NON-REPRESENTABLE DISTRIBUTIVE SEMILATTICES

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Abstract. We present two examples of distributive algebraic lattices which are not isomorphic to the congruence lattice of any lattice. The first such example was discovered by F. Wehrung in 2005. One of our examples is defined topologically, the other one involves majority algebras. In particular, we prove that the conguence lattice of the free majority algebra on (at least) $\aleph_2$ generators is not isomorphic to the congruence lattice of any lattice. Our method is a generalization of Wehrung's approach, so that we are able to apply it to a larger class of distributive semilattices.

1. Introduction

The investigation of congruence lattices is one of the central topics in universal algebra. It is well known that a lattice is isomorphic to the congruence lattice of some algebra if and only if it is algebraic [4]. Congruence lattices of lattices have an additional property: they are distributive. The question, whether the converse of this is true, is referred to as the Congruence Lattice Problem (CLP): *Is every distributive algebraic lattice isomorphic to the congruence lattice of some lattice?* The finite version of this problem has been solved by R. P. Dilworth, who proved that every finite distributive lattice is isomorphic to the congruence lattice of some finite lattice. (The first published proof is due to G. Grätzer and E. T. Schmidt [3].) During the subsequent 60 years of effort (documented in [1](Appendix C) or [13]), various partial positive results have been achieved, but the conjecture has finally been disproved by F. Wehrung in [15]. The impact of this problem to the development of lattice theory has been described in the expository paper [2]. In the present paper we develop further Wehrung's method and provide another two examples, disproving CLP. Our constructions are simpler than the original Wehrung's example and, we believe, can help to understand, which distributive algebraic lattices are isomorphic to congruence lattices of lattices and other kinds of algebras.

We assume familiarity with fundamentals of lattice theory and universal algebra. For all undefined concepts and unreferenced facts we refer to [1] and [7].

For an algebra A let $\operatorname{Con} A$ denote the congruence lattice of A. This lattice is always algebraic and its compact elements form a $\vee$-subsemilattice of $\operatorname{Con} A$, denoted $\operatorname{Con}_c A$. For $x, y \in A$ let $\theta(x, y)$ denote the smallest congruence containing the pair (x, y). (We also write $\theta_A(x, y)$, when A needs to be specified.) The semilattice $\operatorname{Con}_c A$ consists precisely of all finitely generated congruences, i.e. congruences of the form $\theta(x_1, y_1) \vee \dots \vee \theta(x_n, y_n)$. The smallest and the largest element of $\operatorname{Con} A$

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will be denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively. The congruence $\mathbf{0}$ (the equality relation) is considered as compact, so $\mathrm{Con}_c A$ always has a smallest element.

An ideal of a $(\vee, 0)$-semilattice S is a nonempty, $\vee$-closed lower set $I \subset S$. (That is, $a \leq b \in I$ implies $a \in I$.)

A $\vee$-semilattice S is called *distributive* if for every $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in S$ satisfying $\boldsymbol{z} \leq \boldsymbol{x} \vee \boldsymbol{y}$ there are $\boldsymbol{x}' \leq \boldsymbol{x}$, $\boldsymbol{y}' \leq \boldsymbol{y}$ such that $\boldsymbol{x}' \vee \boldsymbol{y}' = \boldsymbol{z}$. It is well known that an algebraic lattice is distributive if and only if its $\vee$-semilattice of compact elements is distributive. Thus, we have an equivalent formulation of CLP: *Is every distributive* $(\vee, 0)$*-semilattice isomorphic to* $\mathrm{Con}_c A$ *for some lattice* A*?* We use this formulation and construct two new examples of distributive $(\vee, 0)$-semilattices not isomorphic to $\mathrm{Con}_c A$ for any lattice A.

A homomorphism of $\vee$-semilattices $\mu : T \to S$ is called *weakly distributive*, if for all $\boldsymbol{x} \in T$ and $\boldsymbol{y}_0, \boldsymbol{y}_1 \in S$ such that $\mu(\boldsymbol{x}) \leq \boldsymbol{y}_0 \vee \boldsymbol{y}_1$, there are $\boldsymbol{x}_0, \boldsymbol{x}_1 \in T$ such that $\boldsymbol{x} \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1$ and $\mu(\boldsymbol{x}_i) \leq \boldsymbol{y}_i$, for all $i \in \{0, 1\}$.

If $\alpha \in \mathrm{Con}\, A$ and B is a subalgebra of A, then the restriction of α to B is the relation $\alpha \cap B^2$ and will usually be denoted by $\alpha \restriction B$. Notice that it is always a congruence on B.

We use standard set-theoretic notation. We identify a natural number n with the set $\{0, 1, \ldots, n-1\}$. The least infinite ordinal is denoted ω. If Ω is a set then $[\Omega]^n$ denotes the family of all n-element subsets of Ω, while $[\Omega]^{<\omega}$ stands for the family of all finite subsets of Ω.

If $f : A \to B$ is a map, then we define its kernel as the relation $\mathrm{Ker}(f) = \{(x, y) \in A^2 \mid f(x) = f(y)\}$. If f is a homomorphism of algebras, then $\mathrm{Ker}(f) \in \mathrm{Con}\, A$.

For any function f let $\mathrm{dom}(f)$ and $\mathrm{rng}(f)$ denote its domain and range, respectively.

2. Free trees

Let k be a positive integer and X a set. For a map $\Phi : [\Omega]^{k-1} \to [\Omega]^{<\omega}$ we say that a k-element set $B \subseteq \Omega$ is *free with respect to* Φ if $b \notin \Phi(B \setminus \{b\})$ for all $b \in B$.

The following statement of infinite combinatorics is a one direction of a theorem due to K. Kuratowski [6]

Theorem 2.1. *Let* Ω *be a set of cardinality at least* $\aleph_{k-1}$*. Then for every map* $\Phi : [\Omega]^{k-1} \to [\Omega]^{<\omega}$ *there is a* k*-element free subset of* Ω*.*

The special case of this principle (for $k = 2$) has been used in several papers ([14], [11], [8] and others) to prove negative results concerning the representability of distributive algebraic lattices as congruence lattices of algebras. The general Kuratowski's theorem played an important role in therecent solution of Congruence Lattice Problem by Wehrung [15]. For our purpose we need a modification of this principle, recently discovered by P. Růžička [12].

Let m, n, k be natural numbers with $k > 0$, $m \leq n$ and let $g : \{m, \ldots, n-1\} \to k$ be a map. We denote

$$T_{n,k}(g) = \{f : n \to k \mid f \text{ extends } g\}. \tag{1}$$

If $0 < m$ and $i \in \{0, \ldots, k-1\}$ then we also use

$$T_{n,k}(g, i) = \{f \in T_{n,k}(g) \mid f(m-1) = i\}, \tag{2}$$

$$T_{n,k}(g, \neg i) = \{f \in T_{n,k}(g) \mid f(m-1) \neq i\}, \tag{3}$$

Definition 2.2. ([12]) *Let Ω be a set and let $\Phi\colon [\Omega]^{<\omega} \to [\Omega]^{<\omega}$ be a map. Let k and n be positive integers. We say that a family $\mathcal{T} = (\alpha(f) \mid f\colon n \to k)$ of elements of Ω is a* free k-tree of height n with respect to Φ *if*

$$\{\alpha(f) \mid f \in T_{n,k}(g,i)\} \cap \Phi(\{\alpha(f) \mid f \in T_{n,k}(g,\neg i)\}) = \emptyset, \tag{4}$$

for every $0 < m \leq n$, every $g\colon \{m,\dots,n-1\} \to k$, and every $i \in k$.

Theorem 2.3. ([12]) *Let k be a positive integer and let Ω be a set of cardinality at least $\aleph_{k-1}$. Then for every map $\Phi\colon [\Omega]^{<\omega} \to [\Omega]^{<\omega}$ and every positive integer n there is a k-free tree of height n with respect to Φ.*

3. Evaporation schemes

Let S be a distributive $(\vee,0)$-semilattice, let $\boldsymbol{e} \in S$. *A decomposition system* at $\boldsymbol{e}$ is a family $\mathcal{F} = ((\boldsymbol{a}_0^\alpha, \boldsymbol{a}_1^\alpha) \mid \alpha \in \Omega)$ such that $\boldsymbol{a}_0^\alpha \vee \boldsymbol{a}_1^\alpha = \boldsymbol{e}$ for every $\alpha \in \Omega$.

Now we introduce the central concept of this paper. Its simplified version has implicitly appeared in [15], called there the "Evaporation Lemma". The denotation "supp" stands for "support".

Definition 3.1. *Let S be a distributive $(\vee,0)$-semilattice. Let $\mathcal{F} = ((\boldsymbol{a}_0^\alpha, \boldsymbol{a}_1^\alpha) \mid \alpha \in \Omega)$ be a decomposition system at $\boldsymbol{e} \in S$. Let $\operatorname{supp}\colon S \to [\Omega]^{<\omega}$ be a function. Let I be an ideal of S. We say that the triple $(\mathcal{F}, \operatorname{supp}, I)$ is* an evaporation scheme *at $\boldsymbol{e}$ if, for all distinct $\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_m,\delta \in \Omega$, all $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in S$, $\boldsymbol{w}_0,\boldsymbol{w}_1 \in I$ and $i \in \{0,1\}$, the conditions*

(i) $\xi_1,\dots,\xi_n \notin \operatorname{supp}(\boldsymbol{y})$, $\eta_1,\dots,\eta_m \notin \operatorname{supp}(\boldsymbol{x})$, $\delta \notin \operatorname{supp}(\boldsymbol{z})$;
(ii) $\boldsymbol{x} \leq \boldsymbol{a}_0^\delta$, $\boldsymbol{y} \leq \boldsymbol{a}_1^\delta$, $\boldsymbol{x} \leq \boldsymbol{a}_i^{\xi_1} \vee \dots \vee \boldsymbol{a}_i^{\xi_n} \vee \boldsymbol{w}_0$, $\boldsymbol{y} \leq \boldsymbol{a}_i^{\eta_1} \vee \dots \vee \boldsymbol{a}_i^{\eta_m} \vee \boldsymbol{w}_1$;
(iii) $\boldsymbol{z} \leq \boldsymbol{x} \vee \boldsymbol{y}$

imply

(iv) $\boldsymbol{z} \in I$.

The next theorem is implicitly contained in [15], with the cardinality restriction $|\Omega| \geq \aleph_{\omega+1}$ and with $I = \{\boldsymbol{0}\}$. We follow the proof from [15], omitting some details. We adopt Růžička's modifications, which allow to optimize the cardinality assumption. The size of the ideal I requires some care, but does not cause any difficulties.

Let ε be the parity function, that is $\varepsilon(i) = 0$ for i even and $\varepsilon(i) = 1$ for i odd.

Theorem 3.2. *If a distributive $(\vee,0,1)$-semilattice S has an evaporation scheme $(\mathcal{F}, \operatorname{supp}, I)$ at $\boldsymbol{1}$ with $|\Omega| \geq \aleph_2$ and $I \neq S$, then for any lattice L there is no weakly distributive $(\vee,0)$-homomorphism $\mu\colon \operatorname{Con}_c L \to S$ with $\boldsymbol{1}$ in its range.*

Proof. For contradiction, suppose that $(\mathcal{F}, \operatorname{supp}, I)$ is an evaporation scheme as required and

$$\mu\colon \operatorname{Con}_c L \to S \tag{5}$$

is a weakly distributive $\vee$-homomorphism for some lattice L with $\boldsymbol{1}$ in its range. Then $\boldsymbol{1} = \mu(\psi)$ for some $\psi \in \operatorname{Con}_c L$, $\psi = \theta(u_1,v_1) \vee \dots \vee \theta(u_k,v_k)$ for some $u_j, v_j \in L$, $u_j \leq v_j$.

We show that $\mu\theta(u_j,v_j) \in I$ for every j. For simplicity, write u, v instead of u_j, v_j. For every $\alpha \in \Omega$ we have $\mu\theta(u,v) \leq \boldsymbol{a}_0^\alpha \vee \boldsymbol{a}_1^\alpha$. Since μ is weakly distributive, there are $\psi_0,\psi_1 \in \operatorname{Con}_c L$ (depending on α) such that $(u,v) \in \psi_0 \vee \psi_1$, $\mu(\psi_i) \leq \boldsymbol{a}_i^\alpha$

$(i = 0, 1)$. Consequently, for every α there are a positive integer $n(\alpha)$ and elements $z_i^\alpha \in L$ for $0 \le i \le n(\alpha)$ such that

$$v = z_0^\alpha \ge z_1^\alpha \ge \cdots \ge z_{n(\alpha)}^\alpha = u \tag{6}$$

and

$$\mu\theta(z_i^\alpha, z_{i+1}^\alpha) \le a_{\varepsilon(i)}^\alpha \tag{7}$$

for every i. (Indeed, from $(u, v) \in \psi_0 \vee \psi_1$ we get elements $v = t_0, t_1, \dots, t_{n(\alpha)} = u$ such that $(t_i, t_{i+1}) \in \psi_{\varepsilon(i)}$ and we set $z_i^\alpha = (t_i \vee \cdots \vee t_{n(\alpha)}) \wedge v$.)

Since $\aleph_2$ is a regular cardinal, there are $\Omega' \subseteq \Omega$ and a positive integer n such that $|\Omega'| = \aleph_2$ and $n(\alpha) = n$ for every $\alpha \in \Omega'$.

For $Y \subseteq \Omega'$ let $S(Y)$ be the $\vee$-subsemilattice of L generated by all elements z_k^ξ with $\xi \in Y$, $0 \le k \le n$. Notice that $S(Y)$ is finite whenever Y is finite.

Now we define a map $\Phi : [\Omega']^{<\omega} \to [\Omega']^{<\omega}$ by

$$\Phi(Y) = \Big(Y \cup \bigcup\{\operatorname{supp}(\mu(\theta(x_1, y_1) \vee \cdots \vee \theta(x_l, y_l))) \mid l \in \omega,\ x_i, y_i \in S(Y)\}\Big) \cap \Omega'.$$

By 2.3 there exists a free 3-tree $\mathcal{T} = (\alpha(f) \mid f : n \to 3)$ of height n with respect to Φ. Observe that the definition of Φ ensures that the map α is one-to-one.

The proof will be completed by the following claim:

Claim. *For every $j \in \{0, \dots, n\}$ and $g : \{j, \dots, n-1\} \to \{0, 1\}$,*

$$\mu\theta\Big(v, \bigvee\{z_j^{\alpha(f)} \mid f \in T_{n,2}(g)\}\Big) \in I. \tag{8}$$

Indeed, for $j = n$ (which means that g is the empty map) we have $z_j^{\alpha(f)} = u$ for every f, so the above claim says that $\mu\theta(v, u) \in I$. Since μ is a $\vee$-homomorphism, we obtain that $\mathbf{1} = \mu(\psi) \in I$, which contradicts the assumption $I \neq S$.

It remains to prove the Claim. We proceed by induction on j. The statement is trivial for $j = 0$, since $z_0^\xi = v$ for every $\xi \in \Omega$ and $\theta(v, v) = \mathbf{0}$.

Suppose now that $0 < j \le n$ and $g : \{j, \dots, n-1\} \to \{0, 1\}$. Let $g_0, g_1 : \{j-1, \dots, n-1\} \to \{0, 1\}$ be the extensions of g with $g_0(j-1) = 0$, $g_1(j-1) = 1$. Observe that $T_{n,2}(g, k) = T_{n,2}(g_k)$ for each $k < 2$. Denote

$$x_0 = \bigvee\{z_j^{\alpha(f)} \mid f \in T_{n,2}(g, 0)\}; \tag{9}$$

$$x_1 = \bigvee\{z_j^{\alpha(f)} \mid f \in T_{n,2}(g, 1)\}, \tag{10}$$

so we need to prove that $\mu\theta(v, x_0 \vee x_1) \in I$.

Choose any $h \in T_{n,3}(g, 2)$ and define elements $\boldsymbol{u}_0, \boldsymbol{u}_1 \in S$ as follows.

$$\boldsymbol{u}_0 = \mu\left(\bigvee\left\{\theta(x_0 \vee z_l^{\alpha(h)}, x_0 \vee z_{l+1}^{\alpha(h)}) \mid 0 \le l < m,\ l \text{ is even}\right\}\right); \tag{11}$$

$$\boldsymbol{u}_1 = \mu\left(\bigvee\left\{\theta(x_1 \vee z_l^{\alpha(h)}, x_1 \vee z_{l+1}^{\alpha(h)}) \mid 0 < l < m,\ l \text{ is odd}\right\}\right). \tag{12}$$

The construction of $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ comes from the "Erosion Lemma" of [15], which plays a central role in Wehrung's proof. In the next few lines we recall essential facts about $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$. The proof of the following statements (13) - (19) follow the lines of proofs in [15], Lemma 6.2 (in a slightly different formalism) and in [10], Lemma 4.3 (in the present formalism).

$$\operatorname{supp}(\boldsymbol{u}_0) \cap \{\alpha(f) \mid f \in T_{n,3}(g,1)\} = \emptyset; \tag{13}$$

$$\operatorname{supp}(\boldsymbol{u}_1) \cap \{\alpha(f) \mid f \in T_{n,3}(g,0)\} = \emptyset; \tag{14}$$

$$\alpha(h) \notin \operatorname{supp}(\mu\theta(v, x_0 \vee x_1)); \tag{15}$$

$$\boldsymbol{u}_0 \leq \boldsymbol{a}_0^{\alpha(h)} \text{ and } \boldsymbol{u}_1 \leq \boldsymbol{a}_1^{\alpha(h)}; \tag{16}$$

$$\boldsymbol{u}_0 \leq \bigvee \left\{\boldsymbol{a}_{\varepsilon(j-1)}^{\alpha(f)} \mid f \in T_{n,2}(g_0)\right\} \vee \mu\theta\left(v, \bigvee\{z_{j-1}^{\alpha(f)} \mid f \in T_{n,2}(g_0)\}\right); \tag{17}$$

$$\boldsymbol{u}_1 \leq \bigvee \left\{\boldsymbol{a}_{\varepsilon(j-1)}^{\alpha(f)} \mid f \in T_{n,2}(g_1)\right\} \vee \mu\theta\left(v, \bigvee\{z_{j-1}^{\alpha(f)} \mid f \in T_{n,2}(g_1)\}\right); \tag{18}$$

$$\boldsymbol{u}_0 \vee \boldsymbol{u}_1 \geq \mu\theta(v, x_0 \vee x_1). \tag{19}$$

The inequality (17) (and similarly (18)) follows from

$$\boldsymbol{u}_0 \leq \mu\theta(v, x_0) \leq \mu\theta\left(v, \bigvee_{f\in T} z_{j-1}^{\alpha(f)}\right) \vee \mu\theta\left(\bigvee_{f\in T} z_{j-1}^{\alpha(f)}, \bigvee_{f\in T} z_j^{\alpha(f)}\right), \tag{20}$$

where $T = T_{n,2}(g_0)$, because $\mu\theta(z_{j-1}^{\alpha(f)}, z_j^{\alpha(f)}) \leq \boldsymbol{a}_{\varepsilon(j-1)}^{\alpha(f)}$ by (7).

By the induction hypothesis, $\mu\theta(v, \bigvee\{z_{j-1}^{\alpha(f)} \mid f \in T_{n,2}(g_k)\}) \in I$, $(k = 0, 1)$. Using the definition of an evaporation scheme with $\boldsymbol{x} := \boldsymbol{u}_0$, $\boldsymbol{y} := \boldsymbol{u}_1$, $\boldsymbol{z} := \mu\theta(v, x_0 \vee x_1)$, $\{\xi_1, \dots, \xi_n\} := \{\alpha(f) \mid f \in T_{n,2}(g_0)\}$, $\{\eta_1, \dots, \eta_m\} := \{\alpha(f) \mid f \in T_{n,2}(g_1)\}$, $\delta := \alpha(h)$, $i := \varepsilon(j-1)$, $\boldsymbol{w}_0 := \mu\theta(v, \bigvee\{z_{j-1}^{\alpha(f)} \mid f \in T_{n,2}(g_0)\})$, $\boldsymbol{w}_1 := \mu\theta(v, \bigvee\{z_{j-1}^{\alpha(f)} \mid f \in T_{n,2}(g_1)\})$, we obtain that $\mu\theta(v, x_0 \vee x_1) \in I$, which completes the proof. $\square$

As every isomorphism is weakly distributive, we obtain the following result.

Theorem 3.3. *If a distributive* $(\vee, 0, 1)$*-semilattice* S *has an evaporation scheme* $(\mathcal{F}, \operatorname{supp}, I)$ *at* $\mathbf{1}$ *with* $|\Omega| \geq \aleph_2$ *and* $I \neq S$*, then it is not isomorphic to* $\operatorname{Con}_c L$ *for any lattice* L*.*

In his solution of CLP, Wehrung found a distributive semilattice with an evaporation scheme $(\mathcal{F}, \operatorname{supp}, I)$, where $I = \{\mathbf{0}\}$. In the next sections we provide two more such examples. However our evaporation schemes will have different I, which justifies a more general definition.

The cardinality bound $\aleph_2$ in Theorem 3.3 is optimal, since any distributive $(\vee, \mathbf{0})$-semilattice of cardinality at most $\aleph_1$ is isomorphic to $\operatorname{Con}_c L$ for some lattice L, as proved by A. P. Huhn. (See [5] or [1], Appendix C.)

Similarly as in [15], the above result can be stated in a stronger form, using the concept of a congruence-compatible function. A finitary function $f : A^n \to A$ on an algebra A is called *congruence-compatible* if, for any congruence $\theta \in \operatorname{Con} A$, $(x_i, y_i) \in \theta$, $i = 1, \dots, n$, implies that $(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \in \theta$. It is not difficult to see that the algebra L in Theorem 3.2 need not actually be a lattice: it is sufficient to assume that L possesses a congruence-compatible lattice structure (i.e. congruence-compatible operations making it a lattice). Furthermore, one can check that the lattice meet operation was only used in proving the existence of elements z_j satisfying (6). However, if L has the largest element 1 (with respect to the join operation), the meet operation is not needed and we obtain the following result.

Theorem 3.4. *If a distributive* $(\vee, 0, 1)$*-semilattice* S *has an evaporation scheme* $(\mathcal{F}, \mathrm{supp}, I)$ *at* $\mathbf{1}$ *with* $|\Omega| \geq \aleph_2$ *and* $I \neq S$*, and* A *is an algebra with a congruence-compatible* $(\vee, 1)$*-structure, then there is no weakly distributive* $\vee$*-homomorphism* $\mathrm{Con}_c\, A \to S$ *with* $\mathbf{1}$ *in its range.*

Proof. In the proof of Theorem 3.2 we can assume that $v_j = 1$ for every j. □

4. Majority algebras

By a *majority algebra* we mean a set M endowed with a ternary operation m such that

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x$$

for every $x, y \in M$. A majority algebra M is called *bounded* if there are constants $0, 1 \in M$ such that

$$m(x, 0, 1) = m(x, 1, 0) = m(0, x, 1) = m(1, x, 0) = m(0, 1, x) = m(1, 0, x) = x$$

for every $x \in M$. It is well known that every majority algebra has a distributive congruence lattice.

Every bounded lattice $(A, \wedge, \vee, 0, 1)$ gives rise to a bounded majority algebra $(A, m, 0, 1)$, where

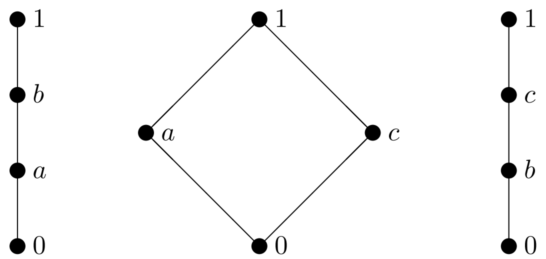
$$m(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$$

is the *(upper) median operation.* In fact, the two algebras are term equivalent, since

$$x \vee y = m(x, y, 1), \quad x \wedge y = m(x, y, 0).$$

Consequently, both algebras have the same congruences, the same subdirect decompositions, etc.

Of course, not all bounded majority algebras arise in this way. In the sequel we shall work with a special 5-element algebra, which is obtained by "gluing" the following three lattices:



A_1 A_2 A_3

Precisely, define the operation m on the set $\{0, 1, a, b, c\}$ by the following rules:

- if $x, y, z \in A_i$ for some $i = 1, 2, 3$, then $m(x, y, z)$ is the lattice upper median evaluated in A_i;

- if $\{x, y, z\} = \{a, b, c\}$ then $m(x, y, z) = 0$.

Denote the resulting algebra by W. It is easy to see that it is a bounded majority algebra. Let $\mathcal{W}$ be the variety generated by W. We regard the constants 0 and 1 as nullary operations. Thus, all members of $\mathcal{W}$ are bounded majority algebras. Further, all homomorphisms between algebras in $\mathcal{W}$ are assumed to preserve 0 and 1.

Let Ω be a set. Let F be the free algebra in $\mathcal{W}$ having Ω as the set of free generators. For every $\xi \in \Omega$ define $\boldsymbol{a}_0^\xi, \boldsymbol{a}_1^\xi \in \operatorname{Con}_c F$ by

$$\boldsymbol{a}_0^\xi = \theta(\xi, 0), \quad \boldsymbol{a}_1^\xi = \theta(\xi, 1). \tag{21}$$

Lemma 4.1. *$\theta(0, 1)$ is the largest congruence on F and $\mathcal{F} = ((\boldsymbol{a}_0^\xi, \boldsymbol{a}_1^\xi) \mid \xi \in \Omega)$ is a decomposition system at $\theta(0, 1)$.*

Proof. For every $x \in F$ we have $(x, 0) = (m(x, 0, 1), m(x, 0, 0)) \in \theta(0, 1)$, hence $(x, y) \in \theta(0, 1)$ for every $x, y \in F$. The second statement is now trivial. $\square$

For every $Y \subseteq \Omega$ let $F(Y)$ denote the subalgebra of F generated by Y. Every $\psi \in \operatorname{Con}_c F$ is generated by a finite subset of F^2 and every element of F belongs to $F(Z)$ for some finite set $Z \subseteq \Omega$. Hence, there exists a finite set $Y \subseteq \Omega$ such that ψ is generated by $\psi{\restriction}F(Y)$. We pick such a set for every ψ, call it the support of ψ, and denote it by $\operatorname{supp}(\psi)$. (We do not require any kind of mimimality, just the finiteness.) The importance of the support lies in the following, rather trivial, observation, which will be frequently used in the sequel.

Lemma 4.2. *Let $\psi \in \operatorname{Con}_c F$, $\varphi \in \operatorname{Con} F$, $\operatorname{supp}(\psi) \subseteq Y \subseteq \Omega$. Then $\psi \subseteq \varphi$ if and only if $\psi{\restriction}F(Y) \subseteq \varphi{\restriction}F(Y)$.*

The variety $\mathcal{W}$ contains the two-element bounded majority algebra $\mathbf{2} = \{0, 1\}$. Now we define

$$I = \{\boldsymbol{w} \in \operatorname{Con}_c F \mid \boldsymbol{w} \subseteq \operatorname{Ker}(f) \text{ for every homomorphism } f\colon\ F \to \mathbf{2}\}.$$

It is clear that I is an ideal of $\operatorname{Con}_c F$.

We need the following technical assertion.

Lemma 4.3. *Let $\boldsymbol{x} \in \operatorname{Con}_c F$, $\boldsymbol{w} \in I$, $\xi_1, \dots, \xi_n \in \Omega$, $i \in \{0, 1\}$ and suppose that $\boldsymbol{x} \subseteq \bigvee_{j=1}^n \boldsymbol{a}_i^{\xi_j} \vee \boldsymbol{w}$. Let $Y \subseteq \Omega$, and let $g\colon\ F(Y) \to \mathbf{2}$ be a homomorphism such that $\boldsymbol{x} \restriction F(Y) \nsubseteq \operatorname{Ker}(g)$. Then $g(\xi_j) = 1 - i$ for some j.*

Proof. Let g' be the homomorphism $F \to \mathbf{2}$ defined by

$$g'(\alpha) = \begin{cases} g(\alpha) & \text{if } \alpha \in Y \\ i & \text{if } \alpha \in \Omega \setminus Y. \end{cases}$$

Since g and g' coincide on $F(Y)$, we have $\boldsymbol{x} \restriction F(Y) \nsubseteq \operatorname{Ker}(g') \restriction F(Y)$, hence $\boldsymbol{x} \nsubseteq \operatorname{Ker}(g')$. Consequently, $\bigvee_{j=1}^n \boldsymbol{a}_i^{\xi_j} \vee \boldsymbol{w} \nsubseteq \operatorname{Ker}(g')$. Since $\boldsymbol{w} \in I$, we obtain that $\boldsymbol{a}_i^{\xi_j} \nsubseteq \operatorname{Ker}(g')$ for some j. Since $\boldsymbol{a}_i^{\xi_j} = \theta(\xi_j, i)$, it follows that $g'(\xi_j) \neq g'(i) = i$, hence $g'(\xi_j) = 1 - i$. By the definition of g', this is only possible if $\xi_j \in Y$ and $g(\xi_j) = 1 - i$. $\square$

Theorem 4.4. *$(\mathcal{F}, \operatorname{supp}, I)$ is an evaporation scheme at $\mathbf{1} \in \operatorname{Con}_c F$.*

Proof. For contradiction, suppose that $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m, \delta \in \Omega$, $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}_0, \boldsymbol{w}_1 \in \operatorname{Con}_c F$, $i \in \{0,1\}$ satisfy (i)-(iii) of Definition 3.1, while $\boldsymbol{z} \notin I$. So,

$$\boldsymbol{z} \nsubseteq \operatorname{Ker}(f) \tag{22}$$

for some homomorphism $f: F \to \mathbf{2}$.

Denote

$$J = \{\xi_j \mid f(\xi_j) = 1 - i\},$$
$$K = \{\eta_k \mid f(\eta_k) = 1 - i\}.$$

We need to separate cases, according to the two possible values of i.

A. Let $i = 0$. Consider the homomorphism $h: F \to W$ determined on the set Ω as follows:

$$h(\alpha) = \begin{cases} a & \text{if } \alpha \in J \\ b & \text{if } \alpha \in K \\ c & \text{if } \alpha = \delta \\ f(\alpha) & \text{otherwise.} \end{cases} \tag{23}$$

We claim that $\boldsymbol{z} \nsubseteq \operatorname{Ker}(h)$, while $\boldsymbol{x} \subseteq \operatorname{Ker}(h)$ and $\boldsymbol{y} \subseteq \operatorname{Ker}(h)$, which means a contradiction with (iii) from Definition 3.1.

Let $p: \{0,1,a,b\} \to \{0,1\}$ be the homomorphism defined on the subalgebra of W by $p(0) = 0$, $p(a) = p(b) = p(1) = 1$. Then clearly

$$ph{\restriction}F(\Omega \setminus \{\delta\}) = f{\restriction}F(\Omega \setminus \{\delta\}), \tag{24}$$

hence

$$\operatorname{Ker}(h{\restriction}F(\Omega \setminus \{\delta\})) \subseteq \operatorname{Ker}(f{\restriction}F(\Omega \setminus \{\delta\})). \tag{25}$$

By Lemma 4.2, $\boldsymbol{z} \nsubseteq \operatorname{Ker}(f)$ implies

$$\boldsymbol{z}{\restriction}F(\Omega \setminus \{\delta\}) \nsubseteq \operatorname{Ker}(h{\restriction}F(\Omega \setminus \{\delta\})), \tag{26}$$

and consequently, $\boldsymbol{z} \nsubseteq \operatorname{Ker}(h)$.

To prove that $\boldsymbol{x} \subseteq \operatorname{Ker} h$, let $p_1, p_2: \{0,1,a,c\} \to \{0,1\}$ be the homomorphisms defined on a subalgebra of W by $p_1(a) = p_2(c) = p_1(0) = p_2(0) = 0$, $p_1(c) = p_2(a) = p_1(1) = p_2(1) = 1$. Then $\operatorname{Ker}(p_1) \cap \operatorname{Ker}(p_2) = \mathbf{0}$ (the smallest equivalence on $\{0,1,a,c\}$), which implies that

$$\operatorname{Ker}(h \restriction F(\Omega \setminus K)) = \operatorname{Ker}(p_1 h{\restriction}F(\Omega \setminus K)) \cap \operatorname{Ker}(p_2 h{\restriction}F(\Omega \setminus K)). \tag{27}$$

Now, if $\boldsymbol{x}{\restriction}F(\Omega \setminus K) \nsubseteq \operatorname{Ker}(p_1 h{\restriction}F(\Omega \setminus K))$, then Lemma 4.3 (using the assumption $\boldsymbol{x} \subseteq \bigvee_{j=1}^{n} \boldsymbol{a}_0^{\xi_j} \vee \boldsymbol{w}_0$) implies that $p_1 h(\xi_j) = 1$ for some j. However, a direct evaluation shows that $p_1 h(\xi_j) = 0$ for every j. Hence,

$$\boldsymbol{x}{\restriction}F(\Omega \setminus K) \subseteq \operatorname{Ker}(p_1 h{\restriction}F(\Omega \setminus K)). \tag{28}$$

Similarly, if $\boldsymbol{x}{\restriction}F(\Omega \setminus K) \nsubseteq \operatorname{Ker}(p_2 h{\restriction}F(\Omega \setminus K))$, then Lemma 4.3 (with $\{\delta\}$ playing the role of $\{\xi_1, \dots, \xi_n\}$ and $\boldsymbol{w} := \mathbf{0}$, using the assumption $\boldsymbol{x} \subseteq \boldsymbol{a}_0^{\delta}$) implies that $p_2 h(\delta) = 1$, which is not true. So,

$$\boldsymbol{x}{\restriction}F(\Omega \setminus K) \subseteq \operatorname{Ker}(p_2 h{\restriction}F(\Omega \setminus K)). \tag{29}$$

Now (27), (28) and (29) imply that $\boldsymbol{x}{\restriction}F(\Omega \setminus K) \subseteq \operatorname{Ker}(h{\restriction}F(\Omega \setminus K))$, hence $\boldsymbol{x} \subseteq \operatorname{Ker}(h)$ by Lemma 4.2.

To show that $y \subseteq \operatorname{Ker}(h)$, consider the maps $q_1, q_2, q_3: \{0,1,b,c\} \to \{0,1\}$ given by the following table.

	0	b	c	1
q_1	0	0	0	1
q_2	0	0	1	1
q_3	0	1	1	1

It is easy to see that q_1, q_2, q_3 are homomorphisms defined on a subalgebra of W and $\mathrm{Ker}(q_1) \cap \mathrm{Ker}(q_2) \cap \mathrm{Ker}(q_3) = \mathbf{0}$, which implies that

$$\mathrm{Ker}(h{\restriction}F(\Omega \setminus J)) = \bigcap_{j=1}^{3} \mathrm{Ker}(q_j h{\restriction}F(\Omega \setminus J)). \tag{30}$$

If $\boldsymbol{y}{\restriction}F(\Omega \setminus J) \nsubseteq \mathrm{Ker}(q_1 h{\restriction}F(\Omega \setminus J))$, then Lemma 4.3 (using the assumption $\boldsymbol{y} \subseteq \bigvee_{j=1}^{m} \boldsymbol{a}_0^{\eta_j} \vee \boldsymbol{w}_1$) yields that $q_1 h(\eta_j) = 1$ for some j. A direct evaluation shows that this is not true, so

$$\boldsymbol{y}{\restriction}F(\Omega \setminus J) \subseteq \mathrm{Ker}(q_1 h{\restriction}F(\Omega \setminus J)). \tag{31}$$

If $\boldsymbol{y}{\restriction}F(\Omega \setminus J) \nsubseteq \mathrm{Ker}(q_k h{\restriction}F(\Omega \setminus J))$, for $k \in \{2,3\}$, then Lemma 4.3 (using the assumption $\boldsymbol{y} \subseteq \boldsymbol{a}_1^{\delta}$) yields that $q_k h(\delta) = 0$. Again, this is not true, so

$$\boldsymbol{y}{\restriction}F(\Omega \setminus J) \subseteq \mathrm{Ker}(q_2 h{\restriction}F(\Omega \setminus J)), \quad k \in \{2,3\}. \tag{32}$$

Now (30), (31) and (32) imply that $\boldsymbol{y}{\restriction}F(\Omega \setminus J) \subseteq \mathrm{Ker}(h{\restriction}F(\Omega \setminus J)$, hence $\boldsymbol{y} \subseteq \mathrm{Ker}(h)$.

B. Let $i = 1$. This case is symmetrical to Case A. The symmetry interchanges a and c, $\boldsymbol{x}$ and $\boldsymbol{y}$, J and K and also the values 0 and 1 of functions p, p_j, q_j at a, b, c. For instance, the homomorphism $h: F \to W$ needs to be defined as follows:

$$h(\alpha) = \begin{cases} b & \text{if } \alpha \in J \\ c & \text{if } \alpha \in K \\ a & \text{if } \alpha = \delta \\ f(\alpha) & \text{otherwise.} \end{cases} \tag{33}$$

The contradiction is again achieved by proving that $\boldsymbol{z} \nsubseteq \mathrm{Ker}(h)$, $\boldsymbol{x} \subseteq \mathrm{Ker}(h)$ and $\boldsymbol{y} \subseteq \mathrm{Ker}(h)$. $\square$

As a consequence, we obtain a new example, showing the negative solution of CLP.

Theorem 4.5. *The congruence lattice of the free (bounded) majority algebra with at least $\aleph_2$ generators is not isomorphic to* $\mathrm{Con}\, A$ *for any lattice A (or for any algebra A with a congruence-compatible* $(\vee, 1)$*-semilattice structure).*

In fact, our proof shows that the same result holds for the free algebra in any variety of majority algebras containing W. Also, the boundedness of the majority algebras is not essential, as the free bounded majority algebra is a homomorphic image of the free (unbounded) majority algebra.

5. A topological construction.

The construction in this section has appeared in [9] as a candidate for the negative solution of CLP. Using Theorem 3.3 we now are able to confirm this conjecture. We define our semilattice as the semilatice of all compact open subsets of a suitable topological space.

Let M denote the 5-element set $\{0,1,a,b,c\}$. Let Ω be any set. Let

$$T_\Omega = \{f \in M^\Omega \mid \text{either } f(\Omega) \subseteq \{0,1\} \text{ or } \{a,b,c\} \subseteq f(\Omega)\}.$$

For for all distinct $u,v \in \{a,b,c\}$ we define functions

$$p_0^{\{u,v\}}, p_1^{\{u,v\}} : \{0,1,u,v\} \to \{0,1\}$$

(shortly written as p_0^{uv}, p_1^{uv}) as follows:
$p_0^{uv}(0) = p_1^{uv}(0) = 0$, $p_0^{uv}(1) = p_1^{uv}(1) = 1$ for every u,v;
$p_0^{ab}(a) = p_0^{ab}(b) = 0$, $p_1^{ab}(a) = p_1^{ab}(b) = 1$;
$p_0^{bc}(b) = p_0^{bc}(c) = 0$, $p_1^{bc}(b) = p_1^{bc}(c) = 1$;
$p_0^{ac}(a) = p_1^{ac}(c) = 0$, $p_0^{ac}(c) = p_1^{ac}(a) = 1$.

Further we denote

$$S_0 = \{r : X_0 \to M \mid X_0 \subseteq \Omega \text{ is finite, } \operatorname{rng}(r) \subseteq \{0,1\}\};$$
$$S_1 = \{r : X_0 \to M \mid X_0 \subseteq \Omega \text{ is finite, } \{a,b,c\} \subseteq \operatorname{rng}(r)\}.$$

For every $r \in S_0$ let

$$\begin{aligned} K_r = \ & \{f \in M^\Omega \mid (\exists \text{ distinct } u,v \in \{a,b,c\})(f(\operatorname{dom}(r)) \subseteq \{0,1,u,v\} \\ & \text{and } (r = p_0^{uv} \cdot (f{\restriction}\operatorname{dom}(r)) \text{ or } r = p_1^{uv} \cdot (f{\restriction}\operatorname{dom}(r))))\}. \end{aligned}$$

In particular, if $f(\operatorname{dom}(r)) \subseteq \{0,1\}$ then $f \in K_r$ iff f extends r.

For every $r \in S_1$ let

$$K_r = \{f \in M^\Omega \mid f \text{ extends } r\}.$$

Finally, for every $r \in S_0 \cup S_1$ let $G_r = K_r \cap T_\Omega$, and let $\mathcal{G} = \{G_r \mid r \in S_0 \cup S_1\}$.

Lemma 5.1. (See [9].) *$\mathcal{G}$ is a basis of a topology on T_Ω. In this topology, the compact open sets are exactly the finite unions of the sets from $\mathcal{G}$.*

Thus, T_Ω has a basis of compact open sets. Let L_Ω be the family of all open subsets of T_Ω ordered by set inclusion. It is clear that L_Ω is a distributive algebraic lattice. The compact elements of L_Ω form a distributive semilattice S_Ω. Observe that the semilattice operation in S_Ω is the set-theoretical union.

Now we construct an evaporation scheme for S_Ω. For every $\alpha \in \Omega$ let

$$\boldsymbol{a}_0^\alpha = G_{\alpha\mapsto 0} = \{f \in T_\Omega \mid f(\alpha) \neq 1\},$$
$$\boldsymbol{a}_1^\alpha = G_{\alpha\mapsto 1} = \{f \in T_\Omega \mid f(\alpha) \neq 0\}.$$

By definition, $\boldsymbol{a}_0^\alpha$ and $\boldsymbol{a}_1^\alpha$ are equal to G_r for the two possible maps $r : \{\alpha\} \to \{0,1\}$ and hence they belong to S_Ω. Clearly, $\boldsymbol{a}_0^\alpha \cup \boldsymbol{a}_1^\alpha = T_\Omega$, so we have a decomposition system $\mathcal{F}$ at T_Ω (which, as the greatest element of S_Ω, will be denoted by $\mathbf{1}$).

For every $\boldsymbol{x} \in S_\Omega$ we pick a representation in the form

$$\boldsymbol{x} = G_{r_1} \cup \cdots \cup G_{r_n}$$

and we set $\operatorname{supp}(\boldsymbol{x}) = \operatorname{dom}(r_1) \cup \cdots \cup \operatorname{dom}(r_n)$. Since the validity of the relationship $f \in G_r$ only depends on the values of f on $\operatorname{dom}(r)$, we obtain the following analogue of Lemma 4.2.

Lemma 5.2. *Let $\boldsymbol{x} \in S_\Omega$ and $f, g \in T_\Omega$ with $f{\restriction}\operatorname{supp}(\boldsymbol{x}) = g{\restriction}\operatorname{supp}(\boldsymbol{x})$. Then $f \in \boldsymbol{x}$ iff $g \in \boldsymbol{x}$.*

To complete the evaporation scheme, let

$$I = \{\boldsymbol{x} \in S_\Omega \mid \{a,b,c\} \subseteq \operatorname{rng}(f) \text{ for every } f \in \boldsymbol{x}\}.$$

We also need the analogue of Lemma 4.3.

Lemma 5.3. *Let $r \in S_0$, $\boldsymbol{w} \in I$, $\xi_1, \dots, \xi_n \in \Omega$, $i \in \{0,1\}$. If $G_r \subseteq \bigcup_{j=1}^n \boldsymbol{a}_i^{\xi_j} \cup \boldsymbol{w}$, then $r(\xi_j) = i$ for some j. In particular, $G_r \subseteq \boldsymbol{a}_i^{\xi}$ implies $r(\xi) = i$.*

Proof. Define $g: \ \Omega \to \{0,1\}$ by

$$g(\alpha) = \begin{cases} r(\alpha) & \text{if } \alpha \in \operatorname{dom}(r) \\ 1-i & \text{if } \alpha \notin \operatorname{dom}(r). \end{cases}$$

Then $g \in G_r$. Since $\boldsymbol{w}$ does not contain any functions $\Omega \to \{0,1\}$, the inclusion $G_r \subseteq \bigcup_{j=1}^n \boldsymbol{a}_i^{\xi_j} \cup \boldsymbol{w}$ implies that $g \in \boldsymbol{a}_i^{\xi_j}$ for some j. Then $g(\xi_j) = i$, which, by the definition of g, is only possible if $\xi_j \in \operatorname{dom}(r)$ and $r(\xi_j) = i$. □

Theorem 5.4. *For any set Ω, $(\mathcal{F}, \operatorname{supp}, I)$ is an evaporation scheme at $\mathbf{1} \in S_\Omega$.*

Proof. For contradiction, suppose that $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in S_\Omega$, all distinct $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m, \delta \in \Omega$, $\boldsymbol{w}_0, \boldsymbol{w}_1 \in I$ and $i \in \{0,1\}$ satisfy (i)-(iii) from Definition 3.1, but not (iv).

Hence, there exists $f: \ \Omega \to \{0,1\}$, $f \in \boldsymbol{z}$. Denote

$$J = \{\xi_j \mid f(\xi_j) = i\},$$

$$K = \{\eta_k \mid f(\eta_k) = i\}.$$

Let us define $g: \ \Omega \to \{0,1\}$ by $g(\delta) = 0$ and $g(\alpha) = f(\alpha)$ for every $\alpha \neq \delta$. Since $\delta \notin \operatorname{supp}(\boldsymbol{z})$, $f \in \boldsymbol{z}$ implies $g \in \boldsymbol{z}$. From $g(\delta) = 0$ it follows that $g \notin \boldsymbol{a}_1^\delta$, thus $g \notin \boldsymbol{y}$. As $g \in \boldsymbol{z} \subseteq \boldsymbol{x} \cup \boldsymbol{y}$, we obtain that $g \in \boldsymbol{x}$, thus $g \in \bigcup_{j=1}^n \boldsymbol{a}_i^{\xi_j} \cup \boldsymbol{w}_0$. From $\boldsymbol{w}_0 \in I$ and $\operatorname{rng}(g) \subseteq \{0,1\}$ it follows that $g \notin \boldsymbol{w}_0$; hence $g \in \boldsymbol{a}_i^{\xi_j}$ for some $j \in \{1, \dots, n\}$. We have $g(\xi_j) = i$, thus $f(\xi_j) = i$ and therefore $\xi_j \in J$. This proves that J is nonempty. Similarly (using g with $g(\delta) = 1$) one can prove that $K \neq \emptyset$.

Now we separate cases according to the two possible values of i.

A. Let $i = 0$. Consider $h: \ \Omega \to M$ defined as follows:

$$h(\alpha) = \begin{cases} a & \text{if } \alpha \in J \\ b & \text{if } \alpha \in K \\ c & \text{if } \alpha = \delta \\ f(\alpha) & \text{otherwise.} \end{cases}$$

Clearly, $h \in T_\Omega$. We claim that $h \in \boldsymbol{z}$, $h \notin \boldsymbol{x}$ and $h \notin \boldsymbol{y}$, which contradicts (iii) from Definition 3.1.

By the definition of the support, there exists $G_r \subseteq \boldsymbol{z}$ such that $f \in G_r$ and $\delta \notin \operatorname{dom}(r)$. Necessarily, $r = f{\restriction}\operatorname{dom}(r)$. Since

$$p_0^{ab} h{\restriction}\operatorname{dom}(r) = f{\restriction}\operatorname{dom}(r) = r, \tag{34}$$

we obtain that $h \in G_r \subseteq \boldsymbol{z}$.

Suppose now that $h \in \boldsymbol{x}$. As $\operatorname{supp}(\boldsymbol{x}) \cap \{\eta_1, \dots, \eta_m\} = \emptyset$, there exists $r \in S_0 \cup S_1$ such that $h \in G_r \subseteq \boldsymbol{x}$ and $\operatorname{dom}(r) \cap \{\eta_1, \dots, \eta_m\} = \emptyset$. Hence, $\operatorname{dom}(r) \cap K = \emptyset$, thus $b \notin h(\operatorname{dom}(r))$, which rules out the case where $r \in S_1$. Thus, $r \in S_0$ and either

$$p_0^{ac} h{\restriction}\operatorname{dom}(r) = r \tag{35}$$

or

$$p_1^{ac} h{\restriction}\operatorname{dom}(r) = r. \tag{36}$$

By (ii) from Definition 3.1 we have $\boldsymbol{x} \subseteq \boldsymbol{a}_0^\delta$, hence $G_r \subseteq \boldsymbol{a}_0^\delta$ and, by Lemma 5.3, $r(\delta) = 0$. However, $p_0^{ac} h(\delta) = 1$, so (35) does not take place. Similarly, $\boldsymbol{x} \subseteq \bigcup_{j=1}^n \boldsymbol{a}_0^{\xi_j} \cup \boldsymbol{w}_0$ implies that $r(\xi_j) = 0$ for some j. If $\xi_j \in J$, then $p_1^{ac} h(\xi_j) =$

$p_1^{ac}(a) = 1$. If $\xi_j \notin J$, then $p_1^{ac}h(\xi_j) = f(\xi_j) = 1$. Therefore, (36) is also impossible. This contradiction proves that $h \notin \boldsymbol{x}$.

Suppose now that $h \in \boldsymbol{y}$. Then $h \in G_r \subseteq \boldsymbol{y}$ for some $G_r \in \mathcal{G}$ with $\mathrm{dom}(r) \cap J = \emptyset$. The only possibility is that $r \in S_0$ and either

$$p_0^{bc}h\restriction \mathrm{dom}(r) = r \tag{37}$$

or

$$p_1^{bc}h\restriction \mathrm{dom}(r) = r. \tag{38}$$

As above, $\boldsymbol{y} \subseteq \boldsymbol{a}_1^\delta$ implies $G_r \subseteq \boldsymbol{a}_1^\delta$, hence $r(\delta) = 1$. On the other hand, $p_0^{bc}h(\delta) = 0$, so (37) does not hold. Further, $\boldsymbol{y} \subseteq \bigcup_{k=1}^m \boldsymbol{a}_0^{\eta_k} \cup \boldsymbol{w}_1$ implies, by Lemma 5.3, that $r(\eta_k) = 0$ for some $k \in \{1, \dots, m\}$. On the other hand, we have $p_1^{bc}h(\eta_k) = p_1^{bc}(b) = 1$ for $\eta_k \in K$ and $p_1^{bc}h(\eta_k) = f(\eta_k) = 1$ for $\eta_k \notin K$. Thus, (38) is also impossible. This contradiction proves that $h \notin \boldsymbol{y}$ and completes the proof for the case $i = 0$.

B. Let $i = 1$. Consider $h :\ \Omega \to M$ defined as follows:

$$h(\alpha) = \begin{cases} b & \text{if } \alpha \in J \\ a & \text{if } \alpha \in K \\ c & \text{if } \alpha = \delta \\ f(\alpha) & \text{otherwise.} \end{cases}$$

We claim again that $h \in \boldsymbol{z}$, $h \notin \boldsymbol{x}$ and $h \notin \boldsymbol{y}$. The argument is the same as in the part A, with the roles of $\boldsymbol{x}$ and $\boldsymbol{y}$ (and of 0 and 1) interchanged. The proof is complete. □

As a consequnce we obtain the following result.

Theorem 5.5. *If $|\Omega| \geq \aleph_2$ then S_Ω is not isomorphic to $\mathrm{Con}_c A$ for any lattice A (or for any algebra A with a congruence-compatible $(\vee, 1)$-semilattice structure).*

The reader may notice a similarity between the proofs in our two examples. This is not a coincidence. The majority algebra W has been built to imitate the behaviour of our topological example. However, the two examples are not isomorphic and the proofs in Sections 4 and 5 use different basic mechanisms. The difference can be seen using the topological representation theory developed in [8]. By this theory, the lattice $\mathrm{Con}\, F$ from Section 4 is isomorphic to the open sets lattice of some topological space H_Ω, which is very similar to the space T_Ω in Section 5. Both spaces have the same underlying set and the only difference in the definition of the open sets is that H_Ω uses two additional maps $p_{01}^{ab} :\ \{0, 1, a, b\} \to \{0, 1\}$ and $p_{01}^{bc} :\ \{0, 1, b, c\} \to \{0, 1\}$ defined by $p_{01}^{ab}(a) = 0$, $p_{01}^{ab}(b) = 1$, $p_{01}^{bc}(b) = 0$, $p_{01}^{bc}(c) = 1$. This is due to the fact that there exist three different homomorphisms from the subalgebras $\{0, 1, a, b\}$ and $\{0, 1, b, c\}$ of W into $\mathbf{2}$. The consequence is that T_Ω and H_Ω have some different topological properties. For instance, H_Ω contains sequences converging to three different limit points, which cannot happen in T_Ω. Using such arguments one can argue that S_Ω and $\mathrm{Con}_c F$ are not isomorphic. Moreover, the example from Section 5 cannot be "translated" to the algebraic form. Notice that $\mathrm{Ker}(p_0^{ab}) \cap \mathrm{Ker}(p_1^{ab}) \neq \mathbf{0}$, so the proof from Section 4 does not extend to S_Ω.

On the other hand, we do not know whether S_Ω is isomorphic to $\mathrm{Con}_c A$ for some other majority algebra A. In fact, it is still an open problem whether every distributive $(\vee, 0)$-semilattice is isomorphic to $\mathrm{Con}_c A$ for some majority algebra A. (See Problem 2 in [15].)

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