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ITERATIVE SEPARATION IN DISTRIBUTIVE CONGRUENCE LATTICES

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ABSTRACT. In [PLOŠČICA, M.: *Separation in distributive congruence lattices*, Algebra Universalis **49** (2003), 1–12] we defined separable sets in algebraic lattices and showed a close connection between the types of non-separable sets in congruence lattices of algebras in a finitely generated congruence distributive variety $\mathscr{V}$ and the structure of subdirectly irreducible algebras in $\mathscr{V}$. Now we generalize these results using the concept of separable mappings (defined on some trees) and apply them to some lattice varieties.

1. Introduction

For a variety (equational class) $\mathscr{V}$ let $\mathrm{Con}(\mathscr{V})$ denote the class of all lattices isomorphic to $\mathrm{Con}\,A$ (the congruence lattice of an algebra A), for some $A \in \mathscr{V}$. Our paper is a contribution to the problem of describing $\mathrm{Con}(\mathscr{V})$. We restrict ourselves to the case when $\mathscr{V}$ is congruence distributive and finitely generated. Even under such restrictions, the problem is very difficult and there are very few relevant varieties for which a satisfactory answer is known.

Let us recall that the congruence distributivity of $\mathscr{V}$ means that the congruence lattice of every algebra in $\mathscr{V}$ is distributive. The most common examples are various varieties of lattices and lattice ordered algebras.

Further, let $\mathrm{SI}(\mathscr{V})$ denote the class of all subdirectly irreducible members of $\mathscr{V}$. A well known consequence of the Jónsson lemma (see [3] or [4]) is that for any congruence distributive and finitely generated (i.e. generated by a finite algebra) variety $\mathscr{V}$ the class $\mathrm{SI}(\mathscr{V})$ consists of (up to isomorphism) finitely

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many finite algebras. In fact, if $\mathscr{V}$ is generated by a finite algebra A, then every subdirectly irreducible member of $\mathscr{V}$ is a homomorphic image of a subalgebra of A.

The aim of this paper (and its predecessor [6]) is to describe the class $\mathrm{Con}(\mathscr{V})$, using the knowledge of $\mathrm{SI}(\mathscr{V})$. One connection is obvious: for any completely meet-irreducible element $x \in L \in \mathrm{Con}(\mathscr{V})$, the interval $\uparrow x = \{y \in L : \ y \geq x\}$ must be isomorphic to $\mathrm{Con}\, A$ for some $A \in \mathrm{SI}(\mathscr{V})$. In [6], we introduced a new condition satisfied by all $L \in \mathrm{Con}(\mathscr{V})$. It turns out that the congruence lattices of subalgebras of subdirectly irreducible algebras play an important role. In this paper we develop further the ideas from [6] and provide even deeper insight into $\mathrm{Con}(\mathscr{V})$. However, a complete description of $\mathrm{Con}(\mathscr{V})$ remains a much more difficult problem.

Our basic reference books are [1] and [4]. All the unexplained concepts and unreferenced facts used in this paper can be found there.

If B is a subalgebra of an algebra A and $\alpha \in \mathrm{Con}\, A$, then $\alpha{\restriction}B = \alpha \cap B^2$ denotes the restriction of α to B. If $f\colon X \to Y$ is a mapping and $Z \subseteq X$, then $f{\restriction}Z$ denotes the restriction of f to Z. Furthermore, $\mathrm{Ker}(f)$ (the kernel of f) is the binary relation on X defined by $(x, y) \in \mathrm{Ker}(f)$ iff $f(x) = f(y)$. If δ is an equivalence relation on a set X and $x \in X$, then $[x]_\delta$ denotes the equivalence class containing x. If γ and δ are equivalence relations on X and $\gamma \subseteq \delta$, then δ/γ denotes the equivalence relation on the quotient set X/γ given by $([x]_\gamma, [y]_\gamma) \in \delta/\gamma$ iff $(x, y) \in \delta$.

2. The iterative separation

Let L be an algebraic lattice. An element $a \in L$ is called *strictly meet-irreducible* (or *completely meet-irreducible*) iff $a = \bigwedge X$ implies that $a \in X$, for every subset X of L. Note that the greatest element of L is not strictly meet-irreducible. Let $\mathrm{M}(L)$ denote the partially ordered set of all strictly meet-irreducible elements of L. Recall that $x = \bigwedge\{a \in \mathrm{M}(L) : \ x \leq a\}$, for every $x \in L$. Thus, every L contains many strictly meet-irreducible elements.

If L is distributive and $x_1 \wedge \cdots \wedge x_n \leq x \in \mathrm{M}(L)$ then $x_i \leq x$ for some i. If L is distributive and finite, then $\mathrm{M}(L)$ characterizes L up to isomorphism. If $L = \mathrm{Con}\, A$ then $\alpha \in \mathrm{M}(L)$ iff the quotient algebra A/α is subdirectly irreducible.

For a partially ordered set P and $x, y \in P$ write $x \prec y$ if $x < y$ and there is no $z \in P$ with $x < z < y$. Further, we denote $\uparrow x = \{p \in P : \ x \leq p\}$, $y^- = \{x \in P : \ x \prec y\}$. The length of a finite partially ordered set P is n if the largest chain in P has $n + 1$ elements.

Definition 2.1. A *tree* is a finite partially ordered set T with a largest element such that $\uparrow x$ is a chain for every $x \in T$.

For a tree T we denote $Z(T) = \{x \in T : \ \text{length}(\uparrow x) = \text{length}(T)\}$ and $T^- = T \setminus Z(T)$. It is clear that if $\text{length}(T) > 0$, then T^- is a tree again and $\text{length}(T^-) = \text{length}(T) - 1$. Further, we denote $T_k = \{x \in T : \ \text{length}(\uparrow x) = k\}$.

Now we can introduce our main concept of a separable mapping. However, it is formally simpler to define first the converse concept.

Definition 2.2. Let L be an algebraic lattice, $T \neq \emptyset$ a tree and φ an injective mapping $Z(T) \to \mathrm{M}(L)$. We define the *non-separability* of the mapping φ by an induction on the length of T as follows.

1° If $\text{length}(T) = 0$ then every φ is non-separable.

2° If $\text{length}(T) > 0$ then φ is non-separable iff for every family $\{x_p : p \in Z(T)\} \subseteq L$ such that $x_p \nleq \varphi(p)$ for every $p \in Z(T)$, there exists a non-separable injective mapping $\varphi^- \colon Z(T^-) \to \mathrm{M}(L)$ satisfying $x_p \nleq \varphi^-(q)$ whenever $p \in Z(T) \cap q^-$.

The mapping φ is called *separable* if it is not non-separable.

For the illustration, consider the special case of a tree of length 1, that is $T = \{p_1, \ldots, p_n, q\}$, $Z(T) = \{p_1, \ldots, p_n\}$, $T^- = \{q\}$. In this case, each mapping $Z(T^-) \to \mathrm{M}(L)$ is non-separable. The rule 2° says that $\varphi\colon Z(T) \to \mathrm{M}(L)$ is non-separable iff for any elements $x_1, \ldots, x_n \in L$ such that $x_i \nleq \varphi(p_i)$ and there is a mapping $\varphi^- \colon Z(T^-) \to \mathrm{M}(L)$ satisfying $x_i \nleq \varphi^-(q)$ for every i. Equivalently, φ is non-separable iff for any $x_i \in L$ such that $x_i \nleq \varphi(p_i)$ there is $u \in \mathrm{M}(L)$ with $\bigwedge\limits_{i=1}^{n} x_i \nleq u$. The existence of such u is equivalent to $\bigwedge\limits_{i=1}^{n} x_i \neq 0$, so we obtain the following statement.

Corollary 2.3. *Let T be a tree of length 1. An injective mapping $\varphi\colon Z(T) \to \mathrm{M}(L)$ is separable iff there are elements $x_p \in L$ $(p \in Z(T))$ such that $x_p \nleq \varphi(p)$ and $\bigwedge\limits_{p \in Z(T)} x_p = 0$.*

By [6], a (possibly infinite) set $P \subseteq \mathrm{M}(L)$ is called separable iff there are elements $x_p \in L$ $(p \in P)$ such that $x_p \nleq p$ and $\bigwedge\limits_{p \in P} x_p = 0$. Hence, for a tree T of length 1, a mapping $\varphi\colon Z(T) \to \mathrm{M}(L)$ is separable iff the set $\{\varphi(p) : \ p \in Z(T)\}$ is separable. So, for finite P, our definition generalizes the concept of separability from [6].

Our definition is easier to understand when we consider the following topological representation. Let L be a distributive algebraic lattice. A set $X \subseteq \mathrm{M}(L)$ is defined to be closed if $X = \mathrm{M}(L) \cap \uparrow x$, for some $x \in L$. It is easy to see (cf. [5]) that this defines a topology on $\mathrm{M}(L)$ and L is isomorphic to $\mathscr{O}(\mathrm{M}(L))$ (the lattice of open subsets of $\mathrm{M}(L)$).

It is not difficult to see that, with respect to this topology, a set $P \subseteq \mathrm{M}(L)$ is separable (in the sense of [6]) iff there are open sets A_p $(p \in P)$ such that

$p \in A_p$ and $\bigcap_{p \in P} A_p = \emptyset$. This is the motivation for using the term "separability". The adjective "iterative" refers to the inductive character of our definition. A topological description of our concept looks as follows.

Lemma 2.4. *Let L be a distributive algebraic lattice, T a tree and $\varphi\colon Z(T) \to \mathrm{M}(L)$ an injective mapping. Then φ is separable iff* $\mathrm{length}(T) > 0$ *and there exist open sets $A_p \subseteq \mathrm{M}(L)$ $(p \in Z(T))$ such that*

(1) *$p \in A_p$, for every p;*
(2) *every injective mapping $\varphi^- : Z(T^-) \to \mathrm{M}(L)$ such that $\varphi^-(q) \in \bigcap_{p \in q^-} A_p$ for every $q \in Z(T^-)$ is separable.*

As an example, consider the tree $T = \{0, 1, 2, 3, 4, 5, 6\}$ depicted in Section 3. An injective mapping $\varphi\colon \{3, 4, 5, 6\} \to L$ is separable iff there are open sets $A_i \subseteq \mathrm{M}(L)$ $(i = 3, 4, 5, 6)$ such that $\varphi(i) \in A_i$ and for every $x \in A_3 \cap A_4$, $y \in A_5 \cap A_6$, $x \neq y$, there are open sets $A_1, A_2 \subseteq \mathrm{M}(L)$ such that $x \in A_1$, $y \in A_2$ and $A_1 \cap A_2 = \emptyset$. Notice that this condition is weaker that the requirement $A_3 \cap A_4 \cap A_5 \cap A_6 = \emptyset$. (Consider the topological space $\mathrm{M}(L)$ consisting of a discrete sequence converging to 4 distinct limit points x_3, x_4, x_5, x_6 and the mapping given by $\varphi(i) = x_i$).

Now we will prove some general results for finitely generated congruence distributive varieties. Recall that any such variety is locally finite. (Finitely generated algebras are finite.)

Our results connect the existence of non-separable mappings into $\mathrm{M}(\mathrm{Con}\, A)$ for some $A \in \mathscr{V}$ with some special representation of the corresponding tree, which we now introduce. Recall the notation $T_k = \{x \in T : \ \mathrm{length}(\uparrow x) = k\}$.

Definition 2.5. Let T be a tree of the length n with the largest element u and $\mathscr{V}$ a variety. We say that T is $\mathrm{SI}(\mathscr{V})$-*representable* if there exist an algebra $B_0 \in \mathrm{SI}(\mathscr{V})$, a chain of its subalgebras $B_n \leq B_{n-1} \leq \cdots \leq B_1$ and congruences $\gamma_p \in \mathrm{M}(\mathrm{Con}\, B_k)$ for $p \in T_k$ such that

(1) if $p \leq q$, $p \in T_k$, then $\gamma_q \restriction B_k \subseteq \gamma_p$;
(2) if $p, q \in T_k$, $p \neq q$, then $\gamma_p \neq \gamma_q$;
(3) $\gamma_u = 0_{B_0}$ (the zero congruence on B_0).

For the proof of our main result we need the following simple technical lemma. It is a small modification of [7, 2.3].

Lemma 2.6. *Suppose that $\mathscr{V}$ is a finitely generated congruence distributive variety, $A \in \mathscr{V}$, $\alpha_1, \ldots, \alpha_n \in \mathrm{M}(\mathrm{Con}\, A)$, $n \in \omega$. Denote $\alpha = \bigcap \{\alpha_i \ : \ i = 1, \ldots, n\}$. Let A_0 be a finite subalgebra of A. Then there exists a finite subalgebra B of A such that $A_0 \subseteq B$ and*

(1) *for every $a \in A$, there is $b \in B$ with $(a,b) \in \alpha$;*

(2) *for every $i = 1, \dots, n$, there is $\beta_i \in \operatorname{Con} A$ such that $\beta_i \restriction B \nsubseteq \alpha_i \restriction B$ and, for every $\beta \in \operatorname{Con} A$, either $\beta_i \subseteq \beta$ or $\beta \restriction B \subseteq \alpha_i$.*

Proof. All algebras A/α_i are subdirectly irreducible and hence finite. Consequently, all congruences α_i have finitely many congruence classes and therefore α has finitely many congruence classes. It is therefore possible to choose a finite set $B_0 \subseteq A$ such that $A_0 \subseteq B_0$ and for every $a \in A$ there is $b \in B$ with $(a,b) \in \alpha$. Let B be the subalgebra of A generated by B_0. Obviously, B is finite and satisfies (1).

To prove (2), let $i \in \{1, \dots, n\}$. By (1), the algebra $B/(\alpha_i \restriction B)$ is subdirectly irreducible, hence $\alpha_i \restriction B \in \mathrm{M}(\operatorname{Con} B)$. Since $\operatorname{Con} B$ is a finite distributive lattice, there is the smallest $\gamma_i \in \operatorname{Con} B$ with $\gamma_i \nsubseteq (\alpha_i \restriction B)$. Let β_i be the congruence on A generated by γ_i. It is easy to see that (2) is satisfied. □

Theorem 2.7. *Let the algebra A belong to a finitely generated congruence distributive variety $\mathscr{V}$. Let T be a tree with a largest element u and let φ be a non-separable mapping $Z(T) \to \mathrm{M}(\operatorname{Con} A)$. Then T is $\mathrm{SI}(\mathscr{V})$-representable.*

Proof. We proceed by induction on $\operatorname{length}(T) = n$. Precisely, we claim that for every finite set $S \subseteq A$ there are $\alpha_q \in \mathrm{M}(\operatorname{Con} A)$ $(q \in T)$ and a chain of finite subalgebras $A_n \leq A_{n-1} \leq \dots \leq A_1 \leq A$ such that

(i) $\alpha_q = \varphi(q)$ for every $q \in Z(T)$;

(ii) $S \subseteq A_k$ for every k;

(iii) if $p \in T_k$, $p \leq q$, then $\alpha_q \restriction A_k \subseteq \alpha_p \restriction A_k$ (for every k);

(iv) if $p, q \in T_k$, $p \neq q$, then $\alpha_p \restriction A_k \neq \alpha_q \restriction A_k$ (for every k);

(v) for every $p \in T_k$, $k > 0$, the natural embedding $A_k/(\alpha_p \restriction A_k) \to A/\alpha_p$ is surjective (for every k).

The claim is clearly true if $\operatorname{length}(T) = 0$. Suppose now that $n = \operatorname{length}(T) > 0$. Let $Z(T) = \{p_1, \dots, p_m\}$. Denote $\alpha_{p_i} = \varphi(p_i)$. By 2.6 there is a finite subalgebra $A_n \leq A$ and $\beta_1, \dots, \beta_m \in \operatorname{Con} A$ such that $S \subseteq A_n$ and

(1) for every $a \in A$, there is $b \in A_n$ with $(a,b) \in \alpha_{p_i}$ for every i;

(2) $\beta_i \restriction A_n \nsubseteq \alpha_{p_i} \restriction A_n$ for every i and, for every $\beta \in \operatorname{Con} A$, either $\beta_i \subseteq \beta$ or $\beta \restriction A_n \subseteq \alpha_{p_i}$.

By the non-separability assumption, there exists a non-separable $\varphi^- \colon Z(T^-) \to \mathrm{M}(\operatorname{Con} A)$ such that $\beta_i \nsubseteq \varphi^-(s)$ whenever $p_i \prec s$. Hence, $\varphi^-(s) \restriction A_n \subseteq \alpha_{p_i} \restriction A_n$. By the induction hypothesis, there are $\alpha_s \in \mathrm{M}(\operatorname{Con} A)$ $(s \in T^-)$ and a chain $A_{n-1} \leq \dots \leq A_1 \leq A$ such that (i)–(v) hold for $k = 1, \dots, n-1$ with A_n playing the role of S. We need to show (i)–(v) for $k = n$. Now, (i) and (ii) are clear from the construction.

For $p_i \in T_n = Z(T)$, $p < q$ we have $p_i \prec s \leq q$ for some $s \in T_{n-1}$. If $s < q$, then $A_n \subseteq A_{n-1}$ and $\alpha_q \restriction A_{n-1} \subseteq \alpha_s \restriction A_{n-1}$ imply that $\alpha_q \restriction A_n \subseteq \alpha_s \restriction A_n$. (If $s = q$ then this is trivial.) Hence, $\alpha_q \restriction A_n \subseteq \alpha_s \restriction A_n = \varphi^-(s) \restriction A_n \subseteq \alpha_{p_i} \restriction A_n$, so (iii) holds.

Further, let $p_i, p_j \in T_n$ be different. Since φ is injective, we have $\alpha_{p_i} \neq \alpha_{p_j}$, so we can assume that there is $(x, y) \in \alpha_{p_i} \setminus \alpha_{p_j}$. By (1), there are $b, c \in A_n$ with $(x, b) \in \bigcap\limits_{p \in Z(T)} \alpha_p$, $(y, c) \in \bigcap\limits_{p \in Z(T)} \alpha_p$. Then $(b, c) \in (\alpha_{p_i} \restriction A_n) \setminus (\alpha_{p_j} \restriction A_n)$, which shows (iv).

It remains to show (v). However, this follows easily from (1).

Thus, we have the congruences α_q and the subalgebras A_k with the required properties. Now we can construct the $\mathrm{SI}(\mathscr{V})$-representation for the tree T. We set $B_0 = A/\alpha_u$, $B_k = A_k/(\alpha_u \restriction A_k)$ $(k = 1, \ldots, n)$ and $\gamma_q = (\alpha_q \restriction A_k)/(\alpha_u \restriction A_k)$. (That is, $([x]_{\alpha_u}, [y]_{\alpha_u}) \in \gamma_q$ iff $(x, y) \in \alpha_q \restriction A_k$.) By the isomorphism theorem, B_k/γ_q is isomorphic to $A_k/(\alpha_q \restriction A_k)$, which by (v) is isomorphic to $A/\alpha_q \in \mathrm{SI}(\mathscr{V})$, hence $\gamma_q \in \mathrm{M}(\mathrm{Con}\, B_k)$. Clearly, $\gamma_u = 0_{B_0}$ and it is easy to see that (iii) and (iv) imply 2.5(1),(2). □

The converse to 2.7 is true for infinite free algebras. Let $F_{\mathscr{V}}(X)$ denote the free algebra in $\mathscr{V}$ with X as the set of free generators.

Theorem 2.8. *Let $\mathscr{V}$ be a finitely generated congruence distributive variety. Let T be a $\mathrm{SI}(\mathscr{V})$-representable tree. Then there exists a non-separable mapping $\varphi\colon Z(T) \to \mathrm{M}(\mathrm{Con}\, F)$, where $F = F_{\mathscr{V}}(X)$, $|X| \geq \aleph_0$.*

Proof. Suppose that we have B_k and γ_q $(k \in \{0, \ldots, n\}$, $q \in T)$ satisfying 2.5.

Every surjective homomorphism $f\colon F \to B_n$ induces surjective homomorphisms $f_p\colon F \to B_n/\gamma_p$ $(p \in Z(T))$ given by $f_p(x) = [f(x)]_{\gamma_p}$. Since B_n/γ_p is subdirectly irreducible, we have $\mathrm{Ker}(f_p) \in M(\mathrm{Con}\, F)$, so we can define $\varphi(p) = \mathrm{Ker}(f_p)$. We claim that φ is non-separable for every such f. We proceed by induction on the length of T. The claim is certainly true for the length 0. Suppose now that $\mathrm{length}(T) > 0$, $Z(T) = \{p_1, \ldots, p_m\}$ and that $\beta_i \in \mathrm{M}(\mathrm{Con}\, F)$, $\beta_i \nsubseteq \varphi(p_i)$ $(i = 1, \ldots, m)$. We need to find a non-separable mapping $\varphi^-\colon Z(T^-) \to \mathrm{M}(\mathrm{Con}\, F)$ with $\beta_i \nsubseteq \varphi^-(q)$ whenever $p_i \prec q$. Let us write γ_i instead of γ_{p_i}.

For every i there exists $(x_i, y_i) \in \beta_i \setminus \varphi(p_i)$. There is a finite set $Y \subseteq X$ such that all x_i and y_i belong to $\langle Y \rangle$ (the subalgebra of F generated by Y). Since B_{n-1} is finite and $B_n \subseteq B_{n-1}$, it is possible to choose a surjective map $g_0\colon X \to B_{n-1}$ such that $g_0(y) = f(y)$, for every $y \in Y$. Since F is free, this map can be extended to a (surjective) homomorphism $g\colon F \to B_{n-1}$. By our induction hypothesis, the mapping φ^- defined by $\varphi^-(q) = \mathrm{Ker}(g_q)$ for every $q \in Z(T^-)$ is non-separable. The injectivity of φ^- follows from the fact that $\gamma_q \neq \gamma_r$ whenever $q, r \in Z(T^-)$, $q \neq r$. Moreover, for every $i = 1, \ldots, m$ we have $(x_i, y_i) \notin \varphi(p_i) = \operatorname{Ker}(f_{p_i})$, hence $f_p(x_i) \neq f_p(y_i)$, therefore $[g(x_i)]_{\gamma_i} = [f(x_i)]_{\gamma_i} \neq [f(y_i)]_{\gamma_i} = [g(y_i)]_{\gamma_i}$. Since $\gamma_q \restriction B_n \subseteq \gamma_i$, we have $[g(x_i)]_{\gamma_q} \neq [g(y_i)]_{\gamma_q}$, hence $(x_i, y_i) \notin \operatorname{Ker}(g_q) = \varphi^-(q)$ and therefore $\beta_i \nsubseteq \varphi^-(q)$. $\square$

Now we formulate a simple consequence of the Definition 2.5, which is often useful when one tries to prove that a particular tree is not $\mathrm{SI}(\mathscr{V})$-representable. (An example will be given in the next section.)

Theorem 2.9. *Let a tree T be $\mathrm{SI}(\mathscr{V})$-representable for a finitely generated congruence distributive variety $\mathscr{V}$. Then there are algebras $A_q \in \mathrm{SI}(\mathscr{V})$ $(q \in T)$, their subalgebras $D_q \leq A_q$ $(q \in T \setminus Z(T))$ and congruences $\delta_{qp} \in \operatorname{Con} D_q$ $(p \prec q)$ such that*

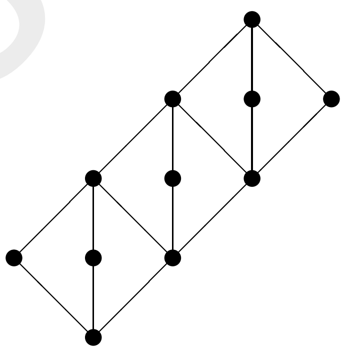
(1) *D_q/δ_{qp} is isomorphic to A_p;*

(2) *if $p_1, p_2 \in q^-$, $p_1 \neq p_2$, then $\delta_{qp_1} \neq \delta_{qp_2}$.*

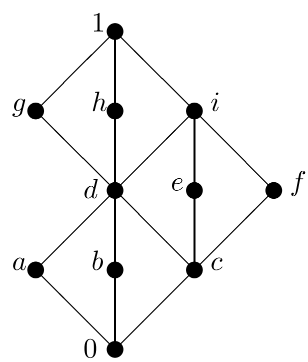
P r o o f. Let B_k and γ_q satisfy the conditions of 2.5. For $q \in T_k$, $p \prec q$, we set $A_q = B_k/\gamma_q$, $D_q = B_{k+1}/(\gamma_q \restriction B_{k+1})$ and $\delta_{qp} = \gamma_p/(\gamma_q \restriction B_{k+1})$. It is easy to see that (1) and (2) are satisfied. $\square$

3. Examples

The results in the previous section provide a generalization of the concept of separable sets developed in [6]. Now we will apply the general results to two concrete varieties of lattices denoted here by $\mathscr{A}$ and $\mathscr{B}$ and show that $\operatorname{Con}(\mathscr{A}) \neq \operatorname{Con}(\mathscr{B})$. The varieties $\mathscr{A}$ and $\mathscr{B}$ are generated by the lattices A and B respectively, depicted below. Our source of information about these (and other) varieties is [2] and [1, App. F].

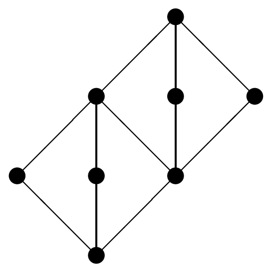


A

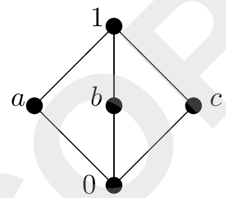


B

Both $\mathscr{A}$ and $\mathscr{B}$ cover the variety $\mathscr{M}_{3,3}$ generated by the lattice $M_{3,3}$. Hence, subdirectly irreducible members of $\mathscr{A}$ are (up to isomorphism) A, $M_{3,3}$, M_3 and $C_2 = \{0,1\}$ (the 2−element chain). For $\mathscr{B}$ the list consists of B, $M_{3,3}$, M_3 and C_2.



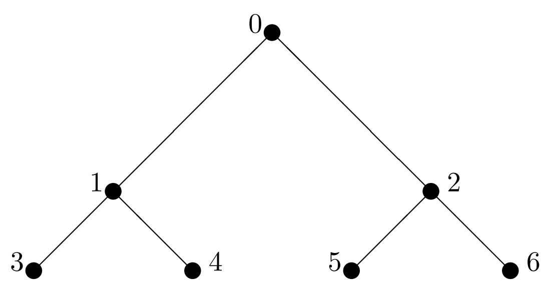
$M_{3,3}$



M_3

Both $\mathscr{A}$ and $\mathscr{B}$ are finitely generated subvarieties of the variety of modular lattices. The results in [6] give the same information for them, namely that $L \in \mathrm{Con}(\mathscr{A})$ (or $L \in \mathrm{Con}(\mathscr{B})$) can contain a 4-element non-separable set, but not a 5-element one. In our terminology: There exists a non-separable mapping $T_4 \to \mathrm{M}(\mathrm{Con}\, F)$, where T_4 is a tree of length 1 with 4 minimal elements and F is an infinite free algebra in $\mathscr{A}$ (or $\mathscr{B}$); every mapping $T_5 \to \mathrm{M}(L)$ (where T_5 is the tree of length 1 with 5 minimal elements and $L \in \mathrm{Con}(\mathscr{A}) \cup \mathrm{Con}(\mathscr{B})$) is separable.

Hence, the mappings from trees of the length 1 cannot distinguish the classes $\mathrm{Con}(\mathscr{A})$ and $\mathrm{Con}(\mathscr{B})$. Now we will show that the following tree T makes a difference.



T

Theorem 3.1. *The tree T is $\mathrm{SI}(\mathscr{B})$-representable, but not $\mathrm{SI}(\mathscr{A})$-representable.*

Proof. Set $B_0 = B$, $B_1 := B \setminus \{e, f\}$, $B_2 = \{0, b, d, h, 1\}$,

$$\begin{aligned} \gamma_1 &= (0abcd)(g)(h)(i)(1), & \gamma_4 &= (0bd)(h1), \\ \gamma_2 &= (0)(a)(b)(c)(dghi1), & \gamma_5 &= (0b)(dh1), \\ \gamma_3 &= (0bdh)(1), & \gamma_6 &= (0)(bdh1). \end{aligned}$$

(The congruences above are given by their equivalence classes.) It is not difficult to check that all conditions of 2.5 are satisfied.

Notice that B_1/γ_1 and B_1/γ_2 are both isomorphic to M_3. In the case of the variety $\mathscr{A}$ the essential difference is that A does not have a subalgebra with two homomorphisms onto M_3 with different kernels. To show that T is not $\mathrm{SI}(\mathscr{A})$-representable, we use 2.9. For contradiction, suppose that we have A_q, D_q and δ_{qp} satisfying the conditions of 2.9.

The subdirectly irreducible algebras A_1 and A_2 have subalgebras with two different meet-irreducible congruences, so they must be M_3 or $M_{3,3}$ or A. Consequently, $A_0 \in \mathrm{SI}(\mathscr{V})$ has a subalgebra D_0 with two different $\delta_{01}, \delta_{02} \in \operatorname{Con} D_0$ such that both D_0/δ_{01} and D_0/δ_{02} are isomorphic to M_3, $M_{3,3}$ or A, and it is possible to check that there is no such A_0 in $\mathscr{A}$. $\square$

As a consequence we obtain that $\mathrm{Con}(\mathscr{A}) \neq \mathrm{Con}(\mathscr{B})$. More precisely, the lattice $\operatorname{Con} F$ (with F being an infinite free algebra in B) is not representable in $\mathscr{A}$. Since both $\mathscr{A}$ and $\mathscr{B}$ are varieties of modular lattices, this solves [6, Problem 5.4]. Actually, this problem is now replaced by the following one.

Problem 3.2. *Let $\mathscr{V}$ and $\mathscr{W}$ be finitely generated modular lattice varieties such that every $\mathrm{SI}(\mathscr{V})$-representable tree is $\mathrm{SI}(\mathscr{W})$-representable and vice versa. Do then $\mathrm{Con}(\mathscr{V})$ and $\mathrm{Con}(\mathscr{W})$ contain the same lattices with countably many compact elements?*

The cardinality restriction in the above problem is essential. Without the restriction on the cardinality, the varieties $\mathscr{M}_n$ for different $n \geq 3$ provide a counterexample. (The variety $\mathscr{M}_n$ is generated by the $(n+2)$-element lattice M_n of length 2.) It is not difficult to prove that (independently of n) a tree T is $\mathrm{SI}(\mathscr{M}_n)$-representable iff $|q^-| \leq 2$ for every $q \in T$ and $|q^-| = 2$ for at most one $q \in T$, and it was proved in [5] that $\mathrm{Con}(\mathscr{M}_n) \neq \mathrm{Con}(\mathscr{M}_k)$ whenever $n \neq k$. (The example showing this inequality has the cardinality $\aleph_2$, so the above problem for the cardinality $\aleph_1$ is also open.)

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