x\}$. For $x \in \mathrm{M}^*(\mathrm{Con}\, A)$ we have $k(x) = h^{-1}(x)$, so $k(\ker \pi_\alpha) = \alpha$.

In Chapter 4 we prove a generalization of Theorem 2.5 for infinite M. Let us recall the Priestley duality for distributive lattices with 0 (but not necessarily with 1). Let L be a distributive lattice with 0. Let $\mathrm{P}(L)$ denote the set of all prime ideals of L (including L itself). For every $x \in L$ we define

$$U_x = \{I \in \mathrm{P}(L) \mid x \in I\}, \qquad V_x = \{I \in \mathrm{P}(L) \mid x \notin I\}.$$

We endow $\mathrm{P}(L)$ with the ordering $\leq$ by the set inclusion and the topology τ generated by all sets of the form U_x and V_x. The resulting structure $(\mathrm{P}(L), \leq, \tau)$ is called the dual Priestley space of L. The ordered topological space $\mathrm{P}(L)$ determines L uniquely. In fact, L is isomorphic to the lattice of all proper clopen down-sets of $\mathrm{P}(L)$. As a topological space, $\mathrm{P}(L)$ is compact, Hausdorff, zero-dimensional. It has a largest element. The compatibility of the order and the topology can be expressed by the following condition of compact totally order-disconnectedness:

(CTOD) If $y, z \in \mathrm{P}(L)$, $y \not\leq z$, then there exists a clopen up-set $U \subseteq \mathrm{P}(L)$ with $y \in U$, $z \notin U$.

Moreover if F, G are closed sets such that $\uparrow F \cap \downarrow G = \emptyset$, then there exists a clopen up-set U such that $F \subseteq U$ and $U \cap G = \emptyset$.

Further denote by $\mathrm{Id}(L)$ an ideal lattice of a lattice L. Prime ideals of L can be also characterized as finitely meet irreducible elements of $\mathrm{Id}\, L$. The next lemma is easy to prove.

Lemma 2.6. *Let L be a distributive lattice and let $I \in \mathrm{Id}\, L$. Then I is prime if and only if I is finitely meet irreducible element of $\mathrm{Id}\, L$ or $I = L$.*

Now let $\mathcal{V}$ be a finitely generated congruence distributive variety. We prove that, for every $A \in \mathcal{V}$, all finitely meet-irreducible elements of $\mathrm{Con}\, A$ are strictly meet-irreducible. We use the following concept from [12].

Definition 2.7. A subset P of an algebraic lattice L is called separable, if $P \subseteq \mathrm{M}(L)$ and there exists a family $\{x_p \mid p \in P\} \subseteq L$ such that

(1) $x_p \nleq p$ for every $p \in P$.
(2) $\bigwedge\{x_p \mid p \in P\} = 0$.

Let $s(\mathcal{V}) = \max\{|\mathrm{M}(\mathrm{Con}\, B)| \mid B \leq A \in \mathrm{SI}(\mathcal{V})\}$. Since $\mathcal{V}$ is finitely generated, every subdirectly irreducible algebra is finite and hence $s(\mathcal{V}) \in \mathbb{N}$.

Lemma 2.8. ([12: Consequence 2.4]) *If $Q \subseteq \mathrm{M}(\mathrm{Con}\, A)$ is non-separable, for some $A \in \mathcal{V}$, then $|Q| \leq s(\mathcal{V})$.*

Lemma 2.9. *Let α be a finitely meet-irreducible element of* $\mathrm{Con}\, A$ *for some $A \in \mathcal{V}$. Then $\alpha \in \mathrm{M}(\mathrm{Con}\, A)$.*

Proof. Let α be a finitely meet-irreducible element of $\mathrm{Con}\, A$ for some $A \in \mathcal{V}$. For contradiction suppose that there exists infinite $R \subseteq \mathrm{M}(\mathrm{Con}\, A)$ such that

$$\alpha = \bigwedge R, \qquad \alpha \notin R.$$

Choose finite $P \subseteq R$ with $|P| > s(\mathcal{V})$. By Lemma 2.8, P is separable, so we have $x_p \nleq p$ (hence $x_p \nleq \alpha$) for every $p \in P$ and $\bigwedge\{x_p \mid p \in P\} = 0 \leq \alpha$, which contradicts the finite meet-irreducibility of α. $\square$

Lemma 2.10. *For any algebra $A \in \mathcal{V}$,*

$$I \in \mathrm{P}(\mathrm{Con_c}\, A) \iff \sup I \in \mathrm{M}^*(\mathrm{Con}\, A).$$

Proof. The equivalence follows from Lemma 2.6 and Lemma 2.9. $\square$

Now recall the Compact Intersection Property of variety $\mathcal{V}$. We say that $\mathcal{V}$ has the Compact Intersection Property (CIP), if for every $A \in \mathcal{V}$ the intersection of any two compact congruences of A is a compact congruence.

Theorem 2.11. ([10: Theorem 3.1]) *Let $\mathcal{V}$ be a locally finite congruence distributive variety. The following conditions are equivalent.*

(1) *$\mathcal{V}$ has CIP.*
(2) *Every finite subalgebra of a subdirectly irreducible algebra of $\mathcal{V}$ is subdirectly irreducible.*
(3) *For every embedding $f\colon A \to B$ of algebras in $\mathcal{V}$ with A finite, the mapping* $\mathrm{Con_c}\, f$ *preserves meets.*

3. Description via direct limits

In this and the next section we assume that $\mathcal{V}$ is a finitely generated congruence distributive variety with CIP.

Theorem 3.1. *Let L be a distributive lattice with 0. The following conditions are equivalent:*

(1) *$L \simeq \operatorname{Con_c} A$ for some $A \in \mathcal{V}$.*

(2) *L is isomorphic to the direct limit of a P-indexed diagram $\vec{L} = (L_p, \varphi_{p,q} \mid p \le q \text{ in } P)$, where each L_p is a finite distributive lattice and each $\varphi_{p,q}$ is a 0-preserving lattice homomorphism such that*

   (a) *For every $p \in P$, the ordered set $\mathrm{M}^*(L_p)$ has an admissible $\mathrm{SI}(\mathcal{V})$-valuation $(v_p(\alpha), f^p_{\alpha,\beta})$.*

   (b) *For every $p, q \in P$, $p \le q$ and for every $\alpha \in \mathrm{M}^*(L_q)$ there exists embedding*
$$e^\alpha_{p,q} \colon v_p(\varphi^{\leftarrow}_{p,q}(\alpha)) \to v_q(\alpha)$$
   such that
$$e^\beta_{p,q} f^p_{\alpha',\beta'} = f^q_{\alpha,\beta} e^\alpha_{p,q},$$
   for every $\alpha \le \beta$ in $\mathrm{M}^*(L_q)$ and $\alpha' := \varphi^{\leftarrow}_{p,q}(\alpha)$, $\beta' := \varphi^{\leftarrow}_{p,q}(\beta)$.

Proof.

(1) $\Longrightarrow$ (2): Let $L \simeq \operatorname{Con_c} A$ for some $A \in \mathcal{V}$. Let P be the family of all finite subsets of A ordered by set inclusion. Let A_p be the subalgebra of A generated by $p \in P$. Since $\mathcal{V}$ is finitely generated, every A_p is finite. For every $p, q \in P$, $p \le q$, we put $L_p = \operatorname{Con_c} A_p$ and $\varphi_{p,q} = \operatorname{Con_c} e_{p,q}$, where $e_{p,q}$ is the inclusion $A_p \to A_q$. By Theorem 2.11, every $\varphi_{p,q}$ is 0-homomorphism of finite lattices. Then $A \simeq \varinjlim A_p$, so $L \simeq \operatorname{Con_c} A \simeq \varinjlim \operatorname{Con_c} A_p = \varinjlim L_p$.

Moreover $\mathrm{M}^*(L_p) = \mathrm{M}^*(\operatorname{Con_c} A_p)$, hence we can define a map
$$v_p \colon M^*(L_p) \to \mathrm{SI}(\mathcal{V})$$
by $v_p(\alpha) = A_p/\alpha$ for every $\alpha \in \mathrm{M}^*(L_p)$. Further, for every $\alpha, \beta \in \mathrm{M}^*(L_p)$, $\alpha \le \beta$ we define a homomorphism
$$f^p_{\alpha,\beta} \colon A_p/\alpha \to A_p/\beta$$
as the natural projection ($f^p_{\alpha,\beta}([x]_\alpha) = [x]_\beta$). It is easy to see that $(v_p, f^p_{\alpha,\beta})$ is a $\mathrm{SI}(\mathcal{V})$-valuation on $M^*(L_p)$. By [11: Lemma 2.2], it is admissible.

Now, let $p, q \in P$, $p \le q$ and let $\alpha \in \mathrm{M}^*(L_q) = \mathrm{M}^*(\operatorname{Con_c} A_q)$. Since A_p is a subalgebra of A_q, we know (see the remark before Lemma 2.1) that $\alpha' = \varphi^{\leftarrow}_{p,q}(\alpha) = \alpha \restriction A_p$. We define an embedding
$$e^\alpha_{p,q} \colon A_p/\alpha' \to A_q/\alpha$$

naturally as $e^{\alpha}_{p,q}([x]_{\alpha'}) = [x]_{\alpha}$. It is easy to see that the following diagram commutes:

$$\begin{array}{ccc} A_p/\alpha' & \xrightarrow{e^{\alpha}_{p,q}} & A_q/\alpha \\ {\scriptstyle f^p_{\alpha',\beta'}}\big\downarrow & & {\scriptstyle f^q_{\alpha,\beta}}\big\downarrow \\ A_p/\beta' & \xrightarrow{e^{\beta}_{p,q}} & A_q/\beta \end{array}$$

(2) $\Longrightarrow$ (1): For every $p \in P$ we have a $M^*(L_p)$-indexed diagram $\vec{D}_p := (v_p(\alpha), f^p_{\alpha,\beta})$. By Theorem 2.5, $\varprojlim \vec{D}_p = A_p \in \mathcal{V}$ such that $\mathrm{M}^*(\mathrm{Con_c}\, A_p) \simeq \mathrm{M}^*(L_p)$.

Let $p, q \in P$, $p \le q$ and let $\mathrm{M}^*(L_p) = \{\beta_1, \dots, \beta_r\}$, $\mathrm{M}^*(L_q) = \{\gamma_1, \dots, \gamma_s\}$. We consider elements of $A_p \le \prod\limits_{\alpha \in M^*(L_p)} v_p(\alpha)$ in the form $a = (a_1, \dots, a_r)$ with $a_j \in v_p(\beta_j)$ and similarly for A_q. Further we write $f^q_{i,k}$ and $f^p_{j,l}$ instead of $f^q_{\gamma_i,\gamma_k}$ and $f^p_{\beta_j,\beta_l}$.

By Lemma 2.1(5) we can define a map $g_{p,q} \colon A_p \to A_q$ such that

$$g_{p,q}((a_1, \dots, a_r)) = (d_1, \dots, d_s),$$

where $d_i = e^{\gamma_i}_{p,q}(a_j)$ such that $\beta_j = \varphi^{\leftarrow}_{p,q}(\gamma_i)$. We have $a_j \in v_p(\beta_j)$ and $d_i \in v_q(\gamma_i)$. We need to show that $(d_1, \dots, d_s) \in A_q$.

Let $\gamma_i \le \gamma_k$, then $\beta_j = \varphi^{\leftarrow}_{p,q}(\gamma_i) \le \varphi^{\leftarrow}_{p,q}(\gamma_k) = \beta_l$. Since A_p is an inverse limit, we have $a_l = f^p_{j,l}(a_j)$. Thus, by the assumption (2)(b) we have

$$f^q_{i,k}(d_i) = f^q_{i,k}(e^{\gamma_i}_{p,q}(a_j)) = e^{\gamma_k}_{p,q}(f^p_{j,l}(a_j)) = e^{\gamma_k}_{p,q}(a_l) = d_k.$$

So $(d_1, \dots, d_s) \in A_q$, hence $g_{p,q}$ is well defined and it is a routine to show that $g_{p,q}$ is a homomorphism. Hence $\vec{A} = (A_p, g_{p,q})$ is a directed P-indexed diagram in $\mathcal{V}$. Denote A the direct limit of this diagram.

Denote by δ_k the kth projection $A_p \to v_P(\beta_k)$ $(k = 1, \dots, r)$ and by ε_l the lth projection $A_q \to v_q(\gamma_l)$ $(l = 1, \dots, s)$. By Theorem 2.5 we have $\mathrm{Con_c}\, A_p \simeq L_p$, where the isomorphism $h_p \colon \mathrm{Con_c}\, A_p \to L_p$ can be defined by $h_p(\ker(\delta_k)) = \beta_k$. Similarly, let h_q be the isomorphism $\mathrm{Con_c}\, A_q \to L_q$ defined by $h_q(\ker(\varepsilon_l)) = \gamma_l$.

Now we claim that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Con_c}\, A_p & \xrightarrow{\mathrm{Con_c}\, g_{p,q}} & \mathrm{Con_c}\, A_q \\ {\scriptstyle h_p}\big\downarrow & & {\scriptstyle h_q}\big\downarrow \\ L_p & \xrightarrow{\varphi_{p,q}} & L_q \end{array}$$

Since h_p, h_q are isomorphisms, we have $h_p^{\leftarrow} = h_p^{-1}$, $h_q^{\leftarrow} = h_p^{-1}$. By Lemma 2.1, we can prove equivalently that $h_p^{\leftarrow}\varphi_{p,q}^{\leftarrow} = (\mathrm{Con_c}\, g_{p,q})^{\leftarrow} h_q^{\leftarrow}$. All maps in diagram preserve $\wedge$, it suffices to show that $h_p^{\leftarrow}\varphi_{p,q}^{\leftarrow}(\gamma_i) = (\mathrm{Con_c}\, g_{p,q})^{\leftarrow} h_q^{\leftarrow}(\gamma_i)$ for every $\gamma_i \in \mathrm{M}^*(L_q)$.

Let $\varphi_{p,q}^{\leftarrow}(\gamma_i) = \beta_j$. Then $h_p^{\leftarrow}\varphi_{p,q}^{\leftarrow}(\gamma_i) = \ker(\delta_j)$. Further, $h_q^{\leftarrow}(\gamma_i) = \ker(\varepsilon_i)$ and

$$\begin{aligned}(x,y) \in (\mathrm{Con_c}\, g_{p,q})^{\leftarrow}(\ker(\varepsilon_i)) &\iff (g_{p,q}(x), g_{p,q}(y)) \in \ker(\varepsilon_i)\\ &\iff g_{p,q}(x)_i = g_{p,q}(y)_i \iff e_{p,q}^{\gamma_i}(x_j) = e_{p,q}^{\gamma_i}(y_j)\\ &\iff x_j = y_j \iff (x,y) \in \ker(\delta_j),\end{aligned}$$

so

$$(\mathrm{Con_c}\, g_{p,q})^{\leftarrow} h_q^{\leftarrow}(\gamma_i) = \ker(\delta_j) = h_p^{\leftarrow}\varphi_{p,q}^{\leftarrow}(\gamma_i).$$

This proves that our diagram commutes. Using this commutativity and the fact that the functor $\mathrm{Con_c}$ preserves direct limits, we have

$$\mathrm{Con_c}\, A = \mathrm{Con_c} \varinjlim \vec{A} \simeq \varinjlim \mathrm{Con_c}\, \vec{A} \simeq \varinjlim \vec{L} \simeq L. \qquad \square$$

In concrete cases, the general description of the direct limit system in (2) can be specified more closely, which sometimes leads to a nice description of the class $\mathrm{Con}\,\mathcal{V}$. (See such examples in our previous paper [10].) However, in many cases the description provided by Theorem 3.1 is not quite satisfactory. That's why in the next section we try another approach.

4. Description via Priestley duality

Let $\mathcal{V}$ be a finitely generated congruence distributive variety with CIP. Hence $\mathrm{Con_c}\, A$ is a distributive lattice with 0 for every $A \in \mathcal{V}$. So it is natural to describe these lattices by means of Priestley duality.

Let L be a distributive lattice with 0 and let $(\mathrm{P}(L), \leq, \tau)$ be its dual Priestley space. Consider the following conditions on $(\mathrm{P}(L), \leq, \tau)$:

(Pr1) $\mathrm{P}(L)$ has an admissible $\mathrm{SI}(\mathcal{V})$-valuation $\big(v(I), f_{I,J}\big)$;

(Pr2) For every $I \in \mathrm{P}(L)$ there exists an open set U such that $I \in U$ and for every $J \in U$ the algebra $v(I)$ is isomorphic to a subalgebra of $v(J)$.

Theorem 4.1. *If $L \simeq \mathrm{Con_c}\, A$ for some $A \in \mathcal{V}$, then the dual Priestley space* $(\mathrm{P}(L), \leq, \tau)$ *satisfies* (Pr1) *and* (Pr2).

Proof. Let $L = \mathrm{Con_c}\, A$ for some $A \in \mathcal{V}$. By Lemma 2.10 we have $\sup I \in \mathrm{M}^*(\mathrm{Con}\, A)$ for every $I \in \mathrm{P}(L)$. So we can define a map $v\colon \mathrm{P}(L) \to \mathrm{SI}(\mathcal{V})$ such that

$$v(I) = A/\sup I.$$

Since for every $I, J \in \mathrm{P}(L)$, $I \leq J$, we have $\sup I \leq \sup J$, we can define a surjective homomorphism

$$f_{I,J} \colon A/\sup I \to A/\sup J$$

as natural projection $f_{I,J}([x]_{\sup I}) = [x]_{\sup J}$. It is easy to see that $(v(I), f_{I,J})$ is an $\mathrm{SI}(\mathcal{V})$-valuation on $\mathrm{P}(L)$. The admissibility follows from [11: Lemma 2.2].

To prove (Pr2), let $I \in \mathrm{P}(L)$. Since the quotient algebra $A/\sup I$ is finite, there are $n \in \mathbb{N}$ and $x_1, \dots, x_n \in A$ such that for every $y \in A$ there exists $i \in \{1, \dots, n\}$ with $x_i \in [y]_{\sup I}$.

Let B be the subalgebra of A generated by $x_1, \dots, x_n$. Hence, B is finite and $B/\sup I{\restriction}B$ is isomorphic to $A/\sup I$. Denote by U the intersection

$$\bigcap_{\substack{x,y \in B \\ \Theta(x,y) \in I}} \big\{J \in \mathrm{P}(L) \mid \Theta(x,y) \in J\big\} \cap \bigcap_{\substack{x,y \in B \\ \Theta(x,y) \notin I}} \big\{J \in \mathrm{P}(L) \mid \Theta(x,y) \notin J\big\}.$$

Since U is an intersection of finitely many clopen sets, it is a clopen set. Moreover, it is easy to see that $I \in U$. For every $J \in U$ we have $\sup I{\restriction}B = \sup J{\restriction}B$. Indeed, the compactness of $\Theta(x,y)$ implies that $\Theta(x,y) \leq \sup I$ iff $\Theta(x,y) \in I$, hence

$$\begin{aligned}
\sup I{\restriction}B &= \big\{(x,y) \in B^2 \mid (x,y) \in \sup I\big\} = \big\{(x,y) \in B^2 \mid \Theta(x,y) \leq \sup I\big\} \\
&= \big\{(x,y) \in B^2 \mid \Theta(x,y) \in I\big\} = \big\{(x,y) \in B^2 \mid \Theta(x,y) \in J\big\} \\
&= \sup J{\restriction}B.
\end{aligned}$$

So, $v(J) = A/\sup J \geq B/\sup J{\restriction}B = B/\sup I{\restriction}B \simeq A/\sup I = v(I)$. $\square$

Unfortunately, the converse to Theorem 4.1 does not hold in general. (See [9].) We are only able to prove the sufficiency of conditions (Pr1) and (Pr2) in some special cases. We will present two such special cases. First we prove a generalization of Theorem 2.5.

Theorem 4.2. *Let L be a distributive lattice with 0 and let $(\mathrm{P}(L), \tau, \leq)$ be its dual Priestley space. Let $(v(I), f_{I,J})$ be a $\mathrm{SI}(\mathcal{V})$-valuation on $\mathrm{P}(L)$. Let A be a subalgebra of $\prod_{I \in \mathrm{P}(L)} v(I)$ such that*

(a) *for every $a \in A$ and for every $I, J \in \mathrm{P}(L)$, $I \leq J$,*

$$a_J = f_{I,J}(a_I);$$

(b) *for every $I \in \mathrm{P}(L)$ and for every $u \in v(I)$ there exists $a \in A$ such that*

$$a_I = u;$$

(c) *for every $I, J \in \mathrm{P}(L)$, $I \nleq J$ there exist $a, b \in A$ such that*

$$a_I = b_I, \qquad a_J \neq b_J;$$

(d) *for every* $a, b \in A$ *the set* $U_{a,b} = \{I \mid a_I = b_I\}$ *is clopen.*

Then the Priestley spaces $\mathrm{P}(L)$ *and* $\mathrm{P}(\mathrm{Con_c}\, A)$ *are isomorphic (and hence* L *and* $\mathrm{Con_c}\, A$ *are isomorphic) and the isomorphism* $\varphi\colon \mathrm{P}(L) \to \mathrm{P}(\mathrm{Con_c}\, A)$ *can be defined by* $\varphi(I) = \{\alpha \in \mathrm{Con_c}\, A \mid \alpha \le \ker p_I\}$*, where* $p_I\colon A \to v(I)$ *is the projection.*

Proof. Let $I \in \mathrm{P}(L)$, by (b), p_I is surjective, hence $\ker p_I \in \mathrm{M}^*(\mathrm{Con}\, A)$, so by Lemma 2.10 we have $\{\alpha \in \mathrm{Con_c}\, A \mid \alpha \le \ker p_I\} \in \mathrm{P}(\mathrm{Con_c}\, A)$. Thus the map $\varphi\colon \mathrm{P}(L) \to \mathrm{P}(\mathrm{Con_c}\, A)$ is well-defined.

We prove that φ is an isomorphism of ordered topological spaces. If $K \in \mathrm{P}(\mathrm{Con_c}\, A)$, then $K = \{\alpha \in \mathrm{Con_c}\, A \mid \alpha \le \gamma\}$ for some $\gamma \in \mathrm{M}^*(\mathrm{Con}\, A)$.

We claim that $\ker p_I \le \gamma$ for some I. For contradiction suppose that $\ker p_I \nleq \gamma$ for every $I \in \mathrm{P}(L)$. Our assumption means that

$$\bigcup_{(a,b)\in A^2\setminus\gamma} U_{a,b} = \mathrm{P}(L).$$

Since $\mathrm{P}(L)$ is compact, there exists $n \in \mathbb{N}$ and elements $a^i, b^i \in A$ ($i \in \{1,\dots,n\}$) such that $(a^i, b^i) \notin \gamma$ and for every $J \in \mathrm{P}(L)$ there exists $j \in \{1,\dots,n\}$ with $a^j_J = b^j_J$, hence $\Theta(a^j, b^j) \le \ker p_J$. Then

$$\bigcap_{1\le i\le n} \Theta(a^i, b^i) \le \bigwedge_{J\in\mathrm{P}(L)} \ker_{p_J} = 0 \le \gamma.$$

This contradicts the $\wedge$-irreducibility of γ (note that if $(a,b)\notin\gamma$, then $\Theta(a,b)\nleq\gamma$). Hence, there exists $I \in \mathrm{P}(L)$ such that $\ker p_I \le \gamma$. Since $p_I\colon A \to v(I)$ is surjective, the lattice $\mathrm{Con}\, v(I)$ is isomorphic to the filter $\uparrow \ker p_I$ of $\mathrm{Con}\, A$. The congruence $\gamma \in \uparrow \ker p_I$ corresponds to the congruence $\gamma' \in \mathrm{Con}\, v(I)$ given by $\gamma' = \{(x_I, y_I) \mid (x,y) \in \gamma\}$. By Definition 2.2, $\gamma' = \ker f_{I,J}$ for some $J \ge I$, so

$$\begin{aligned}(x,y)\in\gamma &\iff (x_I,y_I)\in\gamma' = \ker f_{I,J}\\ &\iff f_{I,J}(x_I) = f_{I,J}(y_I) \iff x_J = y_J\\ &\iff (x,y)\in\ker p_J,\end{aligned}$$

hence $\gamma = \ker p_J$. Thus for every $K \in \mathrm{P}(\mathrm{Con_c}\, A)$ there exists $J \in \mathrm{P}(L)$ such that $\varphi(J) = K$, so φ is surjective. Moreover, by (c), $\varphi(I) \le \varphi(J)$ if and only if $I \le J$. Hence φ is bijective and both φ and φ^{-1} preserve the order.

It remains to show that φ is a topological homeomorphism. We check that $\varphi^{-1}(U)$ is open set for every U from the subbase of $\mathrm{P}(\mathrm{Con_c}\, A)$. Let $\alpha \in \mathrm{Con_c}\, A$, so $\alpha = \bigcup_{i=1}^{k} \Theta(a^i, b^i)$ for some $a^i, b^i \in A$, $i \in \{1,\dots,k\}$. Let

$$U = \{I \in \mathrm{P}(\mathrm{Con_c}\, A) \mid \alpha \in I\},$$

then

$$I \in \varphi^{-1}(U) \iff \varphi(I) \in U \iff \alpha \in \varphi(I) \iff \alpha \leq \ker p_I,$$

so

$$\begin{aligned}\varphi^{-1}(U) = \{I \mid \alpha \leq \ker p_I\} &= \bigcap_{1\leq i\leq k} \{I \mid \Theta(a^i, b^i) \leq \ker p_I\} \\ &= \bigcap_{1\leq i\leq k} \{I \mid a^i_I = b^i_I\}.\end{aligned}$$

Hence $\varphi^{-1}(U) = \bigcap_{1\leq i\leq k} U_{a^i,b^i}$ is clopen. Now let $V = \{I \in \mathrm{P}(\mathrm{Con_c}\, A) \mid \alpha \notin I\}$, hence $\varphi^{-1}(V)$ is a complement of $\varphi^{-1}(U)$, so it is also clopen.

We have proved that φ is continuous. Since both spaces are compact Hausdorff, and φ is bijective, it must be a homeomorphism. Hence, $\mathrm{P}(L) \simeq \mathrm{P}(\mathrm{Con_c}\, A)$, so $L \simeq \mathrm{Con_c}\, A$. □

Note that if L is finite, then the topology is discrete. Hence Theorem 2.5 is a special case of Theorem 4.2.

5. Special cases

Let $\mathcal{V}$ be a finitely generated congruence distributive variety with CIP. Moreover, assume that $\mathrm{Con}\, S$ is a chain for every $S \in \mathrm{SI}(\mathcal{V})$. We denote

$$\mathbf{S}_i := \{A \in \mathrm{SI}(\mathcal{V}) \mid \mathrm{Con}\, A \text{ is an } i\text{-element chain}\}.$$

Further, denote by $\mathcal{P}_n$ the class of all partially ordered sets $(C, \leq)$ with a largest element such that for every $x \in C$, $\uparrow x$ is a k-element chain, $k \in \{1, \ldots, n\}$. Hence, $C \in \mathcal{P}_n$ is a disjoint union of antichains $C_0, \ldots, C_{n-1}$ such that $|\uparrow x| = k+1$ for $x \in C_k$. Let L be a lattice such that $\mathrm{P}(L) \in \mathcal{P}_n$, then denote $P_k = P_k(L) = (\mathrm{P}(L))_k$ for $k = 0, \ldots, n-1$. Notice that P_0 is a one-element set.

We present a detailed analysis of two special cases.

The first case

We suppose that $\mathcal{V}$ satisfies the following additional assumptions:

(A1) $\max\{j \mid \mathbf{S}_j \neq \emptyset\} = n > 1$.

(A2) If $A \leq B \in \mathrm{SI}(\mathcal{V})$, then $\mathrm{Con}\, A \simeq \mathrm{Con}\, B$.

Lemma 5.1. *Let L be a distributive lattice with 0 such that its dual Priestley space $(\mathrm{P}(L), \leq, \tau)$ satisfies* (Pr1) *and* (Pr2)*. Then*

(1) $\mathrm{P}(L) \in \mathcal{P}_n$,

(2) *for every $k \in \{0, \ldots, n-1\}$, the set $P_k(L)$ is clopen.*

P r o o f. By Lemma 2.3, $\uparrow I$ is isomorphic to $\operatorname{Con} v(I)$, which is a chain of length at most n for every $I \in \mathrm{P}(L)$.

Further, by (Pr2), for every $I \in P_k(L)$ there exists an open set U such that $I \in U$ and $v(I)$ is isomorphic to a subalgebra of $v(J)$ for every $J \in U$. By the assumption (A2) we have $\operatorname{Con} v(I) \simeq \operatorname{Con} v(J)$, thus $J \in P_k(L)$. This shows that $P_k(L)$ is open. Since the sets $P_0(L), \dots, P_n(L)$ are mutually disjoint, they must also be closed. $\square$

Theorem 5.2. *Let $\mathcal{V}$ satisfy the assumptions stated above. Let L be a distributive lattice with 0 and let $(\mathrm{P}(L), \leq, \tau)$ be its dual Priestley space. The following conditions are equivalent.*

1. *$L \simeq \operatorname{Con_c} A$ for some $A \in \mathcal{V}$;*
2. *$(\mathrm{P}(L), \leq, \tau)$ satisfies (Pr1) and (Pr2);*
3. *$\mathrm{P}(L) \in \mathcal{P}_n$ and for every $k = 0, 1, \dots, n-1$ the set $P_k(L)$ is clopen.*

P r o o f. We have already proved (1) $\implies$ (2) $\implies$ (3).

(3) $\implies$ (1): By (A1) there exists $F \in \mathrm{SI}(\mathcal{V})$ such that $\operatorname{Con} F$ is an n-element chain $\alpha_{n-1} < \alpha_{n-2} < \cdots < \alpha_0$. For every $i \in \{0, \dots, n-1\}$ denote $F_i = F/\alpha_i$, so $\operatorname{Con} F_i$ is an $(i+1)$-element chain. For every $j \leq i$ we define a map $f_{i,j} \colon F_i \to F_j$ as the natural projection. For every $I, J \in \mathrm{P}(L)$, $I \leq J$ denote $v(I) = F_{|\uparrow I|-1}$ and $f_{I,J} = f_{|\uparrow I|-1, |\uparrow J|-1}$. We define an algebra

$$A \leq \prod_{I \in \mathrm{P}(L)} F_I$$

such that $a \in A$ if

(i) $a_J = f_{I,J}(a_I)$, whenever $I \leq J$.

(ii) for every $i \in \{0, \dots, n-1\}$ and every $u \in F_i$ the set $\{I \in P_i(L) \mid a_I = u\}$ is open.

We can see that $(v(I), f_{I,J})$ is a $\mathrm{SI}(\mathcal{V})$-valuation on $\mathrm{P}(L)$. Moreover, since F_i is finite, all the sets $\{I \in P_i \mid a_I = u\}$ are clopen. For $a, b \in A$, the set $U_{a,b} = \{I \mid a_I = b_I\}$ is a union of sets $\{I \in P_i \mid a_I = u\} \cap \{I \in P_i \mid b_I = u\}$ for every $u \in F_i$, $(i = 0, \dots, n-1)$, hence $U_{a,b}$ is clopen. It remains to check the conditions (b) and (c) of Theorem 4.2.

To prove (b), let $I \in \mathrm{P}_j(L)$ and let $k \in F_I$. Hence $k = [v]_{\alpha_j}$ for some $v \in F$. Let $a = (a_K)_{K \in \mathrm{P}(L)}$, where $a_K = [v]_{\alpha_i}$ for every $K \in P_i(L)$. We claim that $a \in A$. Condition (i) holds trivially. Let $i \in \{0, \dots, n-1\}$, for every $u \in F_i$ we have $u = [w]_{\alpha_i}$ for some $w \in F$. Hence the set

$$\{I \in P_i \mid a_I = [w]_{\alpha_i}\} = \{I \in P_i \mid [v]_{\alpha_i} = [w]_{\alpha_i}\} = \begin{cases} \emptyset & \text{if } [v]_{\alpha_i} \neq [w]_{\alpha_i}, \\ P_i & \text{if } [v]_{\alpha_i} = [w]_{\alpha_i} \end{cases}$$

is in each case clopen. So $a \in A$ and $a_I = k$.

To prove (c), let $I, J \in \mathrm{P}(L)$ such that $I \not\leq J$. Denote $j = |{\uparrow}J| - 1$. Since $j \geq 1$, there exist $u, v \in v(J)$, $u \neq v$ such that $(u, v) \in \ker f_{j,j-1}$. Hence, there exist $t_1, t_2 \in F$ such that

$$u = [t_1]_{\alpha_j} \neq [t_2]_{\alpha_j} = v$$

and

$$[t_1]_{\alpha_s} \neq [t_2]_{\alpha_s}$$

for every $s < j$.

For every $K \in P_l(L)$ denote

$$a_K = [t_1]_{\alpha_l}.$$

We have already shown that every element of the form $a = (a_K)_{K \in \mathrm{P}(L)}$ belongs to A. Further, by CTOD, there exists a clopen up-set $V \subseteq \mathrm{P}(L)$ such that $I \in V$, $J \notin V$. Denote

$$U := {\downarrow}(P_j(L) \setminus V).$$

Both $P_j \setminus V$ and $P_j \cap V$ are clopen, so ${\downarrow}(P_j \setminus V)$ and ${\downarrow}(P_j \cap V)$ are disjoint closed sets and their union is equal to the clopen set $P_j \cup P_{j+1} \cup \cdots \cup P_{n-1}$. Hence U is a clopen set. For every $l \in \{0, \dots, n-1\}$ and every $K \in P_l$ we denote

$$b_K = \begin{cases} [t_1]_{\alpha_l} & \text{if } K \notin U, \\ [t_2]_{\alpha_l} & \text{if } K \in U. \end{cases}$$

Now denote $b = (b_K)_{K \in \mathrm{P}(L)}$ and we prove that $b \in A$.

Let $K, M \in P(L)$, $K \leq M$. If $K, M \in U$ or $K, M \notin U$, then clearly $f_{K,M}(b_K) = b_M$. If $K \in U$ and $M \notin U$, then

$$r = |{\uparrow}K| - 1 \geq j,$$
$$s = |{\uparrow}M| - 1 < j$$

and $f_{K,M}(b_K) = f_{r,s}([t_2]_{\alpha_r}) = [t_2]_{\alpha_s} = [t_1]_{\alpha_s} = b_M$.

Further, let $i \in \{0, \dots, n-1\}$ and $w \in F_i$. The set

$$\{I \in P_i \mid b_I = w\} = \begin{cases} P_i & \text{if } w = [t_2]_{\alpha_i} = [t_1]_{\alpha_i}, \\ P_i \cap U & \text{if } w = [t_2]_{\alpha_i} \neq [t_1]_{\alpha_i}, \\ P_i \setminus U & \text{if } w = [t_1]_{\alpha_i} \neq [t_2]_{\alpha_i}, \\ \emptyset & \text{otherwise} \end{cases}$$

is in each case clopen. Hence $a, b \in A$. Moreover $a_I = [t_1]_{\alpha_{|{\uparrow}I|-1}} = b_I$, $a_J = [t_1]_{\alpha_j} \neq [t_2]_{\alpha_j} = b_J$.

By Theorem 4.2 we have $\mathrm{P}(L) \simeq \mathrm{P}(\mathrm{Con_c}\, A)$, so $L \simeq \mathrm{Con_c}\, A$. $\square$

Thus, in our special case we have proved the converse to Theorem 4.1. Thanks to the result of Katriňák and Mitschke, we can go even further. Recall [8] or [2] for the definition of a dual Stone lattice of order n.

Theorem 5.3. *Let $\mathcal{V}$ satisfy the assumptions stated above. Let L be a distributive lattice with 0 and let $(\mathrm{P}(L), \leq, \tau)$ be its dual Priestley space. The following conditions are equivalent.*

(1) *$L \simeq \mathrm{Con_c}\, A$ for some $A \in \mathcal{V}$.*

(2) *$\mathrm{P}(L) \in \mathcal{P}_n$ and the set $P_k(L)$ is clopen for every $k = 0, 1, \dots, n-1$.*

(3) *$\mathrm{P}(L) \in \mathcal{P}_n$ and for every $i \in \{0, \dots, n-2\}$, there exists an element $e_i \in \bigcap\{I \mid I \in P_0(L) \cup \dots \cup P_i(L)\}$ such that $e_i \notin J$ for every $J \in P_j(L)$ $(j > i)$.*

(4) *L is a dual Stone lattice of order n.*

Proof. We have already proved the equivalence (1) $\iff$ (2). The equivalence (3) $\iff$ (4) was proved in [8: Theorem 4.5] (in a dual form).

(2) $\implies$ (3): Let $i, j \in \{0, \dots, n-1\}$, $i < j$, let $I \in P_0 \cup \dots \cup P_i$, $J \in P_j$. Since $I \nsubseteq J$, there exists $\alpha_{I,J} \in I \setminus J$. Denote

$$U_{I,J} = \{K \in \mathrm{P}(L) \mid \alpha_{I,J} \notin K\},$$
$$\mathcal{U}_I = \{U_{I,J} \mid J \in P_j \text{ for some } j > i\}.$$

It is easy to see that $I \notin U_{I,J}$, $J \in U_{I,J}$. Moreover since $\mathcal{U}_I$ is an open cover of the closed (and hence compact) set $Q_i = \bigcup\limits_{j>i} P_j$, there exist finitely many $J_1, \dots, J_m \in \mathrm{P}(L)$ such that

$$Q_i \subseteq \{K \mid \alpha_{I,J_1} \notin K \text{ or } \dots \text{ or } \alpha_{I,J_m} \notin K\} = \{K \mid \alpha_{I,J_1} \vee \dots \vee \alpha_{I,J_m} \notin K\}.$$

Denote $\beta_I = \alpha_{I,J_1} \vee \dots \vee \alpha_{I,J_m}$. Hence for every $I \in P_0 \cup \dots \cup P_i$ there exists $\beta_I \in L$ such that

(i) $\beta_I \in I$,

(ii) $\beta_I \notin J$ for every $J \in P_j$, $j > i$.

Further, denote $U_I = \{K \in \mathrm{P}(L) \mid \beta_I \in K\}$. The collection of sets U_I, $I \in P_0 \cup \dots \cup P_i$ covers the compact set $P_0 \cup \dots \cup P_i$. By the compactness, there exist $I_1, \dots, I_q \in P_0 \cup \dots \cup P_i$ such that

$$P_0 \cup \dots \cup P_i \subseteq \{K \mid \beta_{I_1} \in K \text{ or } \dots \text{ or } \beta_{I_q} \in K\}.$$

Using the fact that ideals $K \in \mathrm{P}(L)$ are prime we obtain

$$P_0 \cup \dots \cup P_i \subseteq \{K \mid \beta_{I_1} \wedge \dots \wedge \beta_{I_q} \in K\}.$$

Denote $e_i = \beta_{I_1} \wedge \dots \wedge \beta_{I_q}$. Hence for every $I \in P_0 \cup \dots \cup P_i(L)$ and for every $J \in P_j(L)$ $(j > i)$ we have $e_i \in I$ and $e_i \notin J$.

(3) $\implies$ (2): Let $i \in \{0, \ldots, n-2\}$. By (3), $P_{i+1} \cup \cdots \cup P_{n-1} = Q_i = \{I \in \mathrm{P}(L) \mid e_i \notin I\}$, which is a clopen set. Then also $P_i = Q_{i-1} \setminus Q_i$ is clopen, $i = 1, \ldots, n-2$. Moreover, P_0 is the complement of Q_0 and $P_{n-1} = Q_{n-2}$. $\square$

The second case

Similarly as in the first special case, we assume that $\mathcal{V}$ is finitely generated congruence distributive variety and $\operatorname{Con} A$ is a chain for every $A \in \mathrm{SI}(\mathcal{V})$. Instead of $(A1)$, $(A2)$ we consider the following additional assumptions:

(B1) $\max\{j \mid \mathbf{S}_j \neq \emptyset\} = 3$;

(B2) For every $G \leq F \in \mathrm{SI}(\mathcal{V})$ either $\operatorname{Con} G \simeq \operatorname{Con} F$ or $G \in \mathbf{S}_2$, $F \in \mathbf{S}_3$;

(B3) There exists $F^0 \in \mathbf{S}_3$ such that $F^0/\alpha \leq F^0$, where α is the only nontrivial congruence on F^0;

Lemma 5.4. *Let $G \leq F \in \mathrm{SI}(\mathcal{V})$ such that $\operatorname{Con} G$ is a 2-element chain $0 < 1$ and $\operatorname{Con} F$ is a 3-element chain $0 < \alpha < 1$. Let h be an embedding $G \to F$, then $\operatorname{Con} h(1) = 1$.*

P r o o f. We have $\operatorname{Con} h(1) \neq 0$ because h is injective. For contradiction suppose that $\operatorname{Con} h(1) = \alpha$. Hence $h(G)$ is contained in one α-class, so F/α has an one-element subalgebra. We have a contradiction with the assumption (B2). $\square$

Lemma 5.5. *F^0/α is isomorphic to a retract of F^0.*

P r o o f. Let $e\colon F^0/\alpha \to F^0$ be an embedding and $f\colon F^0 \to F^0/\alpha$ be a natural projection. Then by 5.4

$$\operatorname{Con} fe(1) = \operatorname{Con} f \operatorname{Con} e(1) = \operatorname{Con} f(1) = 1,$$

so $\operatorname{Con} fe$ is an isomorphism $\{0,1\} \to \{0,1\}$, thus fe is injective and since F^0 is finite, fe is an automorphism. Hence $G = e(F^0/\alpha)$ is a retract of F^0 isomorphic with F^0/α (with $e(fe)^{-1}f$ as the retraction). $\square$

Lemma 5.6. *Let L be a distributive lattice with 0 such that its dual Priestley space $(\mathrm{P}(L), \leq, \tau)$ satisfies (Pr1) and (Pr2). Then*

1. *$\mathrm{P}(L) \in \mathcal{P}_3$,*
2. *$P_0(L)$ is clopen, $P_2(L)$ is open.*

P r o o f. By the definition of a $\mathrm{SI}(\mathcal{V})$-valuation, $\uparrow I$ is isomorphic to $\operatorname{Con} v(I)$ which is a chain of length at most 3.

Further let $i \in \{0, 2\}$ and let $I \in P_i(L)$. By (Pr2) there exists an open set U with $I \in U$ and for every $J \in U$ we have $v(I) \leq v(J)$, thus $J \in P_i(L)$ by assumption (B2), hence $P_i(L)$ is open. Since $P_0(L)$ is a one-element set, it is also closed. $\square$

Theorem 5.7. *Let $\mathcal{V}$ satisfy the assumptions stated above. Let L be a distributive lattice with 0 and let $(\mathrm{P}(L), \leq, \tau)$ be its dual Priestley space. The following conditions are equivalent.*

(1) $L \simeq \mathrm{Con_c}\, A$ *for some* $A \in \mathcal{V}$*;*

(2) $(\mathrm{P}(L), \leq, \tau)$ *satisfies* (Pr1) *and* (Pr2)*;*

(3) $\mathrm{P}(L) \in \mathcal{P}_3$*,* $P_0(L)$ *is clopen and* $P_2(L)$ *is open.*

P r o o f. We have already proved (1) $\implies$ (2) $\implies$ (3).

(3) $\implies$ (1): Denote $F = F^0$, let G be a retract of F such that $G \simeq F/\alpha$. For every $I \in \mathrm{P}(L)$ denote

$$\begin{aligned} v(I) &:= F \quad \text{if } I \in P_2,\\ v(I) &:= G \quad \text{if } I \in P_1,\\ v(I) &:= 1 \quad \text{if } I \in P_0. \end{aligned}$$

(By 1 we denote both the one-element algebra in $\mathcal{V}$ and its single element.) By Lemma 5.5 there exists a surjective homomorphism $f\colon F \to G$ such that $f{\restriction}G = \mathrm{id}_G$. For every $I, J \in P(L)$, $I < J$ we define a map $f_{I,J} = v(I) \to v(J)$ such that

$$\begin{aligned} f_{I,J}(a) &= f(a), &&\text{if } I \in P_2,\ J \in P_1;\\ f_{I,J}(a) &= 1, &&\text{if } J \in P_0; \end{aligned}$$

(and, of course, $f_{I,I}$ is the identity for every $I \in P(L)$.) We define an algebra

$$A \leq \prod_{I \in \mathrm{P}(L)} v(I)$$

such that $a \in A$ if

(i) $a_J = f_{I,J}(a_I)$, whenever $I \leq J$;

(ii) for every $u \in F$ the set $\bigl\{I \in \mathrm{P}(L) \mid a_I = u\bigr\}$ is clopen.

(Note that the set $\bigl\{I \in \mathrm{P}(L) \mid a_I = u\bigr\}$ may contain elements from both P_1 and P_2.) It is easy to see that $(v(I), f_{I,J})$ is a $\mathrm{SI}(\mathcal{V})$-valuation on $\mathrm{P}(L)$. Let $a, b \in A$. Since $U_{a,b} = \bigl\{I \mid a_I = b_I\bigr\}$ is a union of sets $\bigl\{I \mid a_I = u\bigr\} \cap \bigl\{I \mid b_I = u\bigr\}$ for every possible u, we have that $U_{a,b}$ is clopen. It remains to check the conditions (b) and (c) of Theorem 4.2.

First let $U \subseteq \mathrm{P}_2(L)$ and $V \subseteq P(L)$ be clopen sets. Let $v \in F$, $v_1, v_2 \in G$, $v_1 \neq v_2$. For every $K \in \mathrm{P}(L)$ denote

$$a(U,v)_K = \begin{cases} 1 & \text{if } K \in P_0,\\ v & \text{if } K \in U,\\ f(v) & \text{if } K \in P_1 \cup (P_2 \setminus U), \end{cases}$$

$$b(V,v_1,v_2)_K = \begin{cases} 1 & \text{if } K \in P_0, \\ v_1 & \text{if } K \in \downarrow(P_1 \setminus V), \\ v_2 & \text{if } K \in \downarrow(P_1 \cap V). \end{cases}$$

Since P_1 is closed, we have $P_1 \cap V$ and $P_1 \setminus V$ closed, hence both $\downarrow(P_1 \cap V)$ and $\downarrow(P_1 \setminus V)$ are closed. These sets are disjoint and their union $P_1 \cup P_2$ is clopen. Hence both $\downarrow(P_1 \cap V)$ and $\downarrow(P_1 \setminus V)$ are clopen sets.

Denote $a = (a(U,v)_K)_{K \in \mathrm{P}(L)}$, we prove that $a \in A$. Let $I \in P_2$, $J \in P_1$, $I < J$. Then

$$f_{I,J}(a_I) = f(v) = a_J.$$

Further, for every $u \in F$ the set

$$\{I \in \mathrm{P}(L) \mid a_I = u\} = \begin{cases} U & \text{if } u = v \neq f(v), \\ P_1 \cup (P_2 \setminus U) & \text{if } u = f(v) \neq v, \\ P_1 & \text{if } u = v = f(v), \\ \emptyset & \text{otherwise} \end{cases}$$

is in each case clopen. Thus, $a \in A$.

Now denote $b = (b(V,v_1,v_2)_K)_{K \in \mathrm{P}(L)}$, we prove that $b \in A$. Let $I \in P_2$, $J \in P_1$, $I < J$. Then

$$f_{I,J}(b_I) = \begin{cases} f(v_1) = v_1 = b_J & \text{if } I \in \downarrow(P_1 \setminus V) \cap P_2, \\ f(v_2) = v_2 = b_J & \text{if } I \in \downarrow(P_1 \cap V) \cap P_2. \end{cases}$$

For every $u \in F$ the set

$$\{I \in \mathrm{P}(L) \mid a_I = u\} = \begin{cases} \downarrow(P_1 \cap V) & \text{if } u = v_2 = f(v_2), \\ \downarrow(P_1 \setminus V) & \text{if } u = v_1 = f(v_1), \\ \emptyset & \text{otherwise} \end{cases}$$

is in each case clopen, so $b \in A$.

Now we can deal with the conditions (b) and (c) of Theorem 4.2. Let $J \in P_2$ and let $v \in v(J) = F$. By CTOD there exists a clopen down-set U such that $J \in U$ and $(P_1 \cup P_0) \cap U = \emptyset$. Denote $a = (a(U,v)_K)_{K \in \mathrm{P}(L)}$. We have $a \in A$ and $a_J = v$.

Now let $J \in P_1$ and let $v \in v(J) = G$, hence $f(v) = v$. Denote $a = (a(\emptyset,v)_K)_{K \in \mathrm{P}(L)}$, we have $a \in A$ and $a_J = f(v)$.

Further let $I, J \in \mathrm{P}(L)$ such that $I \nleq J$.

First let $J \in P_2$, then there exist $v_1, v_2 \in v(J) = F$, $v_1 \neq v_2$ such that $f(v_1) = f(v_2)$. By CTOD there exists a clopen down-set $U \subseteq P_2$ such that $J \in U$ and $(P_1 \cup P_0 \cup \{I\}) \cap U = \emptyset$. Denote $a = (a(U,v_1)_K)_{K \in \mathrm{P}(L)}$, $b = (a(U,v_2)_K)_{K \in \mathrm{P}(L)}$. We have $a, b \in A$ and $a_I = f(v_1) = f(v_2) = b_I$, $a_J = v_1 \neq v_2 = b_J$.

Now let $J \in P_1$ and let $v_1, v_2 \in v(J) = G$ such that $v_1 \neq v_2$. By CTOD there exists a clopen up-set $V \subseteq \mathrm{P}(L)$ such that $I \in V$ and $J \notin V$. Denote $a = (a(\emptyset, v_2)_K)_{K \in \mathrm{P}(L)}$, $b = (b(V, v_1, v_2)_K)_{K \in \mathrm{P}(L)}$. We have $a, b \in A$. Moreover $a_I = f(v_2) = v_2 = b_I$, $a_J = v_2 \neq v_1 = b_J$.

By Theorem 4.2, $\mathrm{P}(L) \simeq \mathrm{P}(\mathrm{Con_c}\, A)$, so $L \simeq \mathrm{Con_c}\, A$. □

Similarly as in the first case we can go even further. Recall some basic facts about dual Stone lattices. A bounded lattice L is called dually pseudocomplemented if for every $x \in L$ there exists its dual pseudocomplement $x^+ = \min\{y \in L \mid x \vee y = 1\}$. The elements satisfying $x^+ = 1$ are called *co-dense* and form an ideal of L denoted by $\bar{D}(L)$. *A dual Stone lattice* is a distributive dually pseudocomplemented lattice satisfying the identity $x^+ \wedge x^{++} = 0$.

The next lemma follows from results of Katriňák and Mitschke (see [8]).

Lemma 5.8. *Let L be a dual Stone lattice. Denote $\max(\mathrm{P}(L))$ the set of all maximal elements of $\mathrm{P}(L) \setminus \{L\}$. Then*

(1) *$I \in \max(\mathrm{P}(L))$ if and only if $\bar{D}(L) \in I$;*

(2) *for every $I \in \mathrm{P}(L)$ there exists exactly one $J \in \max(\mathrm{P}(L))$ such that $I \subseteq J$.*

Theorem 5.9. *The following conditions are equivalent:*

(1) *$L \simeq \mathrm{Con_c}\, A$ for some $A \in \mathcal{V}$.*

(2) *$\mathrm{P}(L) \in \mathcal{P}_3$ and $P_0(L)$ is clopen, $P_2(L)$ is open.*

(3) *L is a dual Stone lattice and its co-dense elements form a generalized Boolean lattice.*

Proof. We have already proved the equivalence (1) $\iff$ (2).

(2) $\implies$ (3): We know that L is isomorphic to the lattice of all proper clopen down-sets of $\mathrm{P}(L)$, hence $\emptyset$ is the least and $P_1 \cup P_2$ is the greatest element of L. Further, let U be proper clopen down-set of $\mathrm{P}(L)$. It is easy to see that its dual pseudocomplement is $U^+ = \downarrow(P_1 \setminus U)$. Then $U^{++} = \downarrow(P_1 \cap U)$, so $U^+ \cap U^{++} = \emptyset$. Hence, L is a dual Stone lattice.

Clearly, $U^+ = 1$ if and only if $U \subseteq P_2$ and thus

$$\bar{D}(L) = \{U \mid U \subseteq P_2,\ U \text{ clopen}\}.$$

Obviously, clopen subsets of P_2 form a generalized Boolean lattice. This generalized Boolean lattice is not necessarily a Boolean lattice, since P_2 itself need not be clopen.

(3) $\implies$ (2): It is easy to see that $P_0 = \{I \mid 1 \in I\} = \{L\}$ is clopen. Since $P_1 = \max(\mathrm{P}(L))$, by Lemma 5.8(1) for every $I \notin P_1 \cup P_0$ there exists $x \in \bar{D}(L)$ such that $x \notin I$. Hence $I \in V_x = \{J \in \mathrm{P}(L) \mid x \notin J\}$ and since V_x is open and $P_1 \cap V_x = \emptyset$, we have P_1 closed.

Further we prove that $\mathrm{P}(L) \setminus (P_1 \cup P_0)$ is an antichain. For contradiction suppose that there exist $I, J \in \mathrm{P}(L) \setminus (P_1 \cup P_0)$ such that $I < J$. By CTOD there exists a clopen down-set V such that $J \in V$ and $V \cap (P_1 \cup P_0) = \emptyset$. Also by CTOD, there exists a clopen down-set $U \subseteq V$ such that $I \in U$ and $J \notin U$. Identifying L with the lattice of all clopen down-sets of $\mathrm{P}(L)$, we have $V, U \in \bar{D}(L)$. However, U has no complement in the interval $[\emptyset, V]$. Indeed, let $W \subseteq V$ be a clopen down-set. Now

- if $J \in W$, then $I \in W$, so $U \cap W \neq \emptyset$;
- if $J \notin W$, then $J \notin U \cup W$, so $U \cup W \neq V$.

It is a contradiction with the fact that $\bar{D}(L)$ is a generalized Boolean lattice.

Thus, $P_2 = \mathrm{P}(L) \setminus (P_1 \cup P_0)$ is an antichain. By Lemma 5.8(2), for every $I \in P_2$ the set $\uparrow I$ is a 3-element chain. So, $\mathrm{P}(L) \in \mathcal{P}_3$. $\square$

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