Congruence-preserving functions on distributive lattices

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Dedicated to G. Grätzer and E.T. Schmidt on the occasion of their 70th birthday

Abstract. This paper is a continuation of the research motivated by G. Grätzer's study of affine completeness for Boolean algebras and distributive lattices from 1962 and 1964, respectively and by the 1995 work of G. Grätzer and E.T. Schmidt on unary isotone congruence-preserving functions of distributive lattices. We present a complete list of generators for the clone $C(\mathbf{L})$ of all congruence-preserving functions of any distributive lattice $\mathbf{L}$. We introduce a general problem of finding a nice generating set for the clone $C(\mathbf{A})$ of all congruence-preserving functions of a given algebra $\mathbf{A}$.

1. Introduction

A finitary function $f:\ A^n \to A$ on an algebra $\mathbf{A}$ is called *congruence-preserving* (or *compatible*) if, for any congruence θ of $\mathbf{A}$, $(a_i, b_i) \in \theta$, $i = 1, \dots, n$, implies that

$$(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \in \theta.$$

Typical compatible functions are polynomial functions. A *polynomial function* of an algebra $\mathbf{A}$ is a function that can be obtained by composition of the basic operations of $\mathbf{A}$, the projections and the constant functions. As all of these are congruence-preserving, polynomial functions of $\mathbf{A}$ are congruence-preserving too.

There are algebras, on which polynomial functions are the only compatible functions. Such algebras are called *affine complete*. Hence one can imagine affine complete algebras as algebras having 'many' congruences. The problem of characterizing algebras which are affine complete was first formulated in G. Grätzer's monograph [6]. As every algebra is a reduct of some affine complete algebra (it suffices to add to the basic operations of a given algebra all its compatible functions to make it affine complete), in [1] the problem was restricted into the following formulation: *characterize affine complete algebras in your favourite variety.* Many varieties for which the problem had already been solved are mentioned in [1], [8] and [9]. An important source of inspiration for the present paper is G. Grätzer's solution for Boolean algebras [4] and bounded distributive lattices [5]. The intensive study of various kinds of polynomial completeness from the 1960s to the 1990s had

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resulted in the monograph by K. Kaarli and A.F. Pixley [9] of which a significant part is devoted to affine completeness.

Our aim in this paper is more general than the study of affine completeness. We believe that the concept of compatibility is worth investigating also in algebras that are not affine complete, because it it so closely connected with the fundamental algebraic notions of a congruence and a polynomial. Therefore, we would like to initiate a project of describing the compatible functions of an algebra in a given (favourite) variety no matter whether the algebra is affine complete or not. Since compatible functions form a clone, our wish is to express every compatible function as a composition of functions from some nice and well understood family. In other words, we would like to find a nice generating set for the clone of all compatible functions. (We recall that a *clone* on a set A is a set of finitary functions containing all projections and closed under composition.)

Problem 1. *Given an algebra* $\mathbf{A}$*, find a nice generating set for the clone* $C(\mathbf{A})$ *of all compatible functions of* $\mathbf{A}$*.*

A description of this kind is obvious for the clone of all polynomial functions, where the generating set consists of the basic operations and constants. Thus the answer is known if the algebra $\mathbf{A}$ is affine complete. If $\mathbf{A}$ is not affine complete, we seek for an extension of this generating set by some typical compatible non-polynomial functions.

For an algebra $\mathbf{A}$, a function $f: A^n \to A$ is called a *local polynomial* of $\mathbf{A}$ if for every finite set $S \subseteq A^n$ there is a polynomial function $p: A^n \to A$ of $\mathbf{A}$ which coincides with f on S. It is clear that local polynomials form a clone, which is a subclone of $C(\mathbf{A})$. Finding generators for this clone could be an important step towards the solution of the above problem.

Problem 2. *Given an algebra* $\mathbf{A}$*, find a nice generating set for the clone* $LP(\mathbf{A})$ *of all local polynomial functions of* $\mathbf{A}$*.*

The effort to describe compatible functions on algebras of some special kind is not completely new. The problem was solved for Boolean algebras in [4] (by showing that every Boolean algebra is affine complete). A description of compatible functions on bounded distributive lattices is contained implicitly in the initial papers [4] and [5] of G. Grätzer and explicitly in a more recent paper [3] of J. D. Farley. (However, the latter description, see Lemma 2.2 below, is not of the kind we propose to achieve.)

The local polynomial functions on distributive lattices have been characterized by D. Dorninger and G. Eigenthaler ([2]) as exactly the isotone compatible functions. The first attempt to describe the generators of the clone $LP(\mathbf{A})$ for distributive lattices was made in [7] by G. Grätzer and E.T. Schmidt. This paper came with the idea that local polynomial fuctions might be described as polynomials in a suitable extension of a given algebra. This idea has been further developed in [11] for the cases of Stone and Kleene algebras. In our present paper it is the key to the solution of the above Problems for all (in general unbounded) distributive lattices.

2. The bounded case

Let $\mathbf{L} = \langle L; \vee, \wedge, 0, 1\rangle$ be a bounded distributive lattice and let $f : L^n \to L$ be a compatible function. By *the characteristic function* of f we mean its restriction to $\{0,1\}^n$. We start by listing results on compatible functions of bounded distributive lattices published in [5].

We recall that a *Boolean interval* $[a, b]$ in $\mathbf{L}$ means that the bounded lattice $([a, b]; \vee, \wedge, a, b)$ is Boolean; that is, for every element $c \in [a, b]$ there exists its complement $c' \in [a, b]$ such that $c \vee c' = b$ and $c \wedge c' = a$. An interval $[a, b]$ is said to be *non-trivial* if $a < b$. A special case of a non-trivial Boolean interval is a *covering pair* $a < b$ of elements of $\mathbf{L}$ meaning that there is no element c in $\mathbf{L}$ with $a < c < b$.

Lemma 2.1. ([5], Theorem and Corollaries 2 and 3) *For a bounded distributive lattice $\mathbf{L} = \langle L; \vee, \wedge, 0, 1\rangle$, the following hold:*

(i) *The characteristic function φ of a compatible function $f : L^n \to L$ determines f uniquely.*

(ii) *A function $\varphi : \{0,1\}^n \to L$ is the characteristic function of a compatible function $f : L^n \to L$ if and only if for every $a < b$ in $\{0,1\}^n$, the interval $[\varphi(b), \varphi(a) \vee \varphi(b)]$ of $\mathbf{L}$ is Boolean.*

(iii) *A compatible function f is a polynomial of $\mathbf{L}$ if and only if its characteristic function φ is isotone.*

(iv) *The lattice $\mathbf{L}$ is affine complete if and only if $\mathbf{L}$ does not contain a non-trivial Boolean interval.*

The following description of compatible functions comes from [3]. (It can also be deduced from [4] and [5].) It uses the fact that every bounded distributive lattice $\mathbf{L}$ can be canonically embedded into a Boolean algebra (using a set-theoretical representation of $\mathbf{L}$), the complement operation $'$ refers to this Boolean algebra. Here and in the sequel we write elements of $\{0,1\}^n$ in the form $a = (a_1, \dots, a_n)$.

Lemma 2.2. ([3], Theorem 4.7) *Let $f : L^n \to L$ be a compatible function on a bounded distributive lattice $\mathbf{L}$. Then*

$$f(\overline{x}) = \bigvee_{a \in \{0,1\}^n} (f(a) \wedge \bigwedge_{a_i = 1} x_i \wedge \bigwedge_{a_i = 0} x_i')$$

for every $\overline{x} = (x_1, \dots, x_n) \in L^n$.

Of course, the complement x' for $x \in L$ need not belong to L, so the above formula only makes sense if we consider $\mathbf{L}$ embedded into a Boolean algebra. In this paper we present another expression for compatible functions, in which we only use functions defined on $\mathbf{L}$ itself.

Our goal is to find a reasonably simple set S of functions on $\mathbf{L}$ such that every compatible function of $\mathbf{L}$ can be obtained via composition from the finitary projections and the set S. Good candidates for the generating set have been discovered in [5]. If $[a, b]$ is a Boolean interval in a lattice $\mathbf{L}$, then the unary function

$$c_{[a,b]}(x) = ((x \wedge b) \vee a)',$$

where $'$ denotes the complementation inside the Boolean lattice $[a, b]$, is compatible. If $a \neq b$, then the function is not isotone and hence not polynomial. Now we will prove that these functions, together with the polynomials, generate the clone $C(\mathbf{L})$ of all compatible functions of $\mathbf{L}$.

Let $g : L^n \to L$ be a compatible function on a bounded distributive lattice $\mathbf{L}$. Every n-tuple $a \in \{0,1\}^n$ can be considered as an element of L^n. For every $a, b \in \{0,1\}^n$ with $a \leq b$ the interval $J_{ab} = [g(a) \wedge g(b), g(a)]$ is Boolean by a result analogous to Lemma 2.1(ii), so we can consider the unary function

$$c_{J_{ab}}(x) = ((x \wedge g(a)) \vee (g(a) \wedge g(b)))'$$

(in the formula above, $'$ means the complementation inside J_{ab}). We note that $c_{J_{ab}}(0) = g(a)$ and $c_{J_{ab}}(1) = g(a) \wedge g(b)$.

The following result gives a canonical representation of compatible functions on bounded distributive lattices. (We sometimes substitute the notation $a_i = 1$ with $i \in a^{-1}(1)$ and consider, as usual, $\bigvee_{\varnothing} x_i = 0$ and $\bigwedge_{\varnothing} x_i = 1$.)

Theorem 2.3. *Let $g : L^n \to L$ be a compatible function on a bounded distributive lattice $\mathbf{L}$. Then, for every $\overline{x} = (x_1, \ldots, x_n) \in L^n$,*

$$g(\overline{x}) = \bigvee_{a \in \{0,1\}^n} C_a(\overline{x}),$$

where

$$C_a(\overline{x}) = \bigwedge_{b \in \{0,1\}^n, b \geq a} K_{ab}(\overline{x})$$

and

$$K_{ab}(\overline{x}) = (c_{J_{ab}}(\bigvee_{i \in b^{-1}(1) \cap a^{-1}(0)} x_i) \wedge \bigwedge_{i \in a^{-1}(1)} x_i) \vee \bigvee_{i \in b^{-1}(0)} x_i.$$

Proof. It suffices to prove the equality for $\overline{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$ according to Lemma 2.1(i). Let us consider the conjunct $C_a(\overline{x})$ for an arbitrarily fixed n-tuple $a \in \{0,1\}^n$. We distinguish the following three cases:

1. If $a \not\leq \overline{x}$ then there is $i \in \{1, \ldots, n\}$ such that $a_i = 1,\ x_i = 0$, so we obtain $K_{a,1}(\overline{x}) = 0$, whence $C_a(\overline{x}) = 0$.

2. If $a = \overline{x}$ then for every $b \geq a$,

$$K_{a,b}(\overline{x}) = (c_{J_{ab}}(0) \wedge 1) \vee 0 = g(a).$$

Hence $C_a(\overline{x}) = g(a) = g(\overline{x})$.

3. If $a < \overline{x}$ then there is $i \in \{1, \ldots, n\}$ such that $a_i = 0,\ x_i = 1$, so

$$C_a(\overline{x}) \leq K_{a,\overline{x}}(\overline{x}) = (c_{J_{a\overline{x}}}(1) \wedge 1) \vee 0 = g(a) \wedge g(\overline{x}) \leq g(\overline{x}).$$

Now we easily conclude

$$\bigvee_{a \in \{0,1\}^n} C_a(\overline{x}) = g(\overline{x})$$

as required. □

As a consequence we obtain:

Corollary 2.4. *Every compatible function on a bounded distributive lattice is a composition of polynomials and complementations on Boolean intervals.*

G. Grätzer's characterization of affine complete bounded distributive lattices (see Lemma 2.1(iv)) now immediately follows from our result. If there are no non-trivial Boolean intervals, then every interval complementation is a constant function, so every compatible function is a polynomial. If there is a non-trivial Boolean inteval $[a, b]$, then $c_{[a,b]}$ is a compatible non-polynomial function.

3. The extension theorem

From this section the concepts of almost principal ideals and filters will be important in our investigations. Let us recall that an ideal I of a lattice $\mathbf{L}$ is said to be *principal* if it is of the form $\downarrow u = \{x \in L \,|\, x \leq u\}$, for some $u \in L$. It is said that I is *almost principal* if its intersection with every principal ideal of $\mathbf{L}$ is a principal ideal of $\mathbf{L}$. If $\mathbf{L}$ has a largest element, then every almost principal ideal is principal. In general, there are almost principal ideals which are not principal (see [9] or [12].) The notions of principal and almost principal filter are defined dually. The whole lattice $\mathbf{L}$ is also regarded as an (almost principal) ideal and filter.

The almost principal ideals have been first considered in the context of semilattices in [10]. Their relevance for affine completeness of distributive lattices and Stone algebras has been established in [12] and [8], respectively.

Let $\mathcal{I}(\mathbf{L})$ and $\mathcal{F}(\mathbf{L})$ denote the sets of all almost principal ideals and almost principal filters of the lattice $\mathbf{L}$, respectively.

For every $x \in L$, $I \in \mathcal{I}(\mathbf{L})$ and $F \in \mathcal{F}(\mathbf{L})$, we denote $x_I = \max(I \cap \downarrow x)$ and $x^F = \min(F \cap \uparrow x)$. This defines unary functions f_I and f^F on L, given by $f_I(x) = x_I$, $f^F(x) = x^F$. These functions are called *projections on almost principal ideals (filters).*

The following result comes from [12]:

Lemma 3.1. *For every distributive lattice* $\mathbf{L}$ *and every* $I \in \mathcal{I}(\mathbf{L})$*,* $F \in \mathcal{F}(\mathbf{L})$*, the functions* f_I *and* f^F *are compatible. A distributive lattice* $\mathbf{L}$ *is affine complete if and only if the following conditions hold:*

- (i) $\mathbf{L}$ *does not contain a non-trivial Boolean interval;*
- (ii) *every proper almost principal ideal of* $\mathbf{L}$ *is principal;*
- (iii) *every proper almost principal filter of* $\mathbf{L}$ *is principal.*

Hence we have two new types of compatible functions. As observed in [7] and [13], these functions can be interpolated by polynomials in some extensions of $\mathbf{L}$, which we now recall.

For every distributive lattice $\mathbf{L}$, the set $\mathcal{I}(\mathbf{L})$ ordered by the set inclusion is again a distributive lattice. In fact, it is a sublattice of the lattice $I(\mathbf{L})$ of all ideals of $\mathbf{L}$ (see [13]). There is a natural embedding $\mathbf{L} \to \mathcal{I}(\mathbf{L})$ given by $x \mapsto \downarrow x$. Identifying x with $\downarrow x$ we will assume that $\mathbf{L}$ is a sublattice of $\mathcal{I}(\mathbf{L})$.

Dually, we can consider $\mathbf{L}$ as a sublattice of the distributive lattice $\mathcal{F}(\mathbf{L})$. Note that the natural ordering of $\mathcal{F}(\mathbf{L})$ is given by the inverse set inclusion: $F_1 \leq F_2$ iff $F_1 \supseteq F_2$.

Finally, we can consider $\mathbf{L}$ as a sublattice of the distributive lattice $\mathcal{F}(\mathcal{I}(\mathbf{L}))$.

Lemma 3.2. *For every distributive lattice* $\mathbf{L}$ *the following conditions holds.*

(i) $\mathbf{L}$ *is an ideal of* $\mathcal{I}(\mathbf{L})$ *and a filter of* $\mathcal{F}(\mathbf{L})$.
(ii) $\mathbf{L}$ *is convex in* $\mathcal{F}(\mathcal{I}(\mathbf{L}))$.
(iii) *There is a canonical isomorphism* $\mathcal{F}(\mathcal{I}(\mathbf{L})) \to \mathcal{I}(\mathcal{F}(\mathbf{L}))$*, which preserves* $\mathbf{L}$.

Proof. Part (i) is easy to check (or we refer to [13] for its proof). Part (ii) follows from the fact that ideals and filters are convex, so that $\mathbf{L}$ is convex in $\mathcal{I}(\mathbf{L})$ and $\mathcal{I}(\mathbf{L})$ is convex in $\mathcal{F}(\mathcal{I}(\mathbf{L}))$. The canonical isomorphism $\varphi:\ \mathcal{F}(\mathcal{I}(\mathbf{L})) \to \mathcal{I}(\mathcal{F}(\mathbf{L}))$ is given by

$$\varphi(G) = \{F \in \mathcal{F}(\mathbf{L}) \mid F \cap J \neq \emptyset \text{ for every } J \in G\}.$$

It was proved in [13] that φ is an isomorphism. And it is easy to check that φ preserves $\mathbf{L}$. Indeed, if $x \in L$ and $G = \{I \in \mathcal{I}(\mathbf{L}) \mid I \supseteq {\downarrow}x\}$ is the element of $\mathcal{F}(\mathcal{I}(\mathbf{L}))$ representing x, then $\varphi(G) = \{F \in \mathcal{F}(\mathbf{L}) \mid F \leq {\uparrow}x\}$, which is the element of $\mathcal{I}(\mathcal{F}(\mathbf{L}))$ representing x. This completes the proof. $\square$

Hence, $\mathcal{I}(\mathbf{L})$ can be regarded as the filter of $\mathcal{F}(\mathcal{I}(\mathbf{L}))$ generated by $\mathbf{L}$. Similarly, $\mathcal{F}(\mathbf{L})$ can be considered as the ideal of $\mathcal{I}(\mathcal{F}(\mathbf{L}))$. Since the canonical isomorphism φ preserves $\mathbf{L}$, we can also assume that (up to isomorphism) $\mathcal{I}(\mathbf{L})$ is the filter of $\mathcal{I}(\mathcal{F}(\mathbf{L}))$ generated by $\mathbf{L}$ and $\mathcal{F}(\mathbf{L})$ is the ideal of $\mathcal{F}(\mathcal{I}(\mathbf{L}))$ generated by $\mathbf{L}$.

For every prime ideal P on a distributive lattice $\mathbf{L}$ let θ_P denote the congruence on $\mathbf{L}$ with the equivalence classes P and $L \setminus P$, that is, $\theta_P = P^2 \cup (L \setminus P)^2$.

Lemma 3.3. *Let* P *be a prime ideal on a distributive lattice* $\mathbf{L}$. *Then* $(x_I, y_I) \in \theta_P$ *for every* $x, y \in L \setminus P$ *and* $I \in \mathcal{I}(\mathbf{L})$.

Proof. The statement follows from the fact that θ_P is a congruence and f_I is compatible. $\square$

Lemma 3.4. *Let* $f:\ L^n \to L$ *be a compatible function on a distributive lattice* $\mathbf{L}$. *Then, for every* $I_1, \dots, I_n \in \mathcal{I}(\mathbf{L})$*, the set* $J = \{x \in L \mid x \leq f(x_{I_1}, \dots, x_{I_n})\}$ *is an almost principal ideal of* $\mathbf{L}$ *and, for every* $y \in L$, $y_J = y \wedge f(y_{I_1}, \dots, y_{I_n})$.

Proof. Let $x \in J$ and $y \leq x$. We claim that $y \in J$. Suppose, for contradiction, that $y \not\leq f(y_{I_1}, \dots, y_{I_n})$. Then there exists a prime ideal P of $\mathbf{L}$ such that $y \notin P$ and $f(y_{I_1}, \dots, y_{I_n}) \in P$, whence $x \notin P$ as $y \leq x$. By Lemma 3.3, $(x_{I_i}, y_{I_i}) \in \theta_P$ for every i. As by assumption $f(x_{I_1}, \dots, x_{I_n}) \geq x$, we get $f(x_{I_1}, \dots, x_{I_n}) \notin P$, which implies that $(f(x_{I_1}, \dots, x_{I_n}), f(y_{I_1}, \dots, y_{I_n})) \notin \theta_P$, and contradicts the compatibility of f.

Now let $y \in L$. We claim that $y \wedge f(y_{I_1}, \dots, y_{I_n}) = \max J \cap {\downarrow}y$. Certainly, $z := y \wedge f(y_{I_1}, \dots, y_{I_n}) \leq y$. Let us suppose that $z \notin J$, i.e. $z \not\leq f(z_{I_1}, \dots, z_{I_n})$. Again, there is a prime ideal Q of $\mathbf{L}$ such that $f(z_{I_1}, \dots, z_{I_n}) \in Q$ and $z \notin Q$. Since $y \geq z$, we have $y \notin Q$ and similarly $f(y_{I_1}, \dots, y_{I_n}) \notin Q$. By Lemma 3.3, $(y_{I_i}, z_{I_i}) \in \theta_Q$ for every $i = 1, \dots, n$ and $(f(y_{I_1}, \dots, y_{I_n}), f(z_{I_1}, \dots, z_{I_n})) \notin \theta_Q$, which contradicts the compatibility of f.

We have shown $z \in J \cap {\downarrow}y$. To prove the maximality, let $t \in J \cap {\downarrow}y$. Suppose that $t \not\leq z$. Then $t \notin R$, $z \in R$ for some prime ideal R. From $t \notin R$ we obtain that $y \notin R$ and $t \leq f(t_{I_1}, \dots, t_{I_n}) \notin R$. Since $z = y \wedge f(y_{I_1}, \dots, y_{I_n}) \in R$, the primality of R yields that $f(y_{I_1}, \dots, y_{I_n}) \in R$. By Lemma 3.3, $(y_{I_i}, t_{I_i}) \in \theta_R$ for every $i = 1, \dots, n$, which contradicts the compatibility of f.

Thus, $\max(J \cap \downarrow y)$ exists for every $y \in L$. It remains to prove that J is closed under joins: if $x, y \in J$, then $\max(J \cap \downarrow(x \vee y)) \geq x, y$, so $\max(J \cap \downarrow(x \vee y))$ must be equal to $x \vee y$. □

For every compatible function $f : L^n \to L$ on a distributive lattice $\mathbf{L}$, let $\overline{f} : \mathcal{I}(\mathbf{L})^n \to \mathcal{I}(\mathbf{L})$ be defined by

$$\overline{f}(I_1, \dots, I_n) = \{x \in L \mid x \leq f(x_{I_1}, \dots, x_{I_n})\}.$$

We have just proved the correctness of this definition.

Lemma 3.5. *For every compatible function f on a distributive lattice $\mathbf{L}$, the function $\overline{f}$ is compatible on $\mathcal{I}(\mathbf{L})$.*

Proof. For contradiction, suppose that $\overline{f}$ is not compatible. Then there exists a prime ideal P on $\mathcal{I}(\mathbf{L})$ and ideals $I_1, \dots, I_n, J_1, \dots, J_n \in \mathcal{I}(\mathbf{L})$ such that $(I_i, J_i) \in \theta_P$ for every i and, without loss of generality, $\overline{f}(I_1, \dots, I_n) \in P$ and $\overline{f}(J_1, \dots, J_n) \notin P$. We denote $M = \{i \in \{1, \dots, n\} \mid I_i \in P\}$. (As $(I_i, J_i) \in \theta_P$, we note that $I_i \in P$ iff $J_i \in P$ for every i.) It follows

$$\overline{f}(J_1, \dots, J_n) \wedge \bigwedge_{i \notin M} I_i \wedge \bigwedge_{i \notin M} J_i \not\leq \overline{f}(I_1, \dots, I_n) \vee \bigvee_{i \in M} I_i \vee \bigvee_{i \in M} J_i.$$

(The right hand side belongs to P while the left hand side does not.) So, there exists $y \in L$ such that

$$y \in \overline{f}(J_1, \dots, J_n) \wedge \bigwedge_{i \notin M} I_i \wedge \bigwedge_{i \notin M} J_i; \tag{1}$$

$$y \notin \overline{f}(I_1, \dots, I_n) \vee \bigvee_{i \in M} I_i \vee \bigvee_{i \in M} J_i. \tag{2}$$

We obtain from (2) that

$$y \not\leq y_{\overline{f}(I_1, \dots, I_n)} \vee \bigvee_{i \in M} y_{I_i} \vee \bigvee_{i \in M} y_{J_i}. \tag{3}$$

By Lemma 3.4, $y_{\overline{f}(I_1, \dots, I_n)} = y \wedge f(y_{I_1}, \dots, y_{I_n})$, so (3) and the distributivity imply that

$$y \not\leq f(y_{I_1}, \dots, y_{I_n}) \vee \bigvee_{i \in M} y_{I_i} \vee \bigvee_{i \in M} y_{J_i}. \tag{4}$$

Thus, there exists a prime ideal Q on $\mathbf{L}$ such that $y \notin Q$, $f(y_{I_1}, \dots, y_{I_n}) \in Q$ and y_{I_i}, $y_{J_i} \in Q$ for every $i \in M$. Further, $y \in \overline{f}(J_1, \dots, J_n)$ obviously means that $y \leq f(y_{J_1}, \dots, y_{J_n})$, hence $f(y_{J_1}, \dots, y_{J_n}) \notin Q$.

For every $i \in M$ we have $(y_{I_i}, y_{J_i}) \in \theta_Q$. From (1) it follows that for every $i \notin M$ we have $y \in I_i \cap J_i$, hence $y_{I_i} = y = y_{J_i}$, so $(y_{I_i}, y_{J_i}) \in \theta_Q$ for every $i = 1, \dots, n$. Since $(f(y_{I_1}, \dots, y_{I_n}), f(y_{J_1}, \dots, y_{J_n}) \notin \theta_Q$, we obtain a contradiction with the compatibility of f. □

Lemma 3.6. *For every compatible function $f: L^n \to L$ on a distributive lattice* $\mathbf{L}$ *and every $x_1, \dots, x_n \in L$,*

$$\overline{f}(\downarrow x_1, \dots, \downarrow x_n) = \downarrow f(x_1, \dots, x_n).$$

Proof. By the definition,

$$\overline{f}(\downarrow x_1, \dots, \downarrow x_n) = \{x \in L \mid x \leq f(x \wedge x_1, \dots, x \wedge x_n)\}.$$

Suppose for contradiction that $x \leq f(x \wedge x_1, \dots, x \wedge x_n)$ and $x \not\leq f(x_1, \dots, x_n)$ for some $x \in L$. Then there exists a prime ideal P on $\mathbf{L}$ such that $x \notin P$ (hence also $f(x \wedge x_1, \dots, x \wedge x_n) \notin P$) and $f(x_1, \dots, x_n) \in P$. For every i we have $x_i \in P$ iff $x \wedge x_i \in P$, hence $(x_i, x \wedge x_i) \in \theta_P$, which contradicts the compatibility of f.

Thus, $\overline{f}(\downarrow x_1, \dots, \downarrow x_n) \subseteq \downarrow f(x_1, \dots, x_n)$. To show the inverse inclusion, assume for contradiction that $x \not\leq f(x \wedge x_1, \dots, x \wedge x_n)$ and $x \leq f(x_1, \dots, x_n)$ for some $x \in L$. Then, for some prime ideal P on $\mathbf{L}$, $f(x \wedge x_1, \dots, x \wedge x_n) \in P$ and $x \notin P$, hence $f(x_1, \dots, x_n) \notin P$. Again, $x_i \in P$ iff $x \wedge x_i \in P$, so $(x_i, x \wedge x_i) \in \theta_P$, a contradiction with the compatibility of f. □

If we identify $x \in L$ with $\downarrow x \in \mathcal{I}(\mathbf{L})$, then Lemma 3.6 says that $\overline{f}$ is an extension of f. Hence, every compatible function on $\mathbf{L}$ can be extended to a compatible function on $\mathcal{I}(\mathbf{L})$. (In [13] this was proved for isotone compatible functions.)

Of course, an analogous statement holds for the lattice $\mathcal{F}(\mathbf{L})$ of all almost principal filters on $\mathbf{L}$. By iterating these two constructions we obtain the following result.

Theorem 3.7. *Every compatible function on a distributive lattice* $\mathbf{L}$ *can be extended to a compatible function on* $\mathcal{F}(\mathcal{I}(\mathbf{L}))$.

Since $\mathcal{F}(\mathcal{I}(\mathbf{L}))$ is a bounded distributive lattice, Theorem 3.7 allows us to use Theorem 2.3 for the unbounded case, which will be done in Section 5.

4. Classification of compatible functions

The first four types of compatible functions on distributive lattices $\mathbf{L}$ were mentioned in the previous sections, namely

type 1: lattice polynomials;
type 2: $c_{[a,b]}(x) = ((x \vee a) \wedge b)'$ (complementation in a Boolean interval $[a, b]$);
type 3: $f_I(x) = x_I$ for an almost principal ideal I in $\mathbf{L}$;
type 4: $f^F(x) = x^F$ for an almost principal filter F in $\mathbf{L}$.

Thus, Corollary 2.4 says that every compatible function on a bounded distributive lattice is a composition of functions of types 1 and 2. G. Grätzer and E.T. Schmidt proved in [7] that every *unary isotone* compatible function on any distributive lattice is a composition of functions of types 1-4. However, this is not true for binary functions. A new type was discovered in [13]. For every $a \in \mathcal{F}(\mathcal{I}(\mathbf{L}))$, consider the binary polynomial function g on $\mathcal{F}(\mathcal{I}(\mathbf{L}))$ given by $g(x, y) = (a \vee x) \wedge y$. It is easy to see that this function can be restricted to $\mathbf{L}$. (Indeed, $x \wedge y \leq g(x, y) \leq y$, so the convexity of $\mathbf{L}$ in $\mathcal{F}(\mathcal{I}(\mathbf{L}))$ implies that $g(x,y) \in L$ for every $x, y \in L$.) This restriction is a compatible function on $\mathbf{L}$. This is a simple consequence of congruence extension property for distributive lattices. (Every $\theta \in \mathrm{Con}(\mathbf{L})$ can be extended to a congruence $\theta' \in \mathrm{Con}(\mathcal{F}(\mathcal{I}(\mathbf{L})))$, so it is preserved by the polynomial function g.) Hence, we have

type 5: $g_a(x,y) = (a \vee x) \wedge y$ for any $a \in \mathcal{F}(\mathcal{I}(\mathbf{L}))$.

An example in [13] shows that a function of type 5 is not, in general, a composition of functions of types 1-4. Now we present an example which shows that the above five types are still insufficient to generate the clone of all compatible functions.

Example 4.1. Let $\mathbf{L}$ be the lattice of all finite subsets of an infinite set S (ordered by the set-theoretical inclusion). It is not difficult to check that the binary function $f(X,Y) = X \setminus Y$ (the set-theoretical difference) is compatible since it is a restriction of a compatible function of $\mathbf{P}(S)$ (the Boolean lattice of all subsets of S) and distributive lattices have the congruence extension property.

Note that there is no need to consider the functions of types 4 and 5. Since $\mathbf{L}$ has the least element, all almost principal filters are principal, and also $\mathcal{F}(\mathcal{I}(\mathbf{L})) = \mathcal{I}(\mathbf{L})$. Consequently, the projections on almost principal filters are polynomials and every function g_a of type 5 with $a \in \mathcal{I}(\mathbf{L})$ is a composition of a polynomial and a function of type 3, namely $g_a(x,y) = (a \wedge y) \vee (x \wedge y)$.

For contradiction, suppose that f is a composition of polynomials, interval complementations and almost principal ideal projections. This composition can contain only finitely many interval complementations. Let us denote the corresponding Boolean intervals by $[u_1, v_1], \ldots, [u_n, v_n]$ and choose $Y \in L$ with $v_i \subseteq Y$ for every i. Now consider the following binary relation ρ on L:

$$(U,V) \in \rho \quad \text{iff} \quad (U \setminus Y) \subseteq (V \setminus Y).$$

It is not difficult to check that

(i) every polynomial preserves ρ;
(ii) every almost principal ideal projection preserves ρ;
(iii) complementation on every $[u_i, v_i]$ preserves ρ.

(It can easily be verified that the almost principal ideals are all ideals of the form $\{X \in L \mid X \subseteq T\}$ for $T \subseteq S$, so every almost principal ideal projection can be expressed as $g(X) = X \cap T$.) On the other hand, choose arbitrarily $y \in S \setminus Y$ and set $U = \{y\}$, $V_1 = \emptyset$, $V_2 = \{y\}$. Then $(U,U) \in \rho$, $(V_1, V_2) \in \rho$, but $(f(U,V_1), f(U,V_2)) = (\{y\}, \emptyset) \notin \rho$, which shows that f does not preserve ρ. This contradicts the fact that f is a composition of polynomials, interval complementations and almost principal ideal projections.

Inspired by the above example, we introduce three more types of compatible functions:

type 6: $d_{[a,b]}(x,y) = c_{[a,b]}(y) \wedge x$ for any Boolean interval $[a,b]$ in $\mathcal{I}(\mathbf{L})$;
type 7: $d^{[a,b]}(x,y) = c_{[a,b]}(y) \vee x$ for any Boolean interval $[a,b]$ in $\mathcal{F}(\mathbf{L})$;
type 8: $e_{[a,b]}(x,y,z) = (c_{[a,b]}(z) \vee x) \wedge y$ for any Boolean interval $[a,b]$ in $\mathcal{F}(\mathcal{I}(\mathbf{L}))$.

Lemma 4.2. *The functions of types 6, 7, 8 can be restricted to* $\mathbf{L}$ *and the restrictions are compatible.*

Proof. Since L (as a sublattice) is an ideal of $\mathcal{I}(\mathbf{L})$, we have $d_{[a,b]}(x,y) \in L$ for every $x, y \in L$. Similarly, L is a filter of $\mathcal{F}(\mathbf{L})$, so $d^{[a,b]}(x,y) \in L$ for every $x, y \in L$. Further, $x \wedge y \leq e_{[a,b]}(x,y,z) \leq y$, so the convexity of L yields $e_{[a,b]}(x,y,z) \in L$ for every $x, y, z \in L$. Hence, L is closed under all functions of types 6, 7, 8. The compatibility of restrictions follows from the congruence extension property and the fact that $d_{[a,b]}$, $d^{[a,b]}$ and $e_{[a,b]}$ are compatible functions on $\mathcal{I}(\mathbf{L})$, $\mathcal{F}(\mathbf{L})$ and $\mathcal{F}(\mathcal{I}(\mathbf{L}))$, respectively. □

A typical example of a function of type 6 is given in Example 4.1. More details about the existence and structure of the new functions will be given in Section 6. Especially, we will give an alternative definition of functions of types 6 and 7, which does not refer to the lattice structure of $\mathcal{I}(\mathbf{L})$ and $\mathcal{F}(\mathbf{L})$. Further, we will prove that every function of type 8 is a composition of functions of types 1-7. So, the presence of the type 8 in our list is in fact superfluous. Nevertheless, this type is useful in proving the completeness result in the next section.

5. The completeness result

In this section we prove that every compatible function on a distributive lattice is a composition of functions of types 1-8. For the whole section we shall assume that $f : L^n \to L$ is a compatible function on an arbitrary (possibly unbounded) distributive lattice $\mathbf{L}$. By Theorem 3.7 we can extend f to a compatible function $\overline{f} : \mathcal{F}(\mathcal{I}(\mathbf{L}))^n \to \mathcal{F}(\mathcal{I}(\mathbf{L}))$. Since the lattice $\mathcal{F}(\mathcal{I}(\mathbf{L}))$ is bounded, we can express $g = \overline{f}$ in the form presented in Theorem 2.3. The symbols K_{ab} and $c_{J_{ab}}$ below, as well as the symbols such as $a^{-1}(1)$ or $b^{-1}(0)$, will refer to this representation.

Lemma 5.1. *If* $a^{-1}(1) \neq \emptyset$ *and* $b^{-1}(0) \neq \emptyset$ *then* $K_{ab}{\restriction}L$ *is a composition of a function of type 8 and lattice polynomials.*

Proof. Obvious. □

The statement of Lemma 5.1 is not true if $a^{-1}(1) = \emptyset$ or $b^{-1}(0) = \emptyset$. In such case the function K_{ab} cannot be restricted to L. To overcome this difficulty we need one more trick. Define the unary function $f^- : L \to L$ by $f^-(x) = x \wedge f(x, \dots, x)$. Using Lemma 3.4 with $I_1 = \cdots = I_n = L$ we obtain that $f^-(x) = x_J$, where $J = \{x \in L \mid x \leq f(x, \dots, x)\}$ is an almost principal ideal of $\mathbf{L}$. Hence, f^- is a compatible function on $\mathbf{L}$ of type 3. Similarly, the function $f^+(x) = x \vee f(x, \dots, x)$ is of type 4.

Lemma 5.2. *For every* $x_1, \dots, x_n \in L$,

$$f^-(\bigwedge_{i=1}^{n} x_i) \leq f(\overline{x}) \leq f^+(\bigvee_{i=1}^{n} x_i).$$

Proof. We prove the first inequality. Denote $y = \bigwedge_{i=1}^{n} x_i$ and suppose for contradiction that $f^-(y) \not\leq f(\overline{x})$. Then $f(\overline{x}) \in P$ and $f^-(y) \notin P$ for some prime ideal P on $\mathbf{L}$. Since $f^-(y) = y \wedge f(y, \dots, y)$, we have $y \notin P$ and $f(y, \dots, y) \notin P$. As $y \leq x_i$ for every i, we have $x_i \notin P$, hence $(y, x_i) \in \theta_P$. On the other hand, $(f(y, \dots, y), f(x_1, \dots, x_n)) \notin \theta_P$, which contradicts the compatibility of f. The second inequality is analogous. ☐

By the distributivity, Theorem 2.3 and Lemma 5.2 we obtain that, for every $\overline{x} \in L^n$,

$$f(\overline{x}) = \bigvee_{a \in \{0,1\}^n} C_a^*(\overline{x}),$$

where

$$C_a^*(\overline{x}) = \bigwedge_{b \in \{0,1\}^n, b \geq a} K_{ab}^*(\overline{x}),$$

and

$$K_{ab}^*(\overline{x}) = (K_{ab}(\overline{x}) \vee f^-(\bigwedge_{i=1}^{n} x_i)) \wedge f^+(\bigvee_{i=1}^{n} x_i).$$

Especially, $f^-(\bigwedge_{i=1}^{n} x_i) \leq K_{ab}^*(\overline{x}) \leq f^+(\bigvee_{i=1}^{n} x_i)$. The convexity of $\mathbf{L}$ in $\mathcal{F}(\mathcal{I}(\mathbf{L}))$ implies that $K_{ab}^*(\overline{x}) \in L$ for every $\overline{x} \in L^n$, so K_{ab}^* can be restricted to L.

Lemma 5.3. *For every compatible function $h : L^n \to L$ on a bounded distributive lattice $\mathbf{L}$ and every $x \in L$, the following equalities hold:*

$$h(0, \dots, 0) \vee x = h(x, \dots, x) \vee x;$$
$$h(1, \dots, 1) \wedge x = h(x, \dots, x) \wedge x.$$

Proof. For contradiction, suppose that $h(0, \dots, 0) \vee x \neq h(x, \dots, x) \vee x$. Then there is a prime ideal P on $\mathbf{L}$ such that exactly one of $h(0, \dots, 0) \vee x$ and $h(x, \dots, x) \vee x$ is in P. In both possible cases, $(0, x) \in \theta_P$, $(h(0, \dots, 0), h(x, \dots, x)) \notin \theta_P$, which contradicts the compatibility of h. The proof of the second equality is similar. ☐

Lemma 5.4. *For every $a, b \in \{0,1\}^n$, $a \leq b$, the restriction $K_{ab}^* \restriction L$ is a composition of functions of types 1-8.*

Proof. If $a^{-1}(1) \neq \emptyset$ and $b^{-1}(0) \neq \emptyset$, the statement follows from Lemma 5.1. For the remaining cases we distinguish the following five situations.

I. Let $a^{-1}(1) = \emptyset$, $b^{-1}(0) \neq \emptyset$, $b \neq a$. Then

$$K_{ab}^*(\overline{x}) = ((c_{J_{ab}}(\bigvee_{i \in b^{-1}(1)} x_i) \vee \bigvee_{i \in b^{-1}(0)} x_i) \wedge f^+(\bigvee_{i=1}^{n} x_i)) \vee f^-(\bigwedge_{i=1}^{n} x_i),$$

hence

$$K_{ab}^*(\overline{x}) = e_{J_{ab}}(\bigvee_{i \in b^{-1}(0)} x_i, f^+(\bigvee_{i=1}^{n} x_i), \bigvee_{i \in b^{-1}(1)} x_i) \vee f^-(\bigwedge_{i=1}^{n} x_i),$$

which is clearly a composition of functions 1-8.

II. Let $a^{-1}(1) = \emptyset$, $b = a$. Then

$$K^*_{ab}(\overline{x}) = ((\overline{f}(a) \vee \bigvee_{i=1}^{n} x_i) \wedge f^+(\bigvee_{i=1}^{n} x_i)) \vee f^-(\bigwedge_{i=1}^{n} x_i).$$

By Lemma 5.3, $\overline{f}(a) \vee \bigvee_{i=1}^{n} x_i$ is equal to $f^+(\bigvee_{i=1}^{n} x_i)$, hence $K^*_{ab}(\overline{x}) = f^+(\bigvee_{i=1}^{n} x_i)$, which is a composition of a function of type 4 and the lattice join (type 1).

III. Let $b^{-1}(0) = \emptyset$, $a^{-1}(1) \neq \emptyset$, $b \neq a$. Then

$$K^*_{ab}(\overline{x}) = ((c_{J_{ab}}(\bigvee_{i \in a^{-1}(0)} x_i) \wedge \bigwedge_{i \in a^{-1}(1)} x_i) \vee f^-(\bigwedge_{i=1}^{n} x_i)) \wedge f^+(\bigvee_{i=1}^{n} x_i),$$

hence

$$K^*_{ab}(\overline{x}) = e_{J_{ab}}(f^-(\bigvee_{i=1}^{n} x_i), \bigwedge_{i \in a^{-1}(1)} x_i, \bigvee_{i \in a^{-1}(0)} x_i) \wedge f^+(\bigvee_{i=1}^{n} x_i),$$

which is clearly a composition of functions 1-8.

IV. Let $b^{-1}(0) = \emptyset$, $b = a$. Then

$$K^*_{ab}(\overline{x}) = ((\overline{f}(a) \wedge \bigwedge_{i=1}^{n} x_i) \vee f^-(\bigwedge_{i=1}^{n} x_i)) \wedge f^+(\bigvee_{i=1}^{n} x_i).$$

By Lemma 5.3, $\overline{f}(a) \wedge \bigwedge_{i=1}^{n} x_i$ is equal to $f^-(\bigwedge_{i=1}^{n} x_i)$, hence $K^*_{ab}(\overline{x}) = f^-(\bigwedge_{i=1}^{n} x_i)$, which is a composition of a function of type 3 and the lattice meet (type 1).

V. Finally, let $a^{-1}(1) = b^{-1}(0) = \emptyset$. Then

$$K^*_{ab}(\overline{x}) = (c_{J_{ab}}(\bigvee_{i=1}^{n} x_i) \vee f^-(\bigwedge_{i=1}^{n} x_i)) \wedge f^+(\bigvee_{i=1}^{n} x_i),$$

which is equal to

$$e_{J_{ab}}(f^-(\bigwedge_{i=1}^{n} x_i), f^+(\bigvee_{i=1}^{n} x_i), \bigvee_{i=1}^{n} x_i).$$

This concludes the proof. □

Hence we have proved the following result.

Theorem 5.5. *Every compatible function on a distributive lattice is a composition of functions of types 1-8.*

Now we can easily prove the corresponding result for local polynomials. It generalizes the corresponding result in the unary case proved by G. Grätzer and E.T. Schmidt [7]. Recall that by [2], a function on a distributive lattice is a local polynomial if and only if it is compatible and isotone.

Theorem 5.6. *Every isotone compatible function on a distributive lattice is a composition of functions of types 1,3,4 and 5.*

Proof. Assume that $f : L^n \to L$ is compatible and isotone. It is easy to see that its compatible extension $\overline{f} : \mathcal{F}(\mathcal{I}(\mathbf{L}))^n \to \mathcal{F}(\mathcal{I}(\mathbf{L}))$ is also isotone. All Boolean intervals, used for expressing f as a composition of functions of types 0-8, have the form $J_{ab} = [\overline{f}(a) \wedge \overline{f}(b), \overline{f}(a)]$, where $a, b \in \{0,1\}^n$, $a \leq b$. Since $\overline{f}$ is isotone, every such interval is trivial, i.e. one-element. Therefore, the functions of types 2,6,7 and 8 appearing in the decomposition of f only use trivial Boolean intervals. Under such condition, every function of type 2 is constant. Further, every function of type 6 takes the form $d_a(x, y) = a \wedge x$ for some $a \in \mathcal{I}(\mathbf{L})$, which is a projection on the almost principal ideal a, i.e. a function of type 3 (composed with the projection $p(x, y) = x$, to be precise). Similarly the functions of types 7 and 8 reduce to the types 4 and 5, respectively. □

6. Functions of types 6,7,8

In this section we provide more information about functions of types 6,7 and 8. Especially, we prove that every function of type 8 is a composition of functions of types 1-7. We start with one observation that can be proved by elementary calculations (using, for instance, the set-theoretical representation of $\mathbf{L}$).

Lemma 6.1. *Let $[a, b]$ be a Boolean interval in a distributive lattice $\mathbf{L}$. Then, for every $m \in L$, the interval $[a \wedge m, b \wedge m]$ is Boolean and*

$$c_{[a,b]}(y) \wedge m = c_{[a\wedge m, b\wedge m]}(y).$$

The following lemma is due to G. Grätzer and E.T. Schmidt (see [7]).

Lemma 6.2. *For every $J, K \in \mathcal{I}(\mathbf{L})$ and every $x, y \in L$, the following hold:*

(1) $x_{J\vee K} = x_J \vee x_K$;
(2) $x_{J\wedge K} = x_J \wedge x_K$;
(3) $(x \vee y)_J = x_J \vee y_J$;
(4) $(x \wedge y)_J = x_J \wedge y_J$.

Now we show that Boolean intervals in $\mathcal{I}(\mathbf{L})$ are closely related to Boolean intervals in $\mathbf{L}$.

Lemma 6.3. *For every $J, K \in \mathcal{I}(\mathbf{L})$, $J \subseteq K$, the following statements are equivalent:*

(1) *The interval $[J, K]$ in $\mathcal{I}(\mathbf{L})$ is Boolean.*
(2) *For every $x \in L$, the interval $[x_J, x_K]$ in $\mathbf{L}$ is Boolean.*

Further, if the above conditions are satisfied, then

$$d_{[J,K]}(x, y) = c_{[x_J, x_K]}(y)$$

for every $x, y \in L$.

Proof. Since $x_J = x \wedge J$ and $x_K = x \wedge K$ in $\mathcal{I}(\mathbf{L})$, the implication (1)$\Longrightarrow$(2) follows from Lemma 6.1.

Conversely, assume (2). Let $M \in [J, K]$. We set

$$N = \{y \in K \mid \downarrow y \cap M \subseteq J\} = \{y \in K \mid y_M \in J\}.$$

Clearly, N is an ideal in L, $J \subseteq N \subseteq K$ and we claim that N is almost principal. Let $x \in L$. Then $x_M \in [x_J, x_K]$, so there is $y \in [x_J, x_K]$ with $x_M \wedge y = x_J$, $x_M \vee y = x_K$. We claim that $y = \max N \cap \downarrow x$. Obviously, $y \leq x$, so $y_M \leq x_M$, hence $y_M = x_M \wedge y_M \leq x_M \wedge y = x_J$, so $y \in N$. Suppose now that $z \in N$, $z \leq x$. Then $z_M \in J$ and $z \in K$, so $z \leq x_K = x_M \vee y$ and hence $z = (z \wedge x_M) \vee (z \wedge y)$. Since $z \wedge x_M \leq z_M \in J$, we have $z \wedge x_M \leq x_J \leq y$ and therefore $z \leq y \vee (z \wedge y) = y$.

Thus, N is almost principal. For every $x \in M \cap N$ we have $x = x_M \in J$, so $M \wedge N = J$. For every $x \in K$ we have $x = x_K = x_M \vee x_N \in M \vee N$, so $M \vee N = K$. Hence, N is the complement of M in $[J, K]$.

The last statement also follows from Lemma 6.1. □

The above lemma gives an alternative definition of functions of type 6. This definition is intrinsic in the sense that it does not refer to the lattice structure of $\mathcal{I}(\mathbf{L})$. Of course, an analogous description is possible for functions of type 7. Now we can investigate functions of type 8.

Lemma 6.4. *Let $[a, b]$ be a Boolean interval in the lattice $\mathcal{F}(\mathcal{I}(\mathbf{L}))$ for a distributive lattice $\mathbf{L}$. Let m be an arbitrary element of L and define the functions h_1, h_2 and h_3 as follows:*

$$h_1(x, y, z) = e_{[a,b]}(x, y, z) \wedge m;$$
$$h_2(x, y, z) = e_{[a,b]}(x, y, z) \vee m;$$
$$h_3(x, y) = e_{[a,b]}(x, y, m).$$

Then, for every $x, y, z \in L$,

$$e_{[a,b]}(x, y, z) = h_1(x, y, z) \vee (h_2(x, y, z) \wedge h_3(x, y)).$$

Proof. Let $x, y, z \in L$. Since $c_{[a,b]}(m)$ is the complement of $a \vee (m \wedge b)$ in $[a, b]$, we have

$$a = c_{[a,b]}(m) \wedge (a \vee (m \wedge b)) \geq c_{[a,b]}(m) \wedge (m \wedge b) = c_{[a,b]}(m) \wedge m,$$

which implies that

$$m \wedge c_{[a,b]}(m) \wedge y \leq a \wedge y \leq c_{[a,b]}(z) \wedge y \leq e_{[a,b]}(x, y, z).$$

Further,

$$m \wedge x \wedge y \leq x \wedge y \leq e_{[a,b]}(x, y, z),$$

hence

$$m \wedge h_3(x, y) = m \wedge (c_{[a,b]}(m) \vee x) \wedge y = (m \wedge c_{[a,b]}(m) \wedge y) \vee (m \wedge x \wedge y) \leq e_{[a,b]}(x, y, z).$$

Similarly,

$$b = c_{[a,b]}(m) \vee a \vee (m \wedge b) = c_{[a,b]}(m) \vee (m \wedge b) = (c_{[a,b]}(m) \vee m) \wedge b,$$

hence $b \leq c_{[a,b]}(m) \vee m$ and therefore

$$m \vee h_3(x, y) = (m \vee c_{[a,b]}(m) \vee x) \wedge (m \vee y) \geq (b \vee x) \wedge y \geq e_{[a,b]}(x, y, z).$$

Now we obtain that

$$(h_2(x,y,z) \wedge h_3(x,y)) = (e_{[a,b]}(x,y,z) \vee m) \wedge h_3(x,y) =$$
$$(e_{[a,b]}(x,y,z) \wedge h_3(x,y)) \vee (m \wedge h_3(x,y)) \leq e_{[a,b]}(x,y,z).$$

Obviously, $h_1(x,y,z) \leq e_{[a,b]}(x,y,z)$, hence

$$h_1(x,y,z) \vee (h_2(x,y,z) \wedge h_3(x,y)) \leq e_{[a,b]}(x,y,z).$$

The inverse inequality:

$$h_1(x,y,z) \vee (h_2(x,y,z) \wedge h_3(x,y)) = h_2(x,y,z) \wedge (h_1(x,y,z) \vee h_3(x,y)) =$$
$$h_2(x,y,z) \wedge (e_{[a,b]}(x,y,z) \vee h_3(x,y)) \wedge (m \vee h_3(x,y)) \geq e_{[a,b]}(x,y,z).$$

□

Theorem 6.5. *Every function of type 8 on a distributive lattice is a composition of functions of types 1-7.*

Proof. It suffices to prove that the functions h_1, h_2 and h_3 are compositions of functions of types 1-7. This is clear for h_3, which is of type 5 by its definition. Using the distributivity we obtain that

$$h_1(x,y,z) = ((c_{[a,b]}(z) \wedge m) \vee x) \wedge y \wedge m;$$

$$h_2(x,y,z) = ((c_{[a,b]}(z) \vee m) \wedge y) \vee ((x \wedge y) \vee m).$$

By Lemma 6.1, $(c_{[a,b]}(z) \wedge m) \vee x = c_{[a\wedge m, b\wedge m]}(z) \vee x$. Since $\mathcal{F}(\mathbf{L})$ is an ideal in $\mathcal{F}(\mathcal{I}(\mathbf{L}))$ containing m (see Lemma 3.2 and the following comments) and since $a \wedge m, b \wedge m \leq m$ in $\mathcal{F}(\mathcal{I}(\mathbf{L}))$, we can regard $[a \wedge m, b \wedge m]$ as an interval in $\mathcal{F}(\mathbf{L})$. Hence $c_{[a\wedge m, b\wedge m]}(z) \vee x = d^{[a\wedge m, b\wedge m]}(x,z)$ is a function of type 7.

By the dual statement to Lemma 6.1, $(c_{[a,b]}(z) \vee m) \wedge y = c_{[a\vee m, b\vee m]}(z) \wedge y$. Since $\mathcal{I}(\mathbf{L})$ is a filter in $\mathcal{F}(\mathcal{I}(\mathbf{L}))$ containing m, we can analogously as above regard $[a \vee m, b \vee m]$ as an interval in $\mathcal{I}(\mathbf{L})$. Hence $c_{[a\vee m, b\vee m]}(z) \wedge y = d_{[a\vee m, b\vee m]}(y,z)$ is a function of type 6.

Consequently, h_1 and h_2 are compositions of polynomials and functions of types 7 and 6, which concludes the proof. □

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