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CONGRUENCE LIFTING OF SEMILATTICE DIAGRAMS*

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We consider the problem, whether the algebras in two finitely generated congruence-distributive varieties have isomorphic congruence lattices. According to the results of P. Gillibert, this problem is closely connected with the question, which diagrams of finite distributive semilattices can be represented by the congruence lattices of algebras in a given variety. We study this question for varieties of bounded lattices, generated by different nondistributive lattices of length 2 (denoted M_n). For each pair from this family of varieties we construct a diagram indexed by the product of three finite chains, which is liftable in one variety and nonliftable in the other one. We also discover an interesting link to the four-color theorem of graph theory.

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1. Introduction

For a class $\mathcal{K}$ of algebras we denote $\operatorname{Con}\mathcal{K}$ the class of all lattices isomorphic to $\operatorname{Con}(A)$ (the congruence lattice of an algebra A) for some $A \in \mathcal{K}$. There are many papers investigating $\operatorname{Con}\mathcal{K}$ for various classes $\mathcal{K}$. However, the full description of $\operatorname{Con}\mathcal{K}$ has proved to be a very difficult (and probably intractable) problem, even for the most common classes of algebras, like groups or lattices. The latest evidence of this is the recent solution of the Congruence Lattice Problem (CLP) by F. Wehrung [11].

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The difficulty in describing $\operatorname{Con}\mathcal{K}$ leads to consideration of the following problem.

Problem 1.1. Given classes $\mathcal{K}$ and $\mathcal{L}$ of algebras, decide whether $\operatorname{Con}\mathcal{K} \subseteq \operatorname{Con}\mathcal{L}$.

In the present paper we consider this problem under the additional conditions that both $\mathcal{K}$ and $\mathcal{L}$ are finitely generated, congruence-distributive varieties (equational classes) of algebras. We feel that in this very special case there is a hope for an algorithmic solution of Problem 1.1 and our paper is a step towards this goal.

We use the technique of lifting commutative diagrams of semilattices by the Con_c functor. This approach has been used in several papers investigating the congruence lattices of algebras, for instance [7, 9, 10]. The systematic research of this topic in connection with Problem 1.1 has been carried out by P. Gillibert in [1], which is our main source of motivation.

We assume familiarity with fundamentals of lattice theory and universal algebra. For all undefined concepts and unreferenced facts we refer to [3] and [4].

The congruence lattice $\operatorname{Con}(A)$ of an algebra A is always algebraic and its compact elements form a $(\vee, 0)$-subsemilattice of $\operatorname{Con}(A)$, denoted $\operatorname{Con}_c(A)$. For $x, y \in A$ let $\theta(x, y)$ denote the smallest congruence containing the pair (x, y). (We also write $\theta_A(x, y)$, when A needs to be specified.) The semilattice $\operatorname{Con}_c(A)$ consists precisely of all finitely generated congruences, i.e. congruences of the form $\theta(x_1, y_1) \vee \ldots \vee \theta(x_n, y_n)$. The smallest congruence (the equality relation) is compact, so $\operatorname{Con}_c(A)$ has always a smallest element.

Let $\mathcal{K}$ be a class of algebras of the same type, let $A, B \in \mathcal{K}$. For a homomorphism $f:\ A \to B$ we define the mapping $\operatorname{Con}(f):\ \operatorname{Con}(A) \to \operatorname{Con}(B)$ by the rule that $\operatorname{Con}(f)(\alpha)$ is the congruence on B generated by the set $\{(f(x), f(y)) \mid (x, y) \in \alpha\}$. This map preserves 0 and $\vee$, and Con is a functor from $\mathcal{K}$ with homomorphisms of algebras to the category of all algebraic lattices with compactness-preserving complete join-homomorphisms. Hence Con_c is a functor from $\mathcal{K}$ to the category $\mathcal{S}$ of all $(\vee, 0)$-semilattices with $(\vee, 0)$-homomorphisms.

A *diagram* in $\mathcal{K}$ is a functor $\mathcal{A}:\ J \to \mathcal{K}$, where J is a small category. For every such diagram, the composite $\operatorname{Con}_c \circ \mathcal{A}$ is a diagram in the category of $(\vee, 0)$-semilattices.

Conversely, let $\mathcal{D}:\ J \to \mathcal{S}$ be a diagram in the category of $(\vee, 0)$-semilattices. A *lifting* of $\mathcal{D}$ in the category $\mathcal{K}$ of algebras is a functor $\mathcal{A}:\ J \to \mathcal{K}$ such that the functors $\mathcal{D}$ and $\operatorname{Con}_c \circ \mathcal{A}$ are naturally equivalent. This means that for every $j \in J$ we have an isomorphism $\psi_j:\ \operatorname{Con}_c(\mathcal{A}(j)) \to \mathcal{D}(j)$ such that the diagram

$$\begin{array}{ccc} \operatorname{Con}_c(\mathcal{A}(j)) & \xrightarrow{\operatorname{Con}_c(\mathcal{A}(e))} & \operatorname{Con}_c(\mathcal{A}(k)) \\ {\scriptstyle\psi_j}\downarrow & & {\scriptstyle\psi_k}\downarrow \\ \mathcal{D}(j) & \xrightarrow{\mathcal{D}(e)} & \mathcal{D}(k) \end{array}$$

commutes for every J-morphism $e:\ j \to k$.

In this paper we only consider the diagrams indexed by finite ordered sets, viewed as small categories. A $(J, \leq)$-indexed diagram in $\mathcal{K}$ consists of algebras $\mathcal{A}(j) \in \mathcal{K}$ $(j \in J)$ and homomorphisms $\mathcal{A}(j,k): \mathcal{A}(j) \to \mathcal{A}(k)$ $(j,k \in J,\ j \leq k)$ such that $\mathcal{A}(j,j)$ is the identity for every j and $\mathcal{A}(k,l) \cdot \mathcal{A}(j,k) = \mathcal{A}(j,l)$ whenever $j \leq k \leq l$.

The importance of the diagram lifting for the Problem 1.1 follows from the following result of P. Gillibert.

Theorem 1.2 ([1], Corollary 7.13). *Let $\mathcal{K}$ and $\mathcal{L}$ be finitely generated congruence-distributive varieties. The following conditions are equivalent.*

(i) $\operatorname{Con}\mathcal{K} \nsubseteq \operatorname{Con}\mathcal{L}$;
(ii) *there are a natural number n and a diagram of finite* $(\vee, 0)$*-semilattices indexed by the ordered set* $\{0,1\}^n$ *liftable in $\mathcal{K}$ and not liftable in $\mathcal{L}$.*

This theorem says that the containment $\operatorname{Con}\mathcal{K} \subseteq \operatorname{Con}\mathcal{L}$ can be checked on the finite level. However, it does not provide an algorithm, since there is no estimate on n, and one needs to check infinitely many diagrams.

The type of diagrams liftable in $\mathcal{K}$ and not in $\mathcal{L}$ seems to be connected with the critical points in the sense of the following definition. Let L_c denote the set of all compact elements of an algebraic lattice L.

Definition 1.3. Let $\mathcal{K}$ and $\mathcal{L}$ be classes of algebras. The *critical point of $\mathcal{K}$ under $\mathcal{L}$*, denoted $\operatorname{crit}(\mathcal{K};\mathcal{L})$, is the smallest cardinality of L_c for $L \in \operatorname{Con}\mathcal{K} \setminus \operatorname{Con}\mathcal{L}$ (if $\operatorname{Con}\mathcal{K} \nsubseteq \operatorname{Con}\mathcal{L}$) or ∞ (if $\operatorname{Con}\mathcal{K} \subseteq \operatorname{Con}\mathcal{L}$).

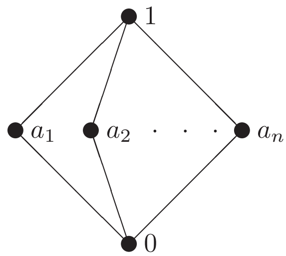
Theorem 1.4 ([1], Corollary 7.6). *Let $\mathcal{K}$ and $\mathcal{L}$ be finitely generated congruence-distributive varieties, let $n \in \omega$. Consider the following conditions*:

(i) *$\operatorname{crit}(\mathcal{K};\mathcal{L}) \leq \aleph_n$;*
(ii) *there exists a diagram of finite* $(\vee, 0)$*-semilattices indexed by a product of $n+1$ finite chains liftable in $\mathcal{K}$ and not liftable in $\mathcal{L}$.*

Then (ii) *implies* (i). *If $n = 0$ then also* (i) *implies* (ii).

In view of the Problem 1.1, it is relevant to know whether the implication (i) $\Longrightarrow$ (ii) holds also for $n > 0$. This seems to be a difficult question. A natural approach is to investigate some concrete cases and then try to generalize the results. However, the known examples of varieties with critical point larger than $\aleph_0$ are quite rare. The first pair of varieties with the critical point $\aleph_1$ was exhibited in [1]. The corresponding diagram of finite semilattices is indexed by $\{0,1\}^2$, i.e. the product of two chains, as predicted by (ii).

For $n \geq 3$ let $\mathcal{M}_n$ denote the variety of lattices generated by the $(n+2)$-element lattice M_n depicted below.



M_n

The bounded members of $\mathcal{M}_n$ also form a variety (denoted $\mathcal{M}_n^{01}$), provided that we consider 0 and 1 as nullary operations.

These varieties are well-known. An especially important fact is that we know all subdirectly irreducible members of $\mathcal{M}_n$ ($\mathcal{M}_n^{01}$). Up to isomorphism, the list consists of $\mathbf{2} = \{0, 1\}$, M_3, $M_4, \ldots, M_n$.

Our varieties form an increasing chain

$$\mathcal{M}_3^{01} \subset \mathcal{M}_4^{01} \subset \mathcal{M}_5^{01} \subset \ldots$$

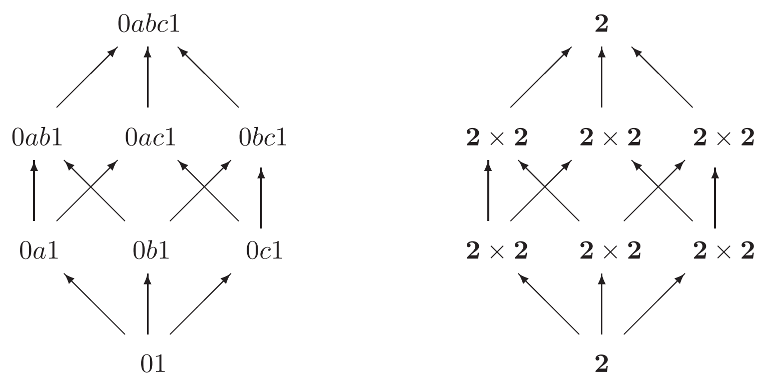
As proved in [5] and [6], $\operatorname{crit}(\mathcal{M}_{n+1}^{01}; \mathcal{M}_n^{01}) = \aleph_2$. More pairs of varieties with critical point $\aleph_2$ can be found in the recent paper [2]. In the present paper we construct for every $n \geq 3$ a diagram of finite $(\vee, 0)$-semilattices indexed by a product of three chains, which is liftable in $\mathcal{M}_{n+1}^{01}$ but not in $\mathcal{M}_n^{01}$. We hope that the methods and ideas in our construction can be used to obtain more general results. Especially, we provide a further support for the conjecture that (i) and (ii) of 1.4 are equivalent.

Let us remark that we do not know an example of a pair of varieties with critical point $\aleph_n$, with $n \geq 3$.

We use standard denotations. The symbol $\boldsymbol{n}$ denotes the linearly ordered set $\{0, 1, 2, \ldots, n-1\}$, which is sometimes regarded as a lattice or a semilattice (depending on the context).

2. $\mathcal{M}_3$ versus $\mathcal{M}_4$

To simplify the notation, we denote the elements of M_3 by $0, 1, a, b, c$, instead of $0, 1, a_1, a_2, a_3$. Consider the following diagram $\mathcal{A}_3 : I \to \mathcal{M}_3$, where I is the cube, i.e. an ordered set isomorphic to $\mathbf{2} \times \mathbf{2} \times \mathbf{2}$. The elements of I will be shortly denoted as $000, 100, 010, \ldots, 111$. For every $i \in I$, $\mathcal{A}_3(i)$ will be subalgebras of M_3, namely $\mathcal{A}_3(000) = \{0, 1\}$, $\mathcal{A}_3(100) = \{0, a, 1\}$, $\mathcal{A}_3(010) = \{0, b, 1\}$, $\mathcal{A}_3(001) = \{0, c, 1\}$, $\mathcal{A}_3(110) = \{0, a, b, 1\}$, $\mathcal{A}_3(101) = \{0, a, c, 1\}$, $\mathcal{A}_3(011) = \{0, b, c, 1\}$, $\mathcal{A}_3(111) = M_3$. For every $i, j \in I$, $i \leq j$, the homomorphism $\mathcal{A}_3(i, j) : \mathcal{A}_3(i) \to \mathcal{A}_3(j)$ will be the set inclusion. Further, let $\mathcal{D}_3 = \operatorname{Con}_c \circ \mathcal{A}_3$ be the corresponding diagram of distributive semilattices. Both diagrams are depicted below.



Let us make several useful observations about the diagram $\mathcal{D}_3$. First, all morphisms in $\mathcal{D}_3$ have the property, that nonzero elements do not map into zero. A consequence is that in every lifting of $\mathcal{D}_3$, all morphisms must be injective. Second, all morphisms $\mathbf{2}\times\mathbf{2} \to \mathbf{2}\times\mathbf{2}$ in $\mathcal{D}_3$ are isomorphisms. Moreover, with respect to these morphisms, the "middle" elements of the semilattices $\mathbf{2}\times\mathbf{2}$ form a "12-cycle", namely

$$\begin{aligned}
\theta_{\mathcal{A}_3(100)}(0,a), \theta_{\mathcal{A}_3(010)}(1,b) &\mapsto \theta_{\mathcal{A}_3(110)}(0,a) = \theta_{\mathcal{A}_3(110)}(1,b),\\
\theta_{\mathcal{A}_3(010)}(1,b), \theta_{\mathcal{A}_3(001)}(0,c) &\mapsto \theta_{\mathcal{A}_3(011)}(1,b) = \theta_{\mathcal{A}_3(011)}(0,c),\\
\theta_{\mathcal{A}_3(001)}(0,c), \theta_{\mathcal{A}_3(100)}(1,a) &\mapsto \theta_{\mathcal{A}_3(101)}(0,c) = \theta_{\mathcal{A}_3(101)}(1,a),\\
\theta_{\mathcal{A}_3(100)}(1,a), \theta_{\mathcal{A}_3(010)}(0,b) &\mapsto \theta_{\mathcal{A}_3(110)}(1,a) = \theta_{\mathcal{A}_3(110)}(0,b),\\
\theta_{\mathcal{A}_3(010)}(0,b), \theta_{\mathcal{A}_3(001)}(1,c) &\mapsto \theta_{\mathcal{A}_3(011)}(0,b) = \theta_{\mathcal{A}_3(011)}(1,c),\\
\theta_{\mathcal{A}_3(001)}(1,c), \theta_{\mathcal{A}_3(100)}(0,a) &\mapsto \theta_{\mathcal{A}_3(101)}(1,c) = \theta_{\mathcal{A}_3(101)}(0,a).
\end{aligned}$$

Trivially, $\mathcal{A}_3$ lifts $\mathcal{D}_3$. We claim that this is essentially the only way on how to lift this diagram in any $\mathcal{M}_n$. Let us be more specific. As all morphisms in any lifting of $\mathcal{D}_3$ must be embeddings, we can assume that they are set inclusions.

Lemma 2.1. *Let $n \geq 3$ and let $\mathcal{B}: I \to \mathcal{M}_n$ lift $\mathcal{D}_3$, with all homomorphisms being the set inclusions. Then*

(i) $\mathcal{B}(111)$ *is isomorphic to M_k for some $k \in \{3,\dots,n\}$;*
(ii) *for every $i \in \{100, 010, 001\}$, $\mathcal{B}(i)$ is isomorphic to* $\mathbf{3}$ *and all three algebras are different.*

Proof. $\mathcal{B}(111)$ must be a simple algebra, which has a subalgebra whose congruence lattice is isomorphic to $\mathbf{2}\times\mathbf{2}$. The only choice in $\mathcal{M}_n$ is (up to isomorphism) $\mathcal{B}(111) = M_k$ for some $k \in \{3,\dots,n\}$.

The algebras $\mathcal{B}(i)$ for $i \in \{100, 010, 001\}$ are subalgebras of M_k whose congruence lattices are isomorphic to $\mathbf{2}\times\mathbf{2}$. Hence, $\mathcal{B}(i) = \{0,x,y,1\}$ for some $x, y \in M_k \setminus \{0,1\}$. We claim that $\mathcal{B}(i) \cap \mathcal{B}(j) = \{0,1\}$ for every $i, j \in \{100, 010, 001\}$, $i \neq j$. We prove it for $i = 100$, $j = 010$.

For contradiction, suppose that $w \in \mathcal{B}(100) \cap \mathcal{B}(010)$, $w \notin \{0,1\}$. Since $\mathcal{B}(110) \supseteq \mathcal{B}(100)$, $\mathcal{B}(101) \supseteq \mathcal{B}(100)$, $\mathcal{B}(011) \supseteq \mathcal{B}(010)$, we have $w \in \mathcal{B}(i)$ for every $i \in \{110, 101, 011\}$. Further, for every $i \in \{100, 010, 110, 101, 011\}$ the algebra $\mathcal{B}(i)$ is isomorphic to $\mathbf{3}$ or $\mathbf{2} \times \mathbf{2}$, so the two nontrivial congruences of $\mathcal{B}(i)$ are $\theta_{\mathcal{B}(i)}(0, w)$ and $\theta_{\mathcal{B}(i)}(1, w)$ (shortly written as $\theta_i(0, w)$, $\theta_i(1, w)$). Further, $\mathcal{B}(001)$ contains some element $x \notin \{0, 1\}$, so its nontrivial congruences are $\theta_{001}(0, x)$ and $\theta_{001}(1, x)$.

Now let us look closer at the "middle layer" homomorphisms (i.e. homomorphisms $\mathcal{D}_3(i, j)$ with $i \in \{001, 010, 100\}$, $j \in \{110, 101, 011\}$). With respect to these isomorphisms, the nontrivial elements of $\mathcal{D}_3(i)$ (nontrivial congruences of $\mathcal{A}_3(i)$), $i \in \{100, 010, 001, 110, 101, 011\}$, form a 12-cycle. We claim that this is not the case in $\mathcal{B}$. We need to consider the following two cases:

(1) If $x = w$ then $\theta_{001}(w, 0)$ maps into $\theta_{101}(w, 0)$ and to $\theta_{011}(w, 0)$, which means that the elements $\theta_i(w, 0)$ with $i \in \{100, 010, 001, 110, 101, 011\}$ form a 6-cycle.
(2) If $x \neq w$ then $\theta_{001}(x, 0)$ maps into $\theta_{101}(w, 1)$ and to $\theta_{011}(w, 1)$, which means that $\theta_{001}(x, 0)$ together with the elements $\theta_i(w, 1)$ with $i \in \{100, 010, 110, 101, 011\}$ form a 6-cycle.

Hence, instead of a 12-cycle, we can only get two 6-cycles. This contradiction shows that $\mathcal{B}(100) \cap \mathcal{B}(010) = \{0, 1\}$. Moreover, both $\mathcal{B}(100)$ and $\mathcal{B}(010)$ are subalgebras of $\mathcal{B}(110)$. As $\mathcal{B}(110)$ is either isomorphic to $\mathbf{2} \times \mathbf{2}$ or $\mathbf{3}$, this is only possible if $\mathcal{B}(100)$ and $\mathcal{B}(010)$ are both three-element chains. □

Now we are ready to deal with a more complicated diagram. It will be a diagram $\mathcal{A}_4 : J \to \mathcal{M}_4$, indexed by the set $J = \mathbf{3} \times \mathbf{3} \times \mathbf{3}$. Again, all algebras are subalgebras of M_4 and all morphisms are the set inclusions. The diagram is depicted below, in the style similar to $\mathcal{A}_3$. The indexing is not indicated on the picture, so we assume that $\mathcal{A}_4(200) = \{0, a, 1\}$, $\mathcal{A}_4(020) = \{0, b, 1\}$, $\mathcal{A}_4(002) = \{0, c, 1\}$. All other indices can be easily deduced from this. The elements of M_4 are denoted by 0, 1, a, b, c, d.

We consider the associated semilattice diagram $\mathcal{D}_4 = \operatorname{Con}_c \circ \mathcal{A}_4$. This is a diagram of finite $(\vee, 0)$-semilattices, liftable in $\mathcal{M}_4$.

Theorem 2.2. *$\mathcal{D}_4$ is not liftable in $\mathcal{M}_3$.*

Proof. For contradiction, suppose that $\mathcal{X} : J \to \mathcal{M}_3$ lifts $\mathcal{D}_4$. Similarly as in Lemma 2.1, we can assume that all algebras $\mathcal{X}(j)$ are subalgebras of $\mathcal{X}(222)$ and all morphisms are set inclusions. Considering the restriction of $\mathcal{X}$ to $\{j \in J \mid 110 \leq j \leq 221\}$ and using Lemma 2.1 we find that

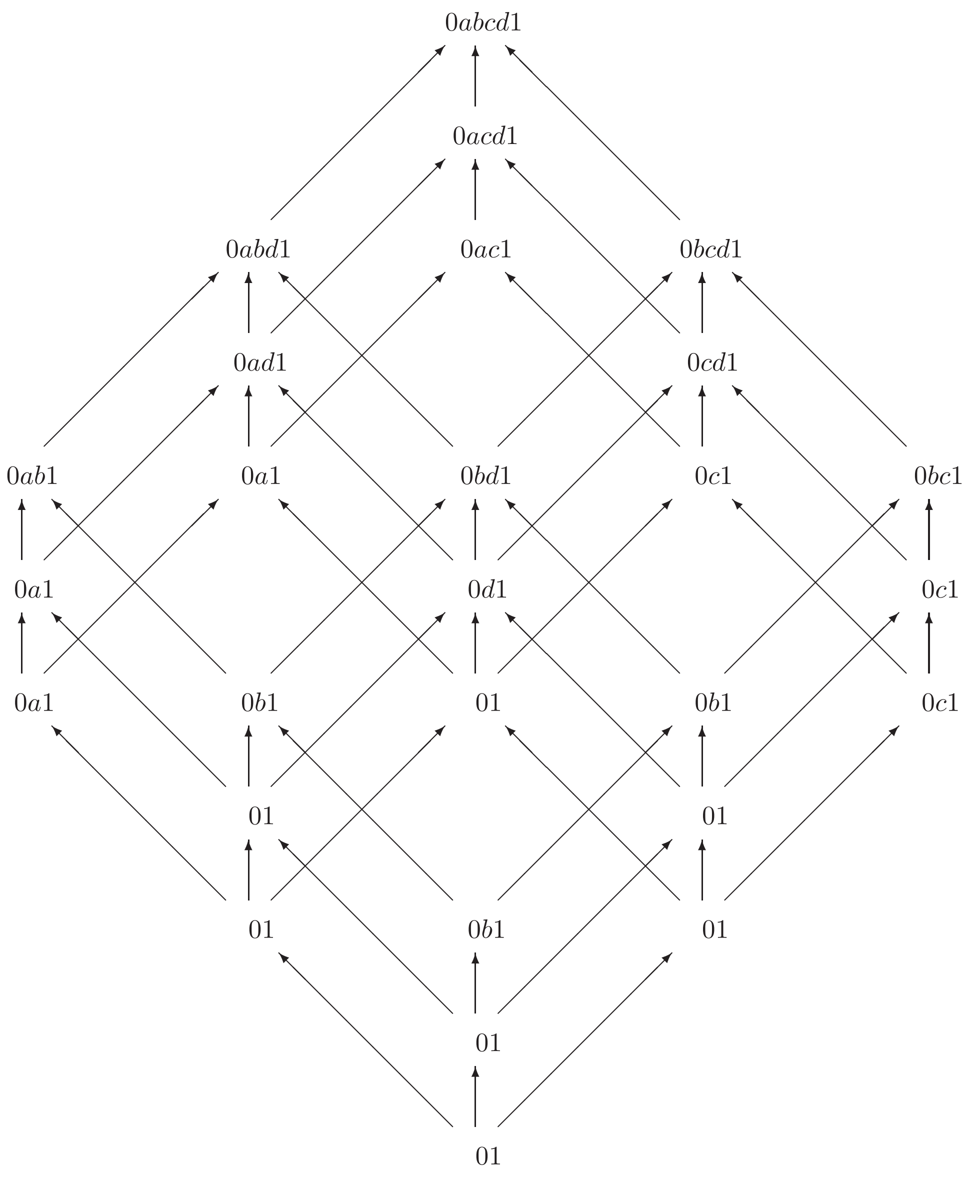
(1) $\mathcal{X}(221)$ is isomorphic to M_3;
(2) $\mathcal{X}(210)$, $\mathcal{X}(120)$ and $\mathcal{X}(111)$ are mutually distinct and isomorphic to $\mathbf{3}$.

Similarly, considering the subdiagrams of $\mathcal{D}_4$ determined by $011 \leq j \leq 122$ and $101 \leq j \leq 212$ respectively, we find that

(3) $\mathcal{X}(122)$ and $\mathcal{X}(212)$ are isomorphic to M_3;
(4) $\mathcal{X}(111)$, $\mathcal{X}(021)$ and $\mathcal{X}(012)$ are mutually distinct and isomorphic to $\mathbf{3}$;
(5) $\mathcal{X}(201)$, $\mathcal{X}(111)$ and $\mathcal{X}(102)$ are mutually distinct and isomorphic to $\mathbf{3}$.

Now, $\mathcal{X}(002)$ is a subalgebra of $\mathcal{X}(102)$. As the two algebras have the same congruence lattice, and $\mathcal{X}(102)$ is isomorphic to **3**, we obtain that $\mathcal{X}(002) = \mathcal{X}(102)$. For the same reasons, $\mathcal{X}(002) = \mathcal{X}(012)$, hence $\mathcal{X}(102) = \mathcal{X}(012)$. Similarly, $\mathcal{X}(021) = \mathcal{X}(020) = \mathcal{X}(120)$ and $\mathcal{X}(201) = \mathcal{X}(200) = \mathcal{X}(210)$.

Hence, $\mathcal{X}(102) \neq \mathcal{X}(201)$ implies that $\mathcal{X}(102) \neq \mathcal{X}(210)$. Further, $\mathcal{X}(012) \neq \mathcal{X}(021)$ implies that $\mathcal{X}(102) \neq \mathcal{X}(021) = \mathcal{X}(120)$. Thus $\mathcal{X}(111)$, $\mathcal{X}(210)$, $\mathcal{X}(120)$ and $\mathcal{X}(102)$ are four different subalgebras of $\mathcal{X}(222)$, each isomorphic to **3**. However, this is impossible, as $\mathcal{X}(222)$ must be simple and in $\mathcal{M}_3$. □



3. A Connection to Graph Theory

The example in the previous section is a very special one. Now we try to find similar examples distinguishing between $\operatorname{Con}\mathcal{M}_n$ for different $n \geq 4$. The elements of M_n will be denoted $0, 1, a_1, \ldots, a_n$.

Let us assume that $\mathcal{A}$ is a diagram of finite algebras in $\mathcal{M}_n^{01}$ $(n \geq 3)$, indexed by the product $J = J_1 \times J_2 \times J_3$ of three finite chains. Denote u the largest element of J and u_k the largest element of J_k $(k = 1, 2, 3)$. Assume that the following conditions are satisfied:

(C1) $\mathcal{A}(u) = M_n$;
(C2) all algebras in $\mathcal{A}$ are subalgebras of $\mathcal{A}(u)$;
(C3) all morphisms in $\mathcal{A}$ are the set inclusions;
(C4) $\mathcal{A}(i \wedge j) = \mathcal{A}(i) \cap \mathcal{A}(j)$ for all $i, j \in J$.

To every such diagram $\mathcal{A}$ we associate an unoriented graph $G(\mathcal{A}) = (V, E)$, whose set of vertices is $V = \{a_1, \ldots, a_n\}$ and $\{x, y\} \in E$ (the set of edges) if $x \neq y$ and $\mathcal{A}(j) = \{0, 1, x, y\}$ for some $j \in J$.

Notice that the diagram $\mathcal{A}_4$ in the previous section satisfies $(C1) - (C4)$ and its associated graph is the complete graph on 4 vertices. In general, we can say the following:

Lemma 3.1. *The graph $G(\mathcal{A})$ is planar.*

Proof. We proceed by induction on n. The statement is true for $n < 5$, as every graph with less than 5 vertices is planar. Let $n \geq 5$. We distinguish two cases.

I. Suppose that there are $x, y \in V$, $x \neq y$, such that $x \in \mathcal{A}(j)$ implies $y \in \mathcal{A}(j)$ for every $j \in J$. Then x can be incident with at most one edge, namely $\{x, y\}$. Without loss of generality we can assume that $x = a_n$. Consider the diagram $\mathcal{A}'$, which arises from $\mathcal{A}$ by omitting $x = a_n$ from every $\mathcal{A}(j)$. Clearly, $\mathcal{A}'$ satisfies (C1)–(C4) with $n-1$ instead of n, so by the induction hypothesis, the corresponding graph $(V \setminus \{x\}, E')$ is planar. Obviously, $E \setminus \{\{x, y\}\} \subseteq E'$, so $(V \setminus \{x\}, E \setminus \{\{x, y\}\})$ is also planar. Since adding a vertex of degree one cannot harm the planarity, (V, E) is planar too.

II. Suppose that for every $x, y \in V$, $x \neq y$, there is $j \in J$ with $x \in \mathcal{A}(j)$, $y \notin \mathcal{A}(j)$. Consider the incidence poset $(P, \leq)$ of $G(\mathcal{A})$, i.e. $P = V \cup E$ and $x < s$ iff $x \in V$, $s \in E$ and $x \in s$. According to Schnyder's theorem [8], $G(\mathcal{A})$ is planar if and only if $(P, \leq)$ has the order-dimension at most 3. Thus, it suffices to define 3 linear orders on P, whose intersection is $\leq$.

First we define the orders on V. For $i \in \{1, 2, 3\}$ let

$$K_i = \{j = (j_1, j_2, j_3) \in J \mid j_k = u_k \text{ whenever } k \neq i\}.$$

For $x, y \in V$, $i \in \{1, 2, 3\}$ we set

$$x <_i' y \text{ iff } x \in \mathcal{A}(j),\ y \notin \mathcal{A}(j) \text{ for some } j \in K_i.$$

It is clear that $<'_i$ is a strict partial ordering of V. Denote by $\leq'_i$ the associated partial ordering. Let us extend it to a linear order $\leq_i$ on V arbitrarily. Now define the relation $\sqsubseteq_i$ on P as follows:

(1) if $x, y \in V$ then $x \sqsubseteq_i y$ iff $x \leq_i y$;
(2) if $x \in V$ and $s = \{y, z\} \in E$ then $x \sqsubseteq_i s$ iff $x \leq_i y$ or $x \leq_i z$ (and $s \sqsubseteq_i x$ iff $y <_i x$ and $z <_i x$);
(3) if $s = \{x, y\} \in E$, $t = \{z, w\} \in E$ with $s \cap t = \emptyset$, then $s \sqsubseteq_i t$ iff $\max\{x, y\} \leq_i \max\{z, w\}$ (the maximums with respect to $\leq_i$);
(4) if $s = \{x, y\} \in E$, $t = \{x, z\} \in E$, then $s \sqsubseteq_i t$ iff $y \leq_i z$.

It is easy to check that each $\sqsubseteq_i$ is a linear order. We claim that $\leq$ is the intersection of $\sqsubseteq_1$, $\sqsubseteq_2$ and $\sqsubseteq_3$. If $x < s$, then $x \in V$ and $s = \{x, y\} \in E$. Then obviously $x \sqsubseteq_i s$ for every i. Conversely, let $p \sqsubseteq_i q$ for every i. We distinguish the following four cases.

(a) Let $p, q \in V$. If $p \neq q$, then, according to our assumption, $q \in \mathcal{A}(j)$ and $p \notin \mathcal{A}(j)$ for some $j = (j_1, j_2, j_3) \in J$. By (C4) we have

$$\mathcal{A}(j) = \mathcal{A}(j_1, u_2, u_3) \cap \mathcal{A}(u_1, j_2, u_3) \cap \mathcal{A}(u_1, u_2, j_3),$$

and we can assume without loss of generality that $p \notin \mathcal{A}(j_1, u_2, u_3)$. Since $(j_1, u_2, u_3) \in K_1$, we obtain $q <'_1 p$, which is a contradiction with $p \sqsubseteq_1 q$, showing that $p = q$.

(b) Let $p \in V$, $q = \{z, w\} \in E$. In order to prove that $p \leq q$ we need to show that $p \in \{z, w\}$. Suppose for contradiction that $p \neq z$, $p \neq w$. Since q is an edge, there exist $j \in J$ with $\mathcal{A}(j) = \{0, 1, z, w\}$. Similarly as in the case (a), j can be written as $(j_1, u_2, u_3) \wedge (u_1, j_2, u_3) \wedge (u_1, u_2, j_3)$. Without loss of generality we can assume that $p \notin \mathcal{A}(j_1, u_2, u_3)$, which implies that $z <'_1 p$ and $w <'_1 p$, contradicting the assumption that $p \sqsubseteq_1 q$.

(c) Let $p = \{x, y\} \in E$, $q \in V$. Then $x, y \sqsubseteq_i p$ for every i. Consequently, $x, y \sqsubseteq_i q$ for every i, which by part (a) of this proof implies that $x = q$ and $y = q$, a contradiction with $x \neq y$. Thus, case (c) is impossible.

(d) Let $p = \{x, y\} \in E$, $q = \{z, w\} \in E$. The same argument as in (c) shows that $x, y \sqsubseteq_i q$ for every i. By the part (b) of this proof, $x \in \{z, w\}$ and $y \in \{z, w\}$, which means that $p = q$. □

Theorem 3.2. *Let $\mathcal{A}$ be a diagram of finite algebras in $\mathcal{M}_n^{01}$ $(n \geq 3)$, indexed by the product $J = J_1 \times J_2 \times J_3$ of three finite chains, satisfying the conditions (C1) – (C4). Then the diagram* $\mathrm{Con}_c \circ \mathcal{A}$ *is liftable in $\mathcal{M}_4^{01}$.*

Proof. By the above lemma, the graph $G(\mathcal{A})$ is planar. According to the well-known four-color theorem, $G(\mathcal{A})$ is 4-colorable. Thus, there exists a map $c : \{a_1, \ldots, a_n\} \to \{a_1, a_2, a_3, a_4\}$ such that $c(x) \neq c(y)$ whenever $\{x, y\} \in E$. We extend c to a mapping $M_n \to M_4$ by setting $c(0) = 0$, $c(1) = 1$.

Now we define a diagram $\mathcal{B} : J \to \mathcal{M}_4^{01}$ as follows:

(1) if $\mathcal{A}(j)$ is isomorphic to M_k for some $k \geq 3$, then $\mathcal{B}(j) = M_4$;
(2) if $\mathcal{A}(j)$ is isomorphic to $\mathbf{2}$ or $\mathbf{3}$ or $\mathbf{2} \times \mathbf{2}$, then $\mathcal{B}(j) = c(\mathcal{A}(j))$;
(3) for every $j, k \in J$, $j \leq k$, the morphism $\mathcal{B}(j,k)$ is the set inclusion.

It is clear that $\mathcal{B}$ is correctly defined. We claim that $\mathcal{B}$ lifts $\operatorname{Con}_c \circ \mathcal{A}$. To prove it, we have to define for every $j \in J$ an isomorphism $\psi_j :\ \operatorname{Con}_c\mathcal{A}(j) \to \operatorname{Con}_c\mathcal{B}(j)$ such that the diagram

$$\begin{array}{ccc} \operatorname{Con}_c(\mathcal{A}(j)) & \xrightarrow{\operatorname{Con}_c \circ \mathcal{A}(j,k)} & \operatorname{Con}_c(\mathcal{A}(k)) \\ {\scriptstyle \psi_j}\downarrow & & {\scriptstyle \psi_k}\downarrow \\ \operatorname{Con}_c(\mathcal{B}(j)) & \xrightarrow{\operatorname{Con}_c \circ \mathcal{B}(j,k)} & \operatorname{Con}_c(\mathcal{B}(k)) \end{array}$$

commutes for every $j \leq k$.

If $\mathcal{A}(j)$ is isomorphic to M_l for some $l \geq 3$, then both $\mathcal{A}(j)$ and $\mathcal{B}(j)$ are simple and the isomorphism $\psi_j :\ \operatorname{Con}_c\mathcal{A}(j) \to \operatorname{Con}_c\mathcal{B}(j)$ is obvious. If $\mathcal{A}(j)$ is isomorphic to $\mathbf{2}$ or $\mathbf{3}$ or $\mathbf{2} \times \mathbf{2}$, then the restriction of c is an isomorphism $\mathcal{A}(j) \to \mathcal{B}(j)$. Indeed, the cases $\mathcal{A}(j) \cong \mathbf{2}$ and $\mathcal{A}(j) \cong \mathbf{3}$ are trivial, in the case $\mathcal{A}(j) \cong \mathbf{2} \times \mathbf{2}$ we use the fact that c is a coloring. So, we can define the isomorphism ψ_j by

$$\psi_j(\theta) = \{(c(x), c(y)) \mid (x,y) \in \theta\}.$$

It is easy to observe that every congruence on every $\mathcal{A}(j)$ is of the form $\theta_{\mathcal{A}(j)}(0,x)$ or $\theta_{\mathcal{A}(j)}(1,x)$ for some $x \in \mathcal{A}(j)$ and ψ_j maps this congruence into $\theta_{\mathcal{B}(j)}(0,c(x))$ or $\theta_{\mathcal{B}(j)}(1,c(x))$, respectively. Now it is easy to check the commutativity of the above diagram. For every $x \in \mathcal{A}(j)$ we have

$$\operatorname{Con}_c\mathcal{B}(j,k)(\psi_j(\theta_{\mathcal{A}(j)}(0,x))) = \operatorname{Con}_c\mathcal{B}(j,k)(\theta_{\mathcal{B}(j)}(0,c(x))) = \theta_{\mathcal{B}(k)}(0,c(x))$$

and

$$\psi_k(\operatorname{Con}_c\mathcal{A}(j,k)(\theta_{\mathcal{A}(j)}(0,x))) = \psi_k(\theta_{\mathcal{A}(k)}(0,x)) = \theta_{\mathcal{B}(k)}(0,c(x)),$$

and similarly for the congruences of the form $\theta_{\mathcal{A}(j)}(1,x)$. This completes the proof. □

4. A General Construction

In this section we construct a diagram liftable in $\mathcal{M}_n$ and not in $\mathcal{M}_{n-1}$ $(n > 3)$.

Before proving the main result we need to introduce one technical tool. Let $\mathcal{A} :\ J \to \mathcal{V}$ be a diagram of algebras, indexed by a poset J with a largest element u. For every $\alpha \in \operatorname{Con}(\mathcal{A}(u))$ and every $j \in J$ we set

$$\alpha_j = \{(x,y) \in \mathcal{A}(j) \mid (\mathcal{A}(j,u)(x), \mathcal{A}(j,u)(y)) \in \alpha\} \in \operatorname{Con}(\mathcal{A}(j)).$$

Let us denote $\mathcal{A}'(j) = \mathcal{A}(j)/\alpha_j$. Further, for every $j \leq k$ there exists a natural homomorphism $\mathcal{A}'(j,k) : \mathcal{A}'(j) \to \mathcal{A}'(k)$ defined by $x/\alpha_j \mapsto \mathcal{A}(j,k)(x)/\alpha_k$. It is a routine to check that $\mathcal{A}'$ is again a diagram of algebras, indexed by J. We also write $\mathcal{A}' = \mathcal{A}/\alpha$.

Lemma 4.1. *Let $\mathcal{A} : J \to \mathcal{V}$ be a diagram of finite algebras indexed by a poset J with a largest element u. Let $\mathcal{B} : J \to \mathcal{W}$ lift $\operatorname{Con}_c \circ \mathcal{A}$ via isomorphisms $\psi_j : \operatorname{Con}_c(\mathcal{A}(j)) \to \operatorname{Con}_c(\mathcal{B}(j))$. Then*

(i) *for every $\alpha \in \operatorname{Con}_c(\mathcal{A}(u))$ the diagram $\mathcal{B}/\psi_u(\alpha)$ lifts $\operatorname{Con}_c \circ \mathcal{A}/\alpha$;*
(ii) *for every $\alpha, \beta \in \operatorname{Con}_c(\mathcal{A}(u))$ and every $j \in J$, if $\alpha_j = \beta_j$ then $\psi_u(\alpha)_j = \psi_u(\beta)_j$.*

Proof. It is clear that

$$\alpha_j = \bigvee\{\delta \in \operatorname{Con}_c(\mathcal{A}(j)) \mid \operatorname{Con}_c\mathcal{A}(j,u)(\delta) \subseteq \alpha\}.$$

Since ψ_j is an isomorphism, we have

$$\begin{aligned}\psi_j(\alpha_j) &= \bigvee\{\psi_j(\delta) \mid \psi_u(\operatorname{Con}_c\mathcal{A}(j,u)(\delta)) \subseteq \psi_u(\alpha)\}\\ &= \bigvee\{\psi_j(\delta) \mid \operatorname{Con}_c\mathcal{B}(j,u)(\psi_j(\delta)) \subseteq \psi_u(\alpha)\}\\ &= \bigvee\{\varepsilon \in \operatorname{Con}_c(\mathcal{B}(j)) \mid \operatorname{Con}_c\mathcal{B}(j,u)(\varepsilon) \subseteq \psi_u(\alpha)\} = \psi_u(\alpha)_j.\end{aligned}$$

Thus, ψ_j can be restricted to the isomorphism $\uparrow\alpha_j \to \uparrow\psi_u(\alpha)_j$, or equivalently, $\operatorname{Con}_c(\mathcal{A}(j)/\alpha_j) \to \operatorname{Con}_c(\mathcal{B}(j)/\psi_u(\alpha)_j)$, showing (i).

Now, let α, β satisfy the assumptions of (ii). Then $\psi_u(\alpha)_j = \psi_j(\alpha_j) = \psi_j(\beta_j) = \psi_u(\beta)_j$. □

If all morphisms of $\mathcal{A}$ are the set inclusions, then, for every $\alpha \in \operatorname{Con}(\mathcal{A}(u))$ and $j \leq u$, the congruence α_j is the restriction of α to $\mathcal{A}(j)$. To simplify the notation, we often write $\mathcal{A}(j)/\alpha$ instead of $\mathcal{A}(j)/\alpha_j$. Further, for $j \leq m$ we regard $\mathcal{A}(j)/\alpha$ as a subalgebra of $\mathcal{A}(m)/\alpha$, identifying the α_j-classes of $\mathcal{A}(j)$ with the corresponding α_m-classes of $\mathcal{A}(m)$. Hence, by saying that $\mathcal{A}(j)/\alpha$ and $\mathcal{A}(k)/\alpha$ are different subalgebras of $\mathcal{A}(m)/\alpha$ $(j, k \leq m)$ we really mean that $\mathcal{A}(j)$ and $\mathcal{A}(k)$ intersect different α_m-classes of $\mathcal{A}(m)$. Notice however, that the inequality $\mathcal{A}(j)/\alpha \neq \mathcal{A}(k)/\alpha$ does not depend on the choice of m.

Now we can proceed with our construction. Let us fix $n > 3$ and consider the following three linear orders on the set $\{1, 2, \ldots, n\}$:

$$\begin{gathered}1 <_1 2 <_1 3 <_1 \cdots <_1 n;\\ 1 <_2 n <_2 n-1 <_2 n-2 <_2 \ldots <_2 2;\\ 2 <_3 n <_3 n-1 <_3 \cdots <_3 3 <_3 1.\end{gathered}$$

Let Z_k^i be the unique k-element lower subset of the ordered set $(\{1, \ldots, n\}, \leq_i)$ $(i \in \{1, 2, 3\}, 1 \leq k \leq n)$.

Now we are ready to define a commutative diagram in $\mathcal{M}_n$ (in fact, in $\mathcal{M}_n^{01}$). As the index set we take the product of three $(n-1)$-element chains $J = \boldsymbol{(n-1)} \times \boldsymbol{(n-1)} \times \boldsymbol{(n-1)}$. Let F be the free algebra in $\mathcal{M}_n^{01}$ freely generated by the set $\{1, \dots, n\}$. For every $e = (j,k,l) \in J$ denote $X(e) = Z^1_{j+2} \cap Z^2_{k+2} \cap Z^3_{l+2}$ and let $\mathcal{A}(e)$ be the subalgebra of F generated by the set $X(e)$. It is easy to see that $X(d \wedge e) = X(d) \cap X(e)$ and $\mathcal{A}(d \wedge e) = \mathcal{A}(d) \cap \mathcal{A}(e)$ for every $d, e \in J$. Further, $\mathcal{A}(d) \subseteq \mathcal{A}(e)$ whenever $d, e \in J$, $d \leq e$, so we can define the homomorphism $\mathcal{A}(d,e) : \mathcal{A}(d) \to \mathcal{A}(e)$ as the set inclusion. Clearly, we have a commutative diagram $\mathcal{A} : J \to \mathcal{M}_n$.

Theorem 4.2. *The diagram* $\operatorname{Con}_c \circ \mathcal{A}$ *is not liftable in* $\mathcal{M}_{n-1}$.

Proof. Suppose for contradiction that $\mathcal{B}$ lifts $\operatorname{Con}_c \circ \mathcal{A}$. Similarly as in 2.1, we can assume that all algebras $\mathcal{B}(j)$ $(j \in J)$ are subalgebras of $\mathcal{B}(u)$, where u is the top element of J, and that all morphisms in $\mathcal{B}$ are set inclusions. So, for every $j \in J$ we have an isomorphism $\psi_j : \operatorname{Con}_c(\mathcal{A}(j)) \to \operatorname{Con}_c(\mathcal{B}(j))$ such that

$$\begin{array}{ccc} \operatorname{Con}_c(\mathcal{A}(j)) & \xrightarrow{\operatorname{Con}_c \circ \mathcal{A}(j,k)} & \operatorname{Con}_c(\mathcal{A}(k)) \\ \psi_j \big\downarrow & & \psi_k \big\downarrow \\ \operatorname{Con}_c(\mathcal{B}(j)) & \xrightarrow{\operatorname{Con}_c \circ \mathcal{B}(j,k)} & \operatorname{Con}_c(\mathcal{B}(k)) \end{array}$$

commutes for every $j \leq k$.

Consider the homomorphism $f : \mathcal{A}(u) = F \to M_n$ defined by $f(i) = a_i$ for every $i \in \{1, \dots, n\}$. Since $\mathcal{A}(u)$ is finite, we have $\operatorname{Ker}(f) \in \operatorname{Con}_c(\mathcal{A}(u))$. By 4.1, the diagram $\mathcal{B}/\psi_u(\operatorname{Ker}(f))$ lifts the diagram $\operatorname{Con}_c \circ \mathcal{A}/\operatorname{Ker}(f)$. Let us denote $\psi_u(\operatorname{Ker}(f)) = \beta$.

Further, for every $k \in \{3, \dots, n\}$ we denote $v(k) = (k-2, n-k, n-k) \in J$, and $v(1) = (0, 0, n-2)$, $v(2) = (0, n-2, 0)$. It is easy to check that $X(v(k)) = \{k\}$, so $\mathcal{A}(v(k))$ as well as $\mathcal{A}(v(k))/\operatorname{Ker}(f)$ are isomorphic to $\mathbf{3}$.

For $k \in \{3, \dots, n-1\}$ consider the subdiagram of $\mathcal{A}/\operatorname{Ker}(f)$ indexed by $\{j \in J \mid (k-2, n-k-1, n-3) \leq j \leq (k-1, n-k, n-2)\}$. One can check directly that $X(k-2, n-k-1, n-2) = \{1\}$, $X(k-2, n-k, n-3) = \{k\}$, $X(k-1, n-k-1, n-3) = \{k+1\}$ and $X(k-1, n-k, n-2) = \{1, k, k+1\}$. Hence, this subdiagram is naturally equivalent to the diagram $\mathcal{A}_3$ and Lemma 2.1 says that $\mathcal{B}(k-1, n-k, n-2)/\beta$ is isomorphic to some M_l and the algebras $\mathcal{B}(k-2, n-k-1, n-2)/\beta$, $\mathcal{B}(k-2, n-k, n-3)/\beta$ and $\mathcal{B}(k-1, n-k-1, n-3)/\beta$ are its distinct subalgebras isomorphic to $\mathbf{3}$.

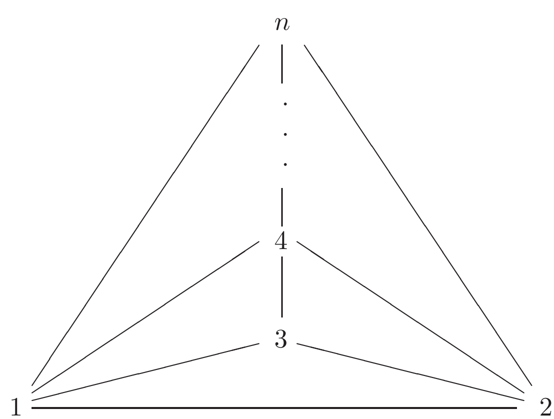
Further, $\mathcal{A}(k-2, n-k-1, n-2) = \mathcal{A}(v(1))$, which implies that $\mathcal{B}(k-2, n-k-1, n-2)/\beta$ and $\mathcal{B}(v(1))/\beta$ have isomorphic congruence lattices. Since $v(1) \leq (k-2, n-k-1, n-2)$, $\mathcal{B}(v(1))/\beta$ is a subalgebra of $\mathcal{B}(k-2, n-k-1, n-2)/\beta$, which is isomorphic to $\mathbf{3}$. This is only possible if $\mathcal{B}(v(1))/\beta = \mathcal{B}(k-2, n-k-1, n-2)/\beta$. The same argument shows that $\mathcal{B}(v(k))/\beta = \mathcal{B}(k-2, n-k, n-3)/\beta$ and $\mathcal{B}(v(k+1))/$

$\beta = \mathcal{B}(k-1, n-k-1, n-3)/\beta$. Hence, $\mathcal{B}(v(1))/\beta$, $\mathcal{B}(v(k))/\beta$ and $\mathcal{B}(v(k+1))/\beta$ are three different subalgebras of $\mathcal{B}(u)/\beta$, each isomorphic to **3**.

Interchanging the roles of the second and third coordinates one can show that $\mathcal{B}(v(2))/\beta$, $\mathcal{B}(v(k))/\beta$ and $\mathcal{B}(v(k+1))/\beta$ are different subalgebras of $\mathcal{B}(u)/\beta$ isomorphic to **3**.

The same reasoning for the subdiagram with $(0, n-3, n-3) \leq j \leq (1, n-2, n-2)$ shows that $\mathcal{B}(v(1))/\beta$, $\mathcal{B}(v(2))/\beta$ and $\mathcal{B}(v(3))/\beta$ are different subalgebras of $\mathcal{B}(u)/\beta$ isomorphic to **3**.

The algebra $\mathcal{B}(u)/\beta$ is simple and has subalgebras isomorphic to **3**. The only possibility in $\mathcal{M}_{n-1}$ is that $\mathcal{B}(u)/\beta$ is isomorphic to some M_l. Its subalgebras $\mathcal{B}(v(k))/\beta$ $(k = 1, \dots n)$ are isomorphic to **3**. The proof will be completed by proving that they all are mutually different (contradicting the assumption $\mathcal{B}(u) \in \mathcal{M}_{n-1}$). However, this cannot be done by considering the diagram $\mathcal{A}/\mathrm{Ker}(f)$ alone. Notice that $\mathcal{A}/\mathrm{Ker}(f)$ satisfies (C1) – (C4) from the previous section, so $\mathrm{Con}_c \circ \mathcal{A}/\mathrm{Ker}(f)$ is liftable in $\mathcal{M}_4$. The graph $G(\mathcal{A}/\mathrm{Ker}(f))$ looks as follows.



In accordance with the previous section, this graph is planar, and hence 4-colorable. So far we have proved that $\mathcal{B}(v(k))/\beta \neq \mathcal{B}(v(m))/\beta$ whenever (k, m) is an edge. To achieve more, we modify the homomorphism f.

So, let $k, m \in \{1, \dots, n\}$, $k < m$. If $k \in \{1, 2\}$ or $m - k = 1$, then the inequality $\mathcal{B}(v(k))/\beta \neq \mathcal{B}(v(m))/\beta$ is already proven. Suppose now that $k \geq 3$ and $m - k \geq 2$. Let $g : F \to M_n$ be the unique homomorphism with $g(m) = a_2$ and $g(i) = a_i$ for $i \neq m$. By Lemma 4.1, the diagram $\mathcal{B}/\psi_u(\mathrm{Ker}(g))$ lifts the diagram $\mathrm{Con}_c \circ \mathcal{A}/\mathrm{Ker}(g)$. Let us denote $\gamma = \psi_u(\mathrm{Ker}(g))$.

By the same way as before we can argue that

(D1) $\mathcal{B}(v(i))/\gamma$ is isomorphic to **3** for every $i \geq 3$;
(D2) $\mathcal{B}(v(2))/\gamma$, $\mathcal{B}(v(i))/\gamma$ and $\mathcal{B}(v(i+1))/\gamma$ are different for every $i \notin \{m-1, m\}$;
(D3) $\mathcal{B}(v(1))/\gamma$, $\mathcal{B}(v(i))/\gamma$ and $\mathcal{B}(v(i+1))/\gamma$ are different for every $i \geq 3$.

Now, for $j = (m-2, n-2, n-m+1)$ we have $X(j) = \{2, m-1, m\}$, so $\mathcal{A}(j)/\mathrm{Ker}(g)$ is isomorphic to $\mathbf{2} \times \mathbf{2}$. The congruence lattice of this algebra, and consequently of $\mathcal{B}(j)/\gamma$, is also isomorphic to $\mathbf{2} \times \mathbf{2}$. Since $\mathcal{B}(j)/\gamma$ is a subalgebra of $\mathcal{B}(u)/\gamma$, which is simple, this is only possible if $\mathcal{B}(j)/\gamma$ is isomorphic to $\mathbf{3}$ or $\mathbf{2} \times \mathbf{2}$. The algebras $\mathcal{B}(v(2))/\gamma$, $\mathcal{B}(v(m-1))/\gamma$ and $\mathcal{B}(v(m))/\gamma$ are subalgebras of $\mathcal{B}(j)/\gamma$, so at least two of them must be equal. By (D2), $\mathcal{B}(v(2))/\gamma$, $\mathcal{B}(v(m-2))/\gamma$ and $\mathcal{B}(v(m-1))/\gamma$ are different. By (D3), $\mathcal{B}(v(1))/\gamma$, $\mathcal{B}(v(m-1))/\gamma$ and $\mathcal{B}(v(m))/\gamma$ are different. Thus, the only possibility is $\mathcal{B}(v(2))/\gamma = \mathcal{B}(v(m))/\gamma$. Hence, $\mathcal{B}(v(2))/\gamma \neq \mathcal{B}(v(k))/\gamma$ implies that $\mathcal{B}(v(m))/\gamma \neq \mathcal{B}(v(k))/\gamma$.

Finally, consider $j = (n-2, n-3, n-2)$. We have $X(j) = \{1, 3, 4, \dots, n\}$, so $\mathcal{B}(j)/\gamma$ contains both $\mathcal{B}(v(m))/\gamma$ and $\mathcal{B}(v(k))/\gamma$ as subalgebras. We obtain that $\mathrm{Ker}(f)_j = \mathrm{Ker}(f) \restriction \mathcal{A}(j) = \mathrm{Ker}(g) \restriction \mathcal{A}(j) = \mathrm{Ker}(g)_j$. By Lemma 4.1, $\beta_j = \gamma_j$, hence $\mathcal{B}(v(k))/\gamma \neq \mathcal{B}(v(m))/\gamma$ implies that $\mathcal{B}(v(k))/\beta \neq \mathcal{B}(v(m))/\beta$. □

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