Czechoslovak Mathematical Journal, 64 (139) (2014), 115–132

CONGRUENCE LATTICES IN VARIETIES WITH COMPACT INTERSECTION PROPERTY

Filip Krajník, Miroslav Ploščica, Košice

(Received November 16, 2012)

Abstract. We say that a variety $\mathcal{V}$ of algebras has the Compact Intersection Property (CIP), if the family of compact congruences of every $A \in \mathcal{V}$ is closed under intersection. We investigate the congruence lattices of algebras in locally finite, congruence-distributive CIP varieties and obtain a complete characterization for several types of such varieties. It turns out that our description only depends on subdirectly irreducible algebras in $\mathcal{V}$ and embeddings between them. We believe that the strategy used here can be further developed and used to describe the congruence lattices for any (locally finite) congruence-distributive CIP variety.

Keywords: compact congruence; congruence-distributive variety

MSC 2010: 08A30, 08B10, 06D15

1. Introduction

It is well known that a lattice is algebraic if and only if it is isomorphic to the congruence lattice of some algebra. Much less is known about congruence lattices of algebras of a specific type.

Let $\mathcal{K}$ be a class of algebras and denote by $\operatorname{Con}\mathcal{K}$ the class of all lattices isomorphic to $\operatorname{Con} A$ (the congruence lattice of an algebra A) for some $A \in \mathcal{K}$. There are many papers investigating $\operatorname{Con}\mathcal{K}$ for various classes $\mathcal{K}$. However, the full description of $\operatorname{Con}\mathcal{K}$ has proved to be a very difficult (and probably intractable) problem, even for the most common classes of algebras, like groups or lattices.

The present paper is motivated by the observation that in most relevant cases when $\operatorname{Con}\mathcal{K}$ is well understood, the algebras in $\mathcal{K}$ have a special property: the intersection of any two compact congruences of $A \in \mathcal{K}$ is compact. This seems

Supported by VEGA Grant 2/0194/10 and VVGS-PF-2012-50

quite natural. Algebraic lattices are determined by their sets of compact elements. There is a considerable evidence that the difficulty in describing congruence lattices is connected with the fact that the compact congruences form a join-semilattice, which in general is not a lattice. For instance, there are several refinement properties that are trivial in lattices, but very nontrivial in semilattices ([15], [12], [11]).

There are nice results using the above intersection property. Let us mention the following two. Every algebraic distributive lattice in which the compact elements are closed under intersection is isomorphic to the congruence lattice of a lattice (E. T. Schmidt [14].) Similarly, every algebraic distributive lattice in which the compact elements are closed under intersection is isomorphic to the congruence lattice of a locally matricial algebra (P. Růžička [13]).

In our paper we first give several characterizations of locally finite, congruence-distributive varieties with CIP. The most difficult part of this theorem has already been proved by W. J. Blok and D. Pigozzi in [3]. However, we present a new approach, which, we believe, provides a valuable insight into the topic and helps to progress in our main aim: to decribe congruence lattices of algebras in congruence-distributive CIP varieties. We provide such a description for three of the simplest types of such varieties. We follow a uniform strategy, which may be effective for solving this problem in general.

2. Basic facts and notation

Let L be a lattice. An element $a \in L$ is called compact if for every $X \subseteq L$ such that $a \leqslant \bigvee X$ there exists a finite $Y \subseteq X$ with $a \leqslant \bigvee Y$. An element $a \in L$ is called strictly meet-irreducible if $a = \bigwedge X$ implies that $a \in X$ for every subset X of L. Note that the greatest element of L is not strictly meet-irreducible. Let $\mathrm{M}(L)$ denote the set of all strictly meet-irreducible elements. The following assertion is well known.

Theorem 2.1. *If L is an algebraic lattice, then for all $a \in L$, $a = \bigwedge X$, where $X = \{b;\ a \leqslant b,\ b \in \mathrm{M}(L)\}$. Further, for every $x, y \in L$ with $x \nleqslant y$ there exists $z \in \mathrm{M}(L)$ such that $z \geqslant y$, $z \ngeqslant x$.*

If f is a mapping, then $\mathrm{dom}(f)$ stands for its domain. By $\ker(f)$ we denote the binary relation on $\mathrm{dom}(f)$ given by $(x, y) \in \ker(f)$ if $f(x) = f(y)$. By $f{\restriction}X$ we mean the restriction of f to X.

Let P be a partially ordered set. For every $x \in P$ we set $\uparrow x = \{y \in P;\ y \geqslant x\}$, $\downarrow x = \{y \in P;\ y \geqslant x\}$.

The congruence lattice of an algebra A will be denoted by $\mathrm{Con}\,A$. The set $\mathrm{Con_c}\,A$ of all compact (finitely generated) congruences of A is a $(0, \vee)$-subsemilattice of

$\operatorname{Con} A$. The smallest element of $\operatorname{Con} A$ will be denoted by Δ. The lattice $\operatorname{Con} A$ is uniquely determined by the semilattice $\operatorname{Con_c} A$ (it is isomorphic to the ideal lattice of $\operatorname{Con_c} A$) and $\operatorname{Con_c} A$ is often easier to describe.

It is a well known fact that for every $\theta \in \operatorname{Con} A$ the lattice $\operatorname{Con} A/\theta$ is isomorphic to $\uparrow\theta$. Hence, $\theta \in \mathrm{M}(\operatorname{Con} A)$ if and only if the quotient algebra A/θ is subdirectly irreducible. Equivalently, $\theta \in \mathrm{M}(\operatorname{Con} A)$ if and only if $\theta = \ker(f)$ for some surjective homomorphism $f\colon A \to S$, with S subdirectly irreducible.

For algebras A and B, $A \leqslant B$ denotes that A is a subalgebra of B. For a subset $B \subseteq A$ let $\langle B\rangle$ denote the subalgebra of A generated by B. If $B \leqslant A$ and $\theta \in \operatorname{Con} A$, then $\theta\restriction B = \theta \cap B^2$ is the restriction of θ to B. For every homomorphism $f\colon A \to B$ we define the mapping

$$\operatorname{Con_c} f\colon \operatorname{Con_c} A \to \operatorname{Con_c} B$$

by the rule that, for every $\alpha \in \operatorname{Con_c} A$, $\operatorname{Con_c} f(\alpha)$ is the congruence generated by the set $\{(f(x), f(y));\ (x,y) \in \alpha\}$. This mapping is a homomorphism of $(\vee, 0)$-semilattices.

Now let $\varphi\colon K \to L$ be a $(0,\vee)$-homomorphism of finite $(0,\vee)$-semilattices. We define a map $\varphi^{\leftarrow}\colon L \to K$ by

$$\varphi^{\leftarrow}(\beta) = \bigvee\{\alpha;\ \varphi(\alpha) \leqslant \beta\}.$$

Note that if $K = \operatorname{Con_c} A$, $L = \operatorname{Con_c} B$ and $\varphi = \operatorname{Con_c} f$ for some algebras A, B and a homomorphism $f\colon A \to B$, then $\varphi^{\leftarrow}(\beta) = \{(x,y) \in A;\ (f(x), f(y)) \in \beta\}$. If A is a subalgebra of B and $f\colon A \to B$ is the inclusion, then $\varphi^{\leftarrow}(\beta)$ is the restriction of $\beta \in \operatorname{Con} B$ to A.

(The construction also works for infinite complete lattices.) Such a pair $(\varphi, \varphi^{\leftarrow})$ is also known as a Galois connection. The following facts are well known.

Lemma 2.2. *Let $\varphi\colon K \to L$ be a $(0,\vee)$-homomorphism of finite lattices.*

(1) *$\varphi^{\leftarrow}$ preserves $\wedge$ and the largest element.*
(2) $\varphi(\alpha) = \bigwedge\{\beta;\ \alpha \leqslant \varphi^{\leftarrow}(\beta)\}$.
(3) $\varphi(\alpha) \leqslant \beta \Leftrightarrow \alpha \leqslant \varphi^{\leftarrow}(\beta)$.
(4) *If $\psi\colon L \to M$ is another $(0,\vee)$-homomorphism of finite lattices, then $(\psi\varphi)^{\leftarrow} = \varphi^{\leftarrow}\psi^{\leftarrow}$.*

Lemma 2.3. *If $\varphi\colon K \to L$ is a $0,1$-preserving homomorphism of finite distributive lattices, then $\varphi^{\leftarrow}(c) \in \mathrm{M}(K)$ for every $c \in \mathrm{M}(L)$.*

We will also use the following simple assertion.

Lemma 2.4. *Let $\varphi\colon L_1 \to L_2$ be a $(0,\vee)$-homomorphism of finite lattices. If $\varphi^{\leftarrow}(\mathrm{M}(L_2)) \subseteq \mathrm{M}(L_1)$, then $\varphi(1) = 1$.*

P r o o f. Suppose that $\varphi(1) < 1$, then $\varphi(1) \leqslant c$ for some $c \in \mathrm{M}(L_2)$. Hence $\varphi^{\leftarrow}(c) = 1$, which is in contradiction with $\varphi^{\leftarrow}(c) \in \mathrm{M}(L_1)$. $\square$

Next we recall the algebraic constructions of the direct and inverse limits. Let P be an ordered set. Let $\mathcal{K}$ be a class of algebras. A P-indexed diagram $\vec{A}$ in $\mathcal{K}$ consists of a family $(A_p, p \in P)$ of algebras in $\mathcal{K}$ and a family $(f_{p,q}, p \leqslant q)$ of homomorphisms $f_{p,q}\colon A_p \to A_q$ such that $f_{p,p}$ is the identity of A_p and $f_{p,r} = f_{q,r} f_{p,q}$ for all $p \leqslant q \leqslant r$.

If the index set P is directed (for every $p, q \in P$ there exists $r \in P$ with $p, q \leqslant r$), then we define the *direct limit* of $\vec{A}$ as

$$\varinjlim \vec{A} := \varinjlim A_p := \bigsqcup_{p \in P} A_p / \sim,$$

where $\bigsqcup_{p \in P} A_p$ is the disjoint union of the family $(A_p, p \in P)$ and the equivalence relation $\sim$ is defined by

$$x \sim y \Leftrightarrow \exists r \in P\colon\ f_{p,r}(x) = f_{q,r}(y).$$

A special case of the direct limit is the directed union, when all the homomorphisms are set inclusions. Note that in the category theory this construction corresponds to the (directed) colimit.

The *inverse limit* of $\vec{A}$ is defined for any partially ordered set P as a subalgebra of the direct product of $\prod_{p \in P} A_p$, namely

$$\varprojlim \vec{A} := \varprojlim A_p := \left\{ a \in \prod_{p \in P} A_p;\ a_q = f_{p,q}(a_p) \text{ for every } p, q \in P,\ p \leqslant q \right\}.$$

(The elements of $\prod_{p \in P} A_p$ are written in the form $a = (a_p)_{p \in P}$.) A special case of this construction is the direct product, which arises when P is an antichain. In the category theory language, this construction is the limit.

It is well known that any variety $\mathcal{K}$ is closed under the formation of direct and inverse limits.

The direct limit construction will be used to obtain the description of $\mathrm{Con}_c A$ for infinite $A \in \mathcal{K}$ from the description of $\mathrm{Con}_c A$ for finite A. This is possible due to the following two facts. First, Con_c is a functor preserving the direct limits, which means that for every directed P-indexed diagram $\vec{A}$ in $\mathcal{K}$ we have the P-indexed diagram $\mathrm{Con}_c \vec{A} = (\mathrm{Con}_c A_p, \mathrm{Con}_c \varphi_{p,q})$ in the category of $(\vee, 0)$-semilattices and $(\vee, 0)$-homomorphisms, and

$$\mathrm{Con}_c \varinjlim \vec{A} \simeq \varinjlim \mathrm{Con}_c \vec{A}.$$

Second, let $\vec{A} = (A_p, \varphi_{p,q})$ and $\vec{B} = (B_p, \psi_{p,q})$ be directed P-indexed diagrams and let $h_p\colon A_p \to B_p$ be an isomorphism for every $p \in P$ such that the following diagram commutes:

$$\begin{array}{ccc} A_p & \xrightarrow{f_{p,q}} & A_q \\ {\scriptstyle h_p}\big\downarrow & & \big\downarrow{\scriptstyle h_q} \\ B_p & \xrightarrow{\psi_{p,q}} & B_q \end{array}$$

Then

$$\varinjlim \vec{A} \simeq \varinjlim \vec{B}.$$

The inverse limits will be used to construct algebras with a prescribed finite (distributive) congruence lattice. This is possible due to the following result.

Theorem 2.5 ([10]). *Let $\mathcal{V}$ be a locally finite congruence distributive variety. Let L be a finite distributive lattice and let $P = \mathrm{M}(L)$. Let $\vec{A} = (A_p, \varphi_{p,q})$ be a P-indexed diagram in $\mathcal{V}$ satisfying the following conditions:*

(1) *For every $p \in P$ and every $u \in A_p$ there exists*

$$a \in \varprojlim A_p$$

such that $a_p = u$.

(2) *For every $p, q \in P$, $p \nleqslant q$ there exist*

$$a, b \in \varprojlim A_p$$

such that $a_p = b_p$ and $a_q \neq b_q$.

(3) *For every $p \in P$, the sets $\{\ker(\varphi_{p,q});\ p \leqslant q\}$ and $\mathrm{M}(\mathrm{Con}\, A_p)$ coincide.*

Then

$$A := \varprojlim A_p$$

is an algebra whose congruence lattice is isomorphic to L. The isomorphism $h\colon \mathrm{M}(L) \to \mathrm{M}(\mathrm{Con}\, A)$ can be defined by $h(p) = \ker(\alpha_p)$, where α_p is the projection $A \to A_p$.

3. Compact intersection property

For any class $\mathcal{V}$ of algebras, let $\mathrm{SI}(\mathcal{V})$ denote the class of all subdirectly irreducible members of $\mathcal{V}$.

Theorem 3.1. *Let $\mathcal{V}$ be a locally finite congruence distributive variety. The following conditions are equivalent.*

(1) *The intersection of two compact congruences of A is compact for every $A \in \mathcal{V}$.*

(2) *Every finite subalgebra of a subdirectly irreducible algebra of $\mathcal{V}$ with more than one element is subdirectly irreducible.*

(3) *If T is a finite subalgebra of a subdirectly irreducible algebra of $\mathcal{V}$ with more than one element, then the ordered set $\mathrm{M}(\mathrm{Con}\, T)$ has a least element.*

(4) *For every embedding $f\colon A \to B$ of algebras in $\mathcal{V}$ with A finite, the mapping $\mathrm{Con_c}\, f$ preserves meets.*

Proof. (2)$\Leftrightarrow$(3) is well known.

(1)$\Rightarrow$(3) Let $T \leqslant S \in \mathrm{SI}(\mathcal{V})$, T finite. Since $\mathrm{Con}\, T$ is finite, it suffices to show that for all $\beta_1, \beta_2 \in \mathrm{M}(\mathrm{Con}\, T)$ there exists $\beta \in \mathrm{M}(\mathrm{Con}\, T)$ such that $\beta \subseteq \beta_1 \cap \beta_2$. Let $A := F(\aleph_0)$ denote the free algebra in $\mathcal{V}$ with $\aleph_0$ as a free generating set. Choose a surjective homomorphism $h_0\colon \langle X_0\rangle \to T$, where $X_0 \subseteq \aleph_0$ is finite and large enough. Since A is free, h_0 can be extended to a homomorphism $h\colon A \to T$. Further, we consider the natural homomorphisms $g_1\colon T \to T/\beta_1$, $g_2\colon T \to T/\beta_2$. Then $\ker(g_i h_0) \in \mathrm{M}(\mathrm{Con}\langle X_0\rangle)$.

$$\begin{tikzcd}
 & \langle X_0\rangle \arrow[d, "h_0"'] & \\
 & T \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\
T/\beta_1 & & T/\beta_2
\end{tikzcd}$$

Since $\mathrm{Con}\langle X_0\rangle$ is finite and distributive, there is a smallest element γ_i in the set $\{\alpha \in \mathrm{Con}\langle X_0\rangle;\ \alpha \nsubseteq \ker(g_i h_0)\}$. Let $\alpha_i \in \mathrm{Con}\, A$ be the congruence generated by γ_i. Then $\alpha_i \restriction \langle X_0\rangle \supseteq \gamma_i$. The inverse inclusion follows from the fact that the projection $\langle X_0\rangle \to \langle X_0\rangle/\gamma_i$ can be extended to a homomorphism $l\colon A \to \langle X_0\rangle/\gamma_i$, thus $\alpha_i \subseteq \ker(l)$ and $\alpha_i \restriction \langle X_0\rangle \subseteq \ker(l \restriction \langle X_0\rangle) = \gamma_i$. So, $\alpha_i \restriction \langle X_0\rangle = \gamma_i$.

Congruences α_1, α_2 are compact, so by our assumption $\alpha_1 \cap \alpha_2$ is compact, too. It means that there exists a finite set $Y \subseteq \aleph_0$, $X_0 \subseteq Y$ such that $\alpha_1 \cap \alpha_2$ is generated by $\alpha_1 \cap \alpha_2 \restriction \langle Y\rangle$.

$$\begin{tikzcd}
\langle X_0\rangle \arrow[r, "h\restriction\langle X_0\rangle"] \arrow[d, "b_0\restriction\langle X_0\rangle"'] & T \arrow[dl, "k"'] \\
W &
\end{tikzcd}$$

Let $f\colon A \to S$ be a surjective homomorphism such that $f{\restriction}\langle Y\rangle = h{\restriction}\langle Y\rangle$, then $\ker(f{\restriction}\langle X_0\rangle) = \ker(h{\restriction}\langle X_0\rangle) \subseteq \ker(g_i h_0)$. Thus $\gamma_i \nleq \ker(f{\restriction}\langle X_0\rangle)$ and hence $\alpha_i \nleq \ker(f)$. Since $\ker(f) \in \mathrm{M}(\mathrm{Con}\, A)$, we have $\alpha_1 \cap \alpha_2 \nleq \ker(f)$ and thus

$$\alpha_1 \cap \alpha_2{\restriction}\langle Y\rangle \nleq \ker(f{\restriction}\langle Y\rangle) = \ker(h{\restriction}\langle Y\rangle).$$

Therefore there exists $\delta \in \mathrm{M}(\mathrm{Con}\langle Y\rangle)$ such that

$$\delta \geqslant \ker(h{\restriction}\langle Y\rangle), \delta \ngeq \alpha_1 \cap \alpha_2{\restriction}\langle Y\rangle.$$

Let $b_0\colon \langle Y\rangle \to \langle Y\rangle|\delta := W$ be the natural map; it can be extended to a homomorphism $b\colon A \to W$. Moreover, for all $y \in Y$ there exists $x_0 \in X_0$ such that $(y, x_0) \in \ker(h)$. Therefore $(y, x_0) \in \ker(b_0)$, so $b_0(y) = b_0(x_0)$. This shows that $b_0(\langle X_0\rangle) = b(\langle Y\rangle) = W$.

Since $\ker(b_0{\restriction}\langle X_0\rangle) = \delta{\restriction}\langle X_0\rangle \supseteq \ker(h{\restriction}\langle X_0\rangle)$, there exists a homomorphism $k\colon T \to W$ such that $kh{\restriction}\langle X_0\rangle = b_0{\restriction}\langle X_0\rangle$. Further, since $b_0(\langle X_0\rangle) = W \in \mathrm{SI}\,(\mathcal{V})$, we have $\ker(k) \in \mathrm{M}(\mathrm{Con}\, T)$. Further, $\alpha_1 \cap \alpha_2{\restriction}\langle Y\rangle \nsubseteq \ker(b_0)$ implies that $\alpha_1 \cap \alpha_2 \nsubseteq \ker(b)$ and thus $\alpha_1, \alpha_2 \nsubseteq \ker(b)$.

Since α_i are generated by γ_i for $i = 1, 2$, we have $\gamma_i \nsubseteq \ker(b)$ and thus

$$\gamma_i \nsubseteq \ker(b_0{\restriction}\langle X_0\rangle).$$

By the definition of γ_i it means $\ker(b_0{\restriction}\langle X_0\rangle) \subseteq \ker(g_i h_0)$. For every $(x, y) \in \ker(k)$ we have $x', y' \in \langle X_0\rangle$ such that $h(x') = x, h(y') = y$. Thus $(x', y') \in \ker(b_0{\restriction}\langle X_0\rangle)$, so $(x', y') \in \ker(g_i h_0)$. It means that $g_i(h_0(x')) = g_i(h_0(y'))$, hence $g_i(x) = g_i(y)$. We have proved that $\ker(k) \leqslant \ker(g_i) = \beta_i$ for $i = 1, 2$.

(3)$\Rightarrow$(1) Let $A \in \mathcal{V}$ and suppose that $\alpha_1, \alpha_2 \in \mathrm{Con}\, A$ are compact, but $\alpha_1 \cap \alpha_2$ is not compact. There exists a finite subalgebra $Y \leqslant A$ such that $\alpha_i{\restriction}Y$ generates α_i $(i = 1, 2)$. Denote $\gamma_i := \alpha_i{\restriction}Y$. Since $\mathrm{Con}\, Y$ is a finite distributive lattice, there exist $\vee$-irreducible $\delta_1, \delta_2, \ldots, \delta_n, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \in \mathrm{Con}\, Y$ such that $\gamma_1 = \bigvee_{j=1}^{n} \delta_j, \gamma_2 = \bigvee_{k=1}^{m} \varepsilon_k$. Let $\bar{\delta_j} \in \mathrm{Con}\, A$ be generated by δ_j, similarly $\bar{\varepsilon}_j$. Since $\delta_j \subseteq \gamma_1 \subseteq \alpha_1 \in \mathrm{Con}\, A$, we have $\bar{\delta_j} \subseteq \alpha_1$. Moreover, $\bigvee_{j=1}^{n} \bar{\delta_j} \supseteq \bigvee_{j=1}^{n} \delta_j = \gamma_1 = \alpha_1 \restriction Y$, thus $\alpha_1 = \bigvee_{j=1}^{n} \bar{\delta_j}$, and similarly $\alpha_2 = \bigvee_{k=1}^{m} \bar{\varepsilon}_k$. By distributivity, $\alpha_1 \cap \alpha_2 = \bigvee_{j,k} (\bar{\delta_j} \cap \bar{\varepsilon}_k)$. Since $\alpha_1 \cap \alpha_2$ is not compact, $\bar{\delta_j} \cap \bar{\varepsilon}_k$ is not compact for some j, k.

Let $\beta \in \mathrm{Con}\, A$ be generated by $\bar{\delta_j} \cap \bar{\varepsilon}_k{\restriction}Y$, thus $\beta \subsetneq \bar{\delta_j} \cap \bar{\varepsilon}_k$, so there exists a surjective homomorphism $h\colon A \to S \in \mathrm{SI}(\mathcal{V})$ such that $\beta \subseteq \ker(h)$, $\bar{\delta_j} \cap \bar{\varepsilon}_k \nsubseteq \ker(h)$. Let $T := h(Y) \subseteq S$, then $\mathrm{Con}\, T$ is isomorphic to $L := \{\alpha \in \mathrm{Con}\, Y;\ \ker(h{\restriction}Y) \subseteq \alpha\}$.

Since δ_j, ε_k are $\vee$-irreducible in $\operatorname{Con} Y$, there exist

$$\eta_1 = \max\{\alpha \in \operatorname{Con} Y;\ \delta_j \nleq \alpha\},$$
$$\eta_2 = \max\{\alpha \in \operatorname{Con} Y;\ \varepsilon_k \nleq \alpha\}.$$

Clearly $\eta_1, \eta_2 \in \mathrm{M}(\operatorname{Con} Y)$. If $\delta_j \subseteq \ker(h{\restriction}Y)$, then $\bar{\delta}_j \subseteq \ker(h)$, which contradicts our definition of the homomorphism h and thus $\delta_j \nsubseteq \ker(h{\restriction}Y)$. Hence $\ker(h{\restriction}Y) \subseteq \eta_1$, thus $\eta_1 \in L$ and similarly $\eta_2 \in L$. Since $\eta_1, \eta_2 \in \mathrm{M}(\operatorname{Con} Y)$, we have $\eta_1, \eta_2 \in \mathrm{M}(L)$. For every $\varrho \in \mathrm{M}(L)$ we have

$$\varrho \supseteq \ker(h{\restriction}Y) \supseteq \beta{\restriction}Y \supseteq \bar{\delta}_j \cap \bar{\varepsilon}_k{\restriction}Y \supseteq \delta_j \cap \varepsilon_k.$$

Either $\varrho \supseteq \delta_j$ or $\varrho \supseteq \varepsilon_k$, by the $\wedge$-irreducibility of ϱ. In the case $\varrho \supseteq \delta_j$ we have $\varrho \nsubseteq \eta_1$, and from $\varrho \supseteq \varepsilon_k$ we deduce $\varrho \nsubseteq \eta_2$. Hence, η_1 and η_2 do not have a common lower bound in L, so L cannot have a least element

(4)$\Rightarrow$(3) Let $A \leqslant B \in \mathrm{SI}(\mathcal{V})$, A finite. Suppose that $\mathrm{M}(\operatorname{Con} A)$ does not have a least element. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the minimal elements of $\mathrm{M}(\operatorname{Con} A)$, $n \geqslant 2$. Denote by $f\colon A \to B$ the inclusion. Then

$$\operatorname{Con_c} f(\alpha_1 \wedge \alpha_2 \wedge \dots \alpha_n) = \operatorname{Con_c} f(\Delta) = \Delta.$$

On the other hand,

$$\operatorname{Con_c} f(\alpha_1) \wedge \operatorname{Con_c} f(\alpha_2) \wedge \operatorname{Con_c} f(\alpha_n) \neq \Delta,$$

since the intersection of nonzero congruences in a subdirectly irreducible algebra cannot be Δ.

(2)$\Rightarrow$(4) Suppose that $\operatorname{Con_c} f\colon \operatorname{Con_c} A \to \operatorname{Con_c} B$ does not preserve meets. We can assume that $f\colon A \to B$ is a set inclusion. Then $\operatorname{Con_c} f(\alpha \wedge \beta) < \operatorname{Con_c} f(\alpha) \wedge \operatorname{Con_c} f(\beta)$ for some $\alpha, \beta \in \operatorname{Con_c} A = \operatorname{Con} A$. Hence, there is $\gamma \in \mathrm{M}(\operatorname{Con} B)$ such that

$$\gamma \geqslant \operatorname{Con_c} f(\alpha \wedge \beta),$$
$$\gamma \ngeqslant \operatorname{Con_c} f(\alpha) \wedge \operatorname{Con_c} f(\beta).$$

Hence,

$$\gamma \ngeqslant \operatorname{Con_c} f(\alpha), \quad \gamma \ngeqslant \operatorname{Con_c} f(\beta).$$

Now A/γ is a finite subalgebra of the subdirectly irreducible algebra B/γ, whose congruence lattice is isomorphic to $L = \{\theta \in \operatorname{Con} A;\ \gamma{\restriction}A \subseteq \theta\}$. To prove that A/γ is not subdirectly irreducible it suffices to find $\alpha^*, \beta^* \in \operatorname{Con} A$ with $\alpha^* \wedge \beta^* = \gamma{\restriction}A$ and $\alpha^*, \beta^* \neq \gamma{\restriction}A$.

We set $\alpha^* = \alpha \vee \gamma{\restriction}A$ and $\beta^* = \beta \vee \gamma{\restriction}A$. By distributivity,

$$\alpha^* \wedge \beta^* = (\alpha \wedge \beta) \vee \gamma{\restriction}A = \gamma{\restriction}A.$$

If $\alpha^* = \gamma{\restriction}A$, then

$$\mathrm{Con_c}\, f(\alpha) \leqslant \mathrm{Con_c}\, f(\alpha^*) = \mathrm{Con_c}\, f(\gamma{\restriction}A) \leqslant \gamma.$$

Hence $\alpha^* \neq \gamma{\restriction}A$ and similarly $\beta^* \neq \gamma{\restriction}A$. $\square$

The above result is not completely new. The equivalence of the first two conditions was proved by W. J. Blok and D. Pigozzi in [3] (and claimed by K. A. Baker on page 139 in [2]), using the concept of equationally definable principal meets. (See also [1].) We provide a new proof which does not refer to polynomials and, we believe, provides an insight helpful in describing the congruence lattices of algebras in congruence-distributive CIP varieties. Our proof follows the lines of reasoning from [9], which connected CIP to the concept of separable sets in $\mathrm{M}(\mathrm{Con}\, A)$ and to topological properties of $\mathrm{M}(\mathrm{Con}\, A)$.

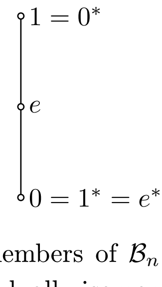
Examples. Let $\mathcal{B}_\omega$ be the variety of bounded distributive lattices with pseudocomplementation. By [8] (see also [5], page 165), the subvarieties of $\mathcal{B}_\omega$ form a chain

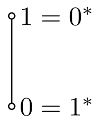
$$\mathcal{B}_{-1} \subset \mathcal{B}_0 \subset \mathcal{B}_1 \subset \ldots \subset \mathcal{B}_n \subset \ldots \subset \mathcal{B}_\omega.$$

Here $\mathcal{B}_{-1}$ is the trivial variety, $\mathcal{B}_0$ is the class of all Boolean algebras and for $n \geqslant 1$ the variety $\mathcal{B}_n$ is determined by the identity

$$(x_1 \wedge \ldots \wedge x_n)^* \vee (x_1^* \wedge \ldots \wedge x_n)^* \vee \ldots \vee (x_1 \wedge \ldots \wedge x_n^*)^* = 1.$$

In particular, $\mathcal{B}_1$ is the class of Stone algebras. The variety $\mathcal{B}_n$ $(n \geqslant 0)$ is generated by the algebra $B_n = \mathbf{2}^n \oplus \mathbf{1}$, that is the power set of an n-element set with a new top element added.

B_1:


B_0:


Subdirectly irreducible members of $\mathcal{B}_n$ are $B_n, B_{n-1}, \ldots, B_0$. (The congruence lattice of B_n is, as a lattice, dually isomorphic to B_n, that is $\mathrm{Con}\, B_n = \mathbf{1} \oplus \mathbf{2}^n$. It is easy to check that all subalgebras of B_n are isomorphic to one of $B_n, B_{n-1}, \dots, B_0$. Hence, by Theorem 3.1, every $\mathcal{B}_n$ has the Compact Intersection Property.

There is an easy way to construct examples of varieties satisfying CIP. Let A be a finite algebra generating a congruence distributive variety $\mathrm{HSP}(A)$. (For instance, A can be any finite lattice.) Enrich the type of A by defining every element $a \in A$ as a constant (nullary operation). Denote the resulting algebra as A^*. Every subdirectly irreducible member of $\mathcal{V} := \mathrm{HSP}(A^*)$ belongs to $\mathrm{HS}(A^*)$ (by Jónsson's lemma). Since A^* has no proper subalgebras, we have $\mathrm{HS}(A^*) = \mathrm{H}(A^*)$. And it is easy to see that the members of $\mathrm{H}(A^*)$ do not have proper subalgebras. Hence subdirectly irreducible algebras in $\mathcal{V}$ have no proper subalgebras, so the condition (2) of Theorem 3.1 is trivially satisfied.

4. Description of congruence lattices

In this section we investigate a few simple types of congruence distributive varieties $\mathcal{V}$ with CIP. We would like to demonstrate how to use Theorem 3.1 to obtain a description of congruence lattices of algebras in $\mathcal{V}$.

The first case. Let $\mathcal{V}$ be a nontrivial locally finite and congruence distributive variety with CIP such that

(1) $\mathrm{Con_c}\, F$ is a two-element chain for every $F \in \mathrm{SI}(\mathcal{V})$;
(2) no $F \in \mathrm{SI}(\mathcal{V})$ has a one-element subalgebra.

As an example of such a variety one can consider the variety of all bounded distributive lattices.

The description of $\mathrm{Con}\, A$ for finite $A \in \mathcal{V}$ is easy: it follows for instance from 2.5. (Note that if B is a finite Boolean lattice, then $\mathrm{M}(B)$ is the set of all coatoms.)

Lemma 4.1. *$L \simeq \mathrm{Con}\, A$ for some finite $A \in \mathcal{V}$ if and only if L is a finite Boolean lattice. Moreover, if $F \in \mathcal{V}$ with $\mathrm{Con}\, F \simeq \mathbf{2}$ and $n > 0$, then $\mathrm{Con}\, F^n \simeq \mathbf{2}^n$ and the coatoms of $\mathrm{Con}\, F^n$ are exactly the kernels of the projections $F^n \to F$.*

Now we prove the description result.

Theorem 4.2. *The following conditions are equivalent.*

(1) *$L \simeq \mathrm{Con_c}\, A$ for some $A \in \mathcal{V}$.*
(2) *L is isomorphic to the direct limit of the system $\vec{B} = (B_p, \varphi_{p,q};\ p \leqslant q \text{ in } P)$, where each B_p is a finite Boolean lattice and each $\varphi_{p,q}$ is a Boolean homomorphism.*
(3) *L is a Boolean lattice.*

P r o o f. (1)$\Rightarrow$(2) Let P be the family of all finite subsets of A ordered by set inclusion. Let A_p be the subalgebra of A generated by $p \in P$. For every $p, q \in P$, $p \leqslant q$, we put $B_p = \operatorname{Con_c} A_p$, $\varphi_{p,q} = \operatorname{Con_c} e_{p,q}$, where $e_{p,q}$ is the inclusion $A_p \to A_q$. Then $A \simeq \varinjlim A_p$, so

$$L \simeq \operatorname{Con_c} A \simeq \varinjlim \operatorname{Con_c} A_p = \varinjlim B_p.$$

By 4.1, every B_p is a finite Boolean lattice. By Theorem 3.1, every $\varphi_{p,q}$ is a 0-preserving lattice homomorphism. Suppose that $\varphi_{p,q}(1) < 1$, then $\varphi_{p,q}(1) \leqslant c$ for some coatom $c \in \mathrm{M}(B_q)$. Hence $\varphi^{\leftarrow}_{p,q}(c) = 1$, which means that $A_p/\varphi^{\leftarrow}_{p,q}(c)$ is a one-element algebra. However, $\varphi^{\leftarrow}_{p,q}(c)$ is a restriction of $c \in \operatorname{Con} A_q$ to A_p, so $A_p/\varphi^{\leftarrow}_{p,q}(c) \leqslant A_q/c$, which means that the subdirectly irreducible algebra A_q/c has a one-element subalgebra, contradicting our assumption on $\mathcal{V}$. Therefore, $\varphi_{p,q}$ is a lattice homomorphism which preserves 0 and 1. It is well known that such a homomorphism must also preserve the complements, so $\varphi_{p,q}$ is a Boolean homomorphism.

(2)$\Rightarrow$(3) L is a direct limit of Boolean lattices and all $\varphi_{p,q}$ are Boolean homomorphisms, thus L is a Boolean lattice.

(3)$\Rightarrow$(2) Every Boolean lattice is the direct limit of its finite Boolean sublattices (with inclusions as homomorphisms).

(2)$\Rightarrow$(1) Choose $F \in \mathrm{SI}(\mathcal{V})$ arbitrarily. So, $\operatorname{Con_c} F = \mathbf{2}$. For every $p \in P$ let $A_p = F^n$, where $n = |\mathrm{M}(B_p)|$. Let $p, q \in P$, $p \leqslant q$. Let $\mathrm{M}(B_p) = \{b_1, b_2, \ldots, b_n\}$, $\mathrm{M}(B_q) = \{c_1, c_2, \ldots, c_m\}$. (So, $A_p = F^n$, $A_q = F^m$.) Let $f_{p,q}$ be a map $A_p \to A_q$ defined by

$$f_{p,q}(a_1, \ldots, a_n) = (d_1, \ldots, d_m),$$

where $d_i = a_j$ such that $b_j = \varphi^{\leftarrow}_{p,q}(c_i)$. By 2.3, $f_{p,q}$ is well defined and it is easy to show that $f_{p,q}$ is a homomorphism. Moreover, $\vec{A} = (A_p, f_{p,q})$ is a directed P-indexed diagram in $\mathcal{V}$. Let A be the direct limit of this diagram.

Denote by α_k the k-th projection $A_p \to F$ $(k = 1, \ldots, n)$ and by β_l the l-th projection $A_q \to F$ $(l = 1, \ldots, m)$.

By 4.1 we have $\operatorname{Con_c} A_p \simeq B_p$ and the isomorphism $h_p\colon \operatorname{Con_c} A_p \to B_p$ can be defined by $h_p(\ker(\alpha_k)) = b_k$. Similarly, let h_q be the isomorphism $\operatorname{Con_c} A_q \to B_q$ defined by $h_q(\ker(\beta_l)) = c_l$. Now we claim that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Con_c} A_p \arrow[r, "\operatorname{Con_c} f_{p,q}"] \arrow[d, "h_p"'] & \operatorname{Con_c} A_q \arrow[d, "h_q"] \\
B_p \arrow[r, "\varphi_{p,q}"] & B_q
\end{tikzcd}$$

By Lemma 2.2, we can prove equivalently that $h^{\leftarrow}_p \varphi^{\leftarrow}_{p,q} = (\operatorname{Con_c} f_{p,q})^{\leftarrow} h^{\leftarrow}_q$. Since all these maps preserve 1 and $\wedge$, it suffices to show that $h^{\leftarrow}_p \varphi^{\leftarrow}_{p,q}(c_i) =$

$(\mathrm{Con_c}\, f_{p,q})^{\leftarrow} h_q^{\leftarrow}(c_i)$ for every coatom c_i of B_q. Let $\varphi_{p,q}^{\leftarrow}(c_i) = b_j$. Then $h_p^{\leftarrow} \varphi_{p,q}^{\leftarrow}(c_i) = \ker(\alpha_j)$. Further, $h_q^{\leftarrow}(c_i) = \ker(\beta_i)$ and

$$\begin{aligned}(x, y) \in (\mathrm{Con_c}\, f_{p,q})^{\leftarrow}(\ker(\beta_i)) \quad &\text{iff} \quad (f_{p,q}(x), f_{p,q}(y)) \in \ker(\beta_i)\\ \text{iff} \quad f_{p,q}(x)_i = f_{p,q}(y)_i \quad \text{iff} \quad &x_j = y_j \quad \text{iff} \quad (x, y) \in \ker(\alpha_j),\end{aligned}$$

so

$$(\mathrm{Con_c}\, f_{p,q})^{\leftarrow} h_q^{\leftarrow}(c_i) = \ker(\alpha_j) = h_p^{\leftarrow} \varphi_{p,q}^{\leftarrow}(c_i),$$

which proves that our diagram commutes. Using this commutativity and the fact that the functor $\mathrm{Con_c}$ preserves direct limits, we have

$$\mathrm{Con_c}\, A \simeq \mathrm{Con_c} \varinjlim \vec{A} \simeq \varinjlim \mathrm{Con_c}\, \vec{A} \simeq \varinjlim \vec{B} \simeq L.$$

□

The second case. Now suppose that $\mathcal{V}$ is a nontrivial, locally finite and congruence distributive variety with CIP such that

(1) $\mathrm{Con_c}\, F$ is the two-element chain for every $F \in \mathrm{SI}(\mathcal{V})$;
(2) there exists $F \in \mathrm{SI}(\mathcal{V})$ such that F has a one-element subalgebra.

As an example of such a variety one can consider the variety of all distributive lattices.

We prove a result similar to the first case. Recall that a *generalized Boolean lattice* B is a distributive lattice with the least element 0 such that for any $b \in B$, the interval $[0, b]$ is a Boolean lattice.

Instead of Lemma 2.3 we use the following assertion (which is equally easy to prove).

Lemma 4.3. *If $\varphi\colon B_1 \to B_2$ is a 0-preserving lattice homomorphism of finite Boolean lattices, then $\varphi^{\leftarrow}(c) \in \mathrm{M}(B_1)$ or $\varphi^{\leftarrow}(c) = 1$ for every $c \in \mathrm{M}(B_2)$.*

Theorem 4.4. *The following conditions are equivalent.*

(1) *$L \simeq \mathrm{Con_c}\, A$ for some $A \in \mathcal{V}$.*
(2) *L is isomorphic to a direct limit of a system $\vec{B} = (B_p, \varphi_{p,q};\ p \leqslant q \text{ in } P)$, where each B_p is a finite Boolean lattice and each $\varphi_{p,q}$ is a 0-preserving lattice homomorphism.*
(3) *L is a generalized Boolean lattice.*

P r o o f. (1)$\Rightarrow$(2) The same as in Theorem 4.2 except that we do not prove $\varphi_{p,q}(1) = 1$.

(2)$\Rightarrow$(3) It is easy to check that the direct limit of a system of generalized Boolean lattices and 0-preserving lattice homomorphisms is a generalized Boolean lattice.

(3)$\Rightarrow$(2) Let B be a generalized Boolean lattice. For every finite $G \subseteq B$ let B_G be the Boolean sublattice of the interval $\langle 0, \bigvee G\rangle$ generated by G. It is easy to see that B is the direct limit of the system of all B_G with the inclusions as the system homomorphisms.

(2)$\Rightarrow$(1) We proceed similarly to Theorem 4.2. Choose $F \in \mathrm{SI}(\mathcal{V})$ with $\mathrm{Con_c}\, F = \mathbf{2}$ which has a 1-element subalgebra $\{u\}$. For every $p \in P$ let $A_p = F^n$, where $n = |\mathrm{M}(B_p)|$. Let $p, q \in P$, $p \leqslant q$. Let $\mathrm{M}(B_p) = \{b_1, b_2, \ldots, b_n\}$, $\mathrm{M}(B_q) = \{c_1, c_2, \ldots, c_m\}$. Let $f_{p,q}$ be a map $A_p \to A_q$ defined by $f_{p,q}(a_1, \ldots, a_n) = (d_1, \ldots, d_m)$, where

$$d_i = \begin{cases} a_j & \text{if } \varphi^{\leftarrow}_{p,q}(c_i) = b_j \\ u & \text{if } \varphi^{\leftarrow}_{p,q}(c_i) = 1. \end{cases}$$

By 4.3, $f_{p,q}$ is well defined and it is easy to show that $f_{p,q}$ is a homomorphism. We consider the same diagram as in 4.2 and prove its commutativity. The only difference is that now we need to consider the additional case $\varphi^{\leftarrow}_{p,q}(c_i) = 1$. Then $h^{\leftarrow}_p \varphi^{\leftarrow}_{p,q}(c_i) = 1 = (\mathrm{Con_c}\, f_{p,q})^{\leftarrow} h^{\leftarrow}_q(c_i)$, because

$$\begin{aligned}(x, y) \in (\mathrm{Con_c}\, f_{p,q})^{\leftarrow}(\ker(\beta_i)) \quad &\text{iff} \quad (f_{p,q}(x), f_{p,q}(y)) \in \ker(\beta_i) \\ &\text{iff} \quad f_{p,q}(x)_i = f_{p,q}(y)_i \quad \text{iff} \quad u = u.\end{aligned}$$

$\square$

The third case. In this case we suppose that $\mathcal{V}$ is a locally finite and congruence distributive variety with CIP such that

(1) $\mathrm{Con_c}\, F$ is a three-element chain or a two-element chain for every $F \in \mathrm{SI}(\mathcal{V})$;
(2) there exists $F \in \mathrm{SI}(\mathcal{V})$ such that $\mathrm{Con_c}\, F$ is a three-element chain;
(3) if $A, B \in \mathrm{SI}(\mathcal{V})$, $A \leqslant B$, then $\mathrm{Con_c}\, A \simeq \mathrm{Con_c}\, B$;
(4) no $A \in \mathrm{SI}(\mathcal{V})$ has a one-element subalgebra.

As an example of such a variety one can consider the variety of all principal Stone algebras. It is the variety generated by the algebra $(\{0, e, 1\}, \vee, \wedge, *, 0, e, 1)$, where $0 < e < 1$ and $*$ denotes the pseudocomplementation.

For the study of this case we need to recall some basic facts about dual Stone lattices. A bounded lattice is called dually pseudocomplemented if for every $x \in L$ there exists its dual pseudocomplement $x^+ = \min\{y \in L;\ x \vee y = 1\}$. The elements satisfying $x^+ = 1$ are called *codense* and form an ideal of L denoted by $\overline{D}(L)$.

A *dual Stone lattice* is a distributive dually pseudocomplemented lattice satisfying the identity $x^+ \wedge x^{++} = 0$. In a dual Stone lattice L, the set $S(L) = \{x^+;\ x \in L\}$ is a Boolean subalgebra and is called the *skeleton* of L.

It is easy to see that every finite distributive lattice is dually pseudocomplemented and its largest codense element is the meet of all maximal $\wedge$-irreducible elements (i.e. coatoms). Denote by $M_1(L)$ the set of all coatoms of L.

The following assertion is well known in the special case of Boolean algebras.

Lemma 4.5. *Let B_1, B_2 be dual Stone lattices with largest codense elements d_1 and d_2, respectively. Let φ be a $0,1$-preserving lattice homomorphism with $\varphi(d_1) = d_2$. Then φ preserves dual pseudocomplements.*

P r o o f. Every $x \in B_1$ satisfies the equality $x = x^{++} \vee (x \wedge d_1)$. Hence,

$$\varphi(x)^+ = (\varphi(x^{++}) \vee (\varphi(x) \wedge d_2))^+ = \varphi(x^{++})^+ \wedge (\varphi(x)^+ \vee 1) = \varphi(x^{++})^+.$$

Since the restriction of φ to $S(B_1)$ is a homomorphism of Boolean algebras and x^{++} is a complement of x^+, we obtain that $\varphi(x^{++})$ is a complement of $\varphi(x^+)$, so $\varphi(x^{++})^+ = \varphi(x^+)$. $\square$

Lemma 4.6. *Let $\varphi\colon\ B_1 \to B_2$ be a $0,1$-preserving lattice homomorphism of finite dual Stone lattices. The following conditions are equivalent.*

(1) *For every $c \in \mathrm{M}(B_2)$, $\varphi^{\leftarrow}(c) \in M_1(B_1)$ if and only if $c \in \mathrm{M}_1(B_2)$.*

(2) *φ preserves the largest codense element.*

P r o o f. Denote $d_i = \bigwedge \mathrm{M}_1(B_i)$, the largest codense element of B_i $(i = 1, 2)$. Clearly, $d_1 \leqslant b \in \mathrm{M}(B_1)$ if and only if $b \in \mathrm{M}_1(B_1)$.

(1)$\Rightarrow$(2) Since in B_2 every element is a meet of $\wedge$-irreducible elements, we have

$$\varphi(d_1) = \bigwedge\{c \in \mathrm{M}(B_2);\ \varphi(d_1) \leqslant c\}.$$

Now, $\varphi(d_1) \leqslant c$ is equivalent to $d_1 \leqslant \varphi^{\leftarrow}(c)$ and hence to $\varphi^{\leftarrow}(c) \in \mathrm{M}_1(B_1)$. By (1), this is equivalent to $c \in \mathrm{M}_1(B_2)$, hence

$$\varphi(d_1) = \bigwedge\{c \in \mathrm{M}(B_2);\ c \in \mathrm{M}_1(B_2)\} = d_2.$$

So φ preserves the largest codense element.

(2)$\Rightarrow$(1) Let $c \in \mathrm{M}(B_2)$. Then $c \in \mathrm{M}_1(B_2)$ if and only if

$$c \geqslant d_2 = \varphi(d_1) = \bigwedge\{\varphi(b);\ b \in \mathrm{M}_1(B_1)\}.$$

Since c is $\wedge$-irreducible, this is equivalent to $c \geqslant \varphi(b)$ for some $b \in \mathrm{M}_1(B_1)$, hence to $\varphi^{\leftarrow}(c) \geqslant b$, which is only possible if $\varphi^{\leftarrow}(c) = b$. $\square$

Similarly to the previous cases we first describe finite L with $L \simeq \operatorname{Con_c} A$ for some $A \in \mathcal{V}$.

Denote by $\mathcal{P}$ the class of all finite partially ordered sets $(P, \leqslant)$ such that for every $x \in P$, $\uparrow x$ is a one-element or two-element chain. Hence, $P \in \mathcal{P}$ is a disjoint union of two antichains P_1 and P_2 such that $|\uparrow x| = 1$ for $x \in P_1$, $|\uparrow x| = 2$ for $x \in P_2$. If L is a lattice such that $\mathrm{M}(L) \in \mathcal{P}$, then denote $\mathrm{M}_i(L) = (\mathrm{M}(L))_i$ for $i = 1, 2$.

Lemma 4.7. *For every finite distributive lattice L the following conditions are equivalent.*

(1) $L \simeq \operatorname{Con_c} A$ *for some* $A \in \mathcal{V}$.
(2) $\mathrm{M}(L) \in \mathcal{P}$.
(3) *L is a dual Stone lattice and its codense elements form a Boolean lattice.*

P r o o f. (1)⇔(2) This equivalence follows from [4], Theorem 8. We recall the following details. Let $\mathrm{M}(L) \in \mathcal{P}$. There exist $F \in \mathcal{V}$ such that $\operatorname{Con_c} F$ is a three element chain $\alpha_0 > \alpha_1 > \alpha_2$. For $i = 1, 2$ the quotient algebra $F_i = F/\alpha_i$ is subdirectly irreducible and $\operatorname{Con_c} F_i$ is the $(i+1)$-element chain. For every $j \leqslant i$ we have a natural homomorphism $g_{i,j}\colon F_i \to F_j$.

We define the diagram indexed by $P = \mathrm{M}(L)$. For every $p \in P$ denote by $i(p)$ the cardinality of the chain $\uparrow p$. If $p \in \mathrm{M}_1(L)$ then set $A_p = F_1$ and if $p \in \mathrm{M}_2(L)$ then set $A_p = F_2$. For every $p, q \in P$, $p \leqslant q$ we set $f_{p,q} = g_{i(p),i(q)}$. Let A be the limit of this diagram. Hence, A is a subalgebra of the direct product $\Pi\{A_p;\ p \in \mathrm{M}(L)\}$. The assumptions of 2.5 are satisfied. (See [4].) Thus, $L \simeq \operatorname{Con_c} A$, and the isomorphism $h\colon \operatorname{Con_c} A \to L$ satisfies $h(\ker(\alpha_p)) = p$ for every $p \in \mathrm{M}(L)$.

(2)⇔(3) Equivalence was proved in [7] Theorem 4.5 (in a dual form). □

Lemma 4.8. *Let L be a dual Stone lattice and let its codense elements form a Boolean lattice. For every finite set $Y \subseteq L$ there exists a finite subalgebra $L_Y \leqslant L$ such that $Y \subseteq L_Y$ and $\overline{D}(L_Y)$ is a Boolean subalgebra of $\overline{D}(L)$.*

P r o o f. By the triple representation of Stone algebras and its simplified version due to Katriňák [6] we can assume that there exist a Boolean lattice B, a bounded distributive lattice D and a $(0,1)$-preserving lattice homomorphism $h\colon B \to D$ such that

$$L = \{(b, d) \in B \times D;\ h(b) \leqslant d\},$$

with the lattice operations defined componentwise and $(b, d)^+ = (b', h(b'))$, where b' denotes the complement of b in B. Moreover, $S(L) = \{(b, h(b));\ b \in B\}$ is isomorphic to B and $\overline{D}(L) = \{(0, d);\ d \in D\}$ is isomorphic to D, so by our assumption D is

Boolean. Now let Y be a finite subset of L. Let B_Y be the Boolean subalgebra of B generated by

$$\{b \in B;\ (b,d) \in X \text{ for some } d \in D\}.$$

Further, let D_Y be the Boolean subalgebra of D generated by

$$\{d \in D;\ (b,d) \in X \text{ for some } b \in B\} \cup \{h(b);\ b \in B_Y\}.$$

Clearly, B_Y and D_Y are finite. Denote

$$L_Y := \{(b,d);\ b \in B_Y, d \in D_Y\}.$$

It is easy to check that L_Y is a finite subalgebra of L, $Y \subseteq L_Y$ and

$$\overline{D}(L_Y) = \{(0,d) \in L;\ d \in D_Y\} \simeq D_Y.$$

$\square$

Theorem 4.9. *The following conditions are equivalent.*

(1) $L \simeq \mathrm{Con_c}\, A$ *for some* $A \in \mathcal{V}$.

(2) L *is isomorphic to the direct limit of a system* $\vec{B} = (B_p, \varphi_{p,q};\ p \leqslant q \text{ in } P)$, *where each* B_p *is a finite distributive lattice with* $\mathrm{M}(B_p) \in \mathcal{P}$ *and each* $\varphi_{p,q}$ *is a dual Stone algebras homomorphism, preserving the largest codense element.*

(3) L *is a dual Stone lattice and its codense elements form a Boolean lattice.*

Proof. (1)⇒(2) Similarly to the above let P be the family of all finite subsets of A ordered by set inclusion. Let A_p be the subalgebra of A generated by $p \in P$, let $B_p = \mathrm{Con_c}\, A_p$ and $\varphi_{p,q} = \mathrm{Con_c}\, e_{p,q}$ for every $p, q \in P$, $p \leqslant q$. By Lemma 4.7, $\mathrm{M}(B_p) \in \mathcal{P}$. We know that every $\varphi_{p,q}$ is a 0-preserving lattice homomorphism. We check the assumptions of 4.6. For every $c \in \mathrm{M}(B_q)$ the algebra $A_p/\varphi^{\leftarrow}_{p,q}(c)$ is a subalgebra of A_q/c. Our assumptions on $\mathcal{V}$ imply that

$$\uparrow \varphi^{\leftarrow}_{p,q}(c) \cong \mathrm{Con_c}\, A_p/\varphi^{\leftarrow}_{p,q}(c) \cong \mathrm{Con_c}\, A_q/c \cong {\uparrow} c.$$

By 2.4, $\varphi_{p,q}$ preserves 1. Further, $c \in \mathrm{M}(B_q)$ is a coatom if and only if $\varphi^{\leftarrow}_{p,q}(c)$ is a coatom. Hence, by 4.6, $\varphi_{p,q}$ preserves the largest codense element. By 4.5, φ also preserves the dual pseudocomplements, so it is a homomorphism of dual Stone algebras.

(2)⇒(3) Since every B_p is a dual Stone lattice, and every $\varphi_{p,q}$ is a lattice homomorphism which also preserves pseudocomplements, the limit algebra L is also a dual

Stone lattice. Moreover, restriction of $\varphi_{p,q}$ to $\overline{D}(B_p)$ is a homomorphism of Boolean lattices, so $\overline{D}(L)$ is a Boolean lattice (the limit of Boolean lattices $\overline{D}(B_p)$).

(2)⇒(1) For every $p \in P$ let $A_p \in \mathcal{V}$ be the algebra with $\mathrm{Con_c}\, A_p \cong B_p$ constructed in 4.7. So, A_p is a subalgebra of the direct product $\Pi\{F_b;\ b \in \mathrm{M}(B_p)\}$, where $F_b = F_{i(b)}$. The isomorphisms $h_p\colon \mathrm{Con_c}\, A_p \to B_p$ are defined in the same way as before.

Let $p, q \in P$, $p \leqslant q$. The assumptions on $\mathcal{V}$ imply that $F_b = F_c$ whenever $b = \varphi^{\leftarrow}_{p,q}(c)$. We define the homomorphism $f_{p,q}\colon A_p \to A_q$ in the same way as in 4.2: $f(a_1, \ldots, a_n) = (d_1, \ldots, d_m)$, where $d_i = a_j$ such that $b_j = \varphi^{\leftarrow}_{p,q}(c_i)$. However, now A_q is not equal to the direct product $\Pi\{F_b;\ b \in \mathrm{M}(B_q)\}$, so we have to check that $(d_1, \ldots, d_m) \in A_q$. Let $c_i \leqslant c_k$ in $M(B_q)$, we need to show that $f_{c_i,c_k}(d_i) = d_k$. Let $b_j = \varphi^{\leftarrow}_{p,q}(c_i)$, $b_l = \varphi^{\leftarrow}_{p,q}(c_k)$. Then $b_j \leqslant b_l$, so $(a_1, \ldots, a_n) \in A_p$ implies that $a_l = f_{b_j,b_l}(a_j)$. Since the homomorphisms f_{c_i,c_k} and f_{b_j,b_l} are the same (see the proof of 4.7), we obtain $d_k = a_l = f_{b_j,b_l}(a_j) = f_{c_i,c_k}(a_j) = f_{c_i,c_k}(d_i)$.

So, $f_{p,q}$ is defined correctly and similarly to 4.2 we can argue that $\mathrm{Con_c}\, A \cong L$, where A is the limit of the system $(A_p, f_{p,q};\ p \leqslant q \text{ in } P)$.

(3)⇒(2) Let P be the family of all finite subsets of L ordered by set inclusion. Using Lemma 4.8 we can see that L is the direct limit of the system $(L_Y, \varphi_{X,Y};\ X \subseteq Y \text{ in } P)$, where $\varphi_{X,Y}$ is the set inclusion. As L_X is a subalgebra of the dually Stone lattice L_Y containing the largest codense element, $\varphi_{X,Y}$ has the required properties. $\square$

References

[1] *P. Agliano, K. A. Baker*: Congruence intersection properties for varieties of algebras. J. Aust. Math. Soc., Ser. A *67* (1999), 104–121. zbl MR

[2] *K. A. Baker*: Primitive satisfaction and equational problems for lattices and other algebras. Trans. Am. Math. Soc. *190* (1974), 125–150. zbl MR

[3] *W. J. Blok, D. Pigozzi*: On the structure of varieties with equationally definable principal congruences. I. Algebra Univers. *15* (1982), 195–227. zbl MR

[4] *P. Gillibert, M. Ploščica*: Congruence FD-maximal varieties of algebras. Int. J. Algebra Comput. *22* (2012), 1250053, 14 pages. zbl MR

[5] *G. Grätzer*: General Lattice Theory. (With appendices with B. A. Davey, R. Freese, B. Ganter, et al.) Paperback reprint of the 1998 2nd edition, Birkhäuser, Basel, 2003. zbl MR

[6] *T. Katriňák*: A new proof of the construction theorem for Stone algebras. Proc. Am. Math. Soc. *40* (1973), 75–78. zbl MR

[7] *T. Katriňák, A. Mitschke*: Stonesche Verbände der Ordnung n und Postalgebren. Math. Ann. *199* (1972), 13–30. (In German.) zbl MR

[8] *K. B. Lee*: Equational classes of distributive pseudo-complemented lattices. Can. J. Math. *22* (1970), 881–891. zbl MR

[9] *M. Ploščica*: Separation in distributive congruence lattices. Algebra Univers. *49* (2003), 1–12. zbl MR

[10] *M. Ploščica*: Finite congruence lattices in congruence distributive varieties. Proceedings of the 64th workshop on general algebra "64. Arbeitstagung Allgemeine Algebra",

Olomouc, Czech Republic, May 30–June 2, 2002 and of the 65th workshop on general algebra “65. Arbeitstagung Allgemeine Algebra”, Potsdam, Germany, March 21–23, 2003 (I. Chajda, et al., ed.). Verlag Johannes Heyn, Klagenfurt. Contrib. Gen. Algebra *14*, 2004, pp. 119–125. zbl MR

[11] *M. Ploščica*: Local separation in distributive semilattices. Algebra Univers. *54* (2005), 323–335. zbl MR

[12] *M. Ploščica, J. Tůma, F. Wehrung*: Congruence lattices of free lattices in non-distributive varieties. Colloq. Math. *76* (1998), 269–278. zbl MR

[13] *P. Růžička*: Lattices of two-sided ideals of locally matricial algebras and the Γ-invariant problem. Isr. J. Math. *142* (2004), 1–28. zbl MR

[14] *E. T. Schmidt*: The ideal lattice of a distributive lattice with 0 is the congruence lattice of a lattice. Acta Sci. Math. (Szeged) *43* (1981), 153–168. zbl MR

[15] *F. Wehrung*: A uniform refinement property for congruence lattices. Proc. Am. Math. Soc. *127* (1999), 363–370. zbl MR

Authors' address: Miroslav Ploščica, Filip Krajník, Institute of Mathematics, Šafárik's University, Jesenná 5, 041 54 Košice, Slovakia, e-mail: `miroslav.ploscica@upjs.sk`, `filip.krajnik@student.upjs.sk`.