CONGRUENCE LATTICES OF LATTICES WITH m-PERMUTABLE CONGRUENCES

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ABSTRACT. We prove that there exist a lattice whose congruence lattice is not isomorphic to the congruence lattice of any lattice with m-permutable congruences. Our proof also extends to a wider class of algebras with m-permutable congruences. In order to do this we use and further develop the method invented by F. Wehrung for solving Dilworth's congruence lattice problem. To minimize the cardinality of our construction, we use the free trees combinatorial principle of P. Růžička.

1. INTRODUCTION

The investigation of congruences is one of the central topics in universal algebra. It is well known that every algebraic lattice is isomorphic to the congruence lattice of some algebra. Much less is known about congruence lattices of algebras of particular type. We do not know, which lattices are isomorphic to the congruence lattices of the most common kinds of algebras, like groups, rings or lattices. In the case of lattices, a longstanding conjecture of R. P. Dilworth is also referred to as the Congruence Lattice Problem (CLP): *Is every distributive algebraic lattice isomorphic to the congruence lattice of some lattice?* After more than 60 years of effort (documented in [1](Appendix C) or [10]), the conjecture was finally disproved by F. Wehrung in [12].

In the intensive research that eventually led to Wehrung's solution, many other related problems became interesting. One of them concerns the congruence lattices of lattices (or more general algebras) with m-permutable congruences.

Problem 1.1. *(See* [12]*.) Prove that there exists a lattice K such that for every integer $m > 1$ there is no lattice L with m-permutable congruences such that* $\operatorname{Con} K \cong \operatorname{Con} L$.

In this paper we show that such a lattice K can be taken as the free algebra in any variety (equational class) of bounded lattices containing the lattice M_3 with at least $\aleph_2$ generators. The cardinality $\aleph_2$ is the lowest possible, since we know that every distributive algebraic lattice with at most $\aleph_1$ compact elements is representable as a congruence lattice of a lattice with permutable (that is, 2-permutable) congruences. (See, for instance, [2].)

Our proof is an modification of Wehrung's method for solving CLP. To achieve the cardinality of $\aleph_2$ we use Růžička's theorem about the existence of free trees from [8].

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It is worth mentioning that in the case $m = 2$ a stronger result is available. By [9], there exists a lattice K such that $\operatorname{Con} K$ is not isomorphic to $\operatorname{Con} A$ for any algebra A with permutable congruences. The example of such lattice is the same as in the present paper (and in [7]), but the method of proof is different.

2. Basic concepts

We assume familiarity with fundamentals of lattice theory and universal algebra. For all undefined concepts and unreferenced facts we refer to [1] and [5].

We do not distinguish between an algebra and its underlying set, hoping that no confusion arises. For an algebra A let $\operatorname{Con} A$ denote the congruence lattice of A. This lattice is always algebraic and its compact elements form a $\vee$-subsemilattice of $\operatorname{Con} A$, denoted $\operatorname{Con}_c A$. For $x, y \in A$ let $\theta(x, y)$ denote the smallest congruence containing the pair (x, y). The semilattice $\operatorname{Con}_c A$ consists precisely of all finitely generated congruences, i.e. congruences of the form $\theta(x_1, y_1) \vee \dots \vee \theta(x_n, y_n)$. The smallest and the largest element of $\operatorname{Con} A$ (and of any other lattice in this paper) will always be denoted by 0 and 1, respectively. The congruence 0 (the equality relation) is considered as compact, so $\operatorname{Con}_c A$ always has a smallest element.

If $\alpha \in \operatorname{Con} A$ and B is a subalgebra of A, then the restriction of α to B is the relation $\alpha \cap B^2$ and will usually be denoted by $\alpha \restriction B$. Notice that it is always a congruence of B.

Let $m > 1$ be an integer. The algebra A is called congruence m-permutable, if $\alpha \circ_m \beta = \beta \circ_m \alpha$ for every $\alpha, \beta \in \operatorname{Con} A$, where $\alpha \circ_m \beta$ denotes the relational product $\alpha \circ \beta \circ \alpha \circ \beta \dots$ ($m - 1$ occurences of $\circ$). If A is congruence m-permutable, then $\alpha \vee \beta = \alpha \circ_m \beta$ for every $\alpha, \beta \in \operatorname{Con} A$.

Although our original motivation was to solve a problem for lattices, our proof works in a more general situation. It is not necessary that A is actually a lattice. We only need that A admits a compatible lattice structure in the following sense.

Definition 2.1. ([3]) *A finitary function $f : A^n \to A$ on an algebra A is called* compatible *(or* congruence compatible*) if, for any congruence $\theta \in \operatorname{Con} A$, $(x_i, y_i) \in \theta$, $i = 1, \dots, n$, implies that $(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \in \theta$.*

For instance, the binary join operation $\vee$ is compatible if, for every $\theta \in \operatorname{Con} A$,

$$(x, y) \in \theta, \text{ and } (u, v) \in \theta \text{ imply } (x \vee u, y \vee v) \in \theta. \tag{1}$$

It is clear that every basic operation of A is compatible and, more generally, every polynomial function of A is compatible. In general, however, there are compatible functions that are not polynomial.

We say that A admits a compatible lattice structure, if there are compatible functions $\vee$ and $\wedge$ on A such that $(A, \vee, \wedge)$ is a lattice. This structure induces a natural order relation $\leq$ on A.

Lemma 2.2. *Let the algebra A admit a compatible lattice structure. Suppose that $1 \in \operatorname{Con}_c A$. Then*

(i) *$1 = \theta(u, v)$ for some $u, v \in A$, $u \leq v$;*

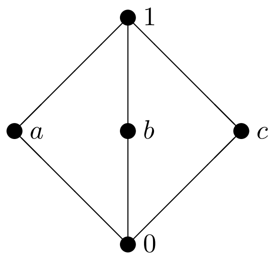
(ii) *If A is congruence m-permutable and $\alpha \vee \beta = 1$ ($\alpha, \beta \in \operatorname{Con} A$), then there exist $z_0, \dots, z_m \in A$ such that $v = z_0 \geq z_1 \geq \dots \geq z_m = u$, $(z_k, z_{k+1}) \in \alpha$ for all even k and $(z_k, z_{k+1}) \in \beta$ for all odd k.*

Proof. By our assumption, $1 = \theta(x_1, y_1) \vee \cdots \vee \theta(x_n, y_n)$ for some $x_k, y_k \in A$ ($k = 1, \ldots, n$). Since A is a lattice, there are elements $u, v \in A$ such that $u \leq x_k \leq v$, $u \leq y_k \leq v$ for every k. Since $(x_k, v) = (x_k \vee u, x_k \vee v)$ and $(v, y_k) = (v \vee u, y_k \vee u)$, the compatibility of $\vee$ implies that $(x_k, v), (v, y_k) \in \theta(u, v)$, hence $(x_k, y_k) \in \theta(u, v)$ for every k, and consequently, $\theta(u, v) = 1$.

If $\alpha \vee \beta = 1$, then $(v, u) \in \alpha \vee \beta$, and the m-permutability yields that $(v, u) \in \alpha \circ_m \beta$. By the definition of the relational product, there are $w_0 = v, w_1, \ldots, w_m = u$ such that $(w_k, w_{k+1}) \in \alpha$ for every even k and $(w_k, w_{k+1}) \in \beta$ for every odd k. We set $z_k = (w_k \vee w_{k+1} \vee \cdots \vee w_m) \wedge v$ ($k = 0, \ldots, m$). For k even $(w_k, w_{k+1}) \in \alpha$ implies that $(z_k, z_{k+1}) = ((\bigvee_{i=k+1}^m w_i \vee w_k) \wedge v, (\bigvee_{i=k+1}^m w_i \vee w_{k+1}) \wedge v) \in \alpha$, and similarly for k odd.

$\square$

The lattice M_3 depicted below will play an essential role in our considerations. (Also the denotation of its elements is important.) This lattice generates the variety (equational class) $\mathcal{M}_3$. The bounded members of $\mathcal{M}_3$ also form a variety provided that we consider 0 and 1 as nullary basic operations. This variety will be denoted by $\mathcal{M}_3^{01}$.



M_3

We use standard set-theoretic notation. We identify a natural number n with the set $\{0, 1, \ldots, n-1\}$. The least infinite ordinal is denoted ω. If Ω is a set then $[\Omega]^n$ denotes the family of all n-element subsets of Ω, while $[\Omega]^{<\omega}$ stands for the family of all finite subsets of Ω.

If $f : A \to B$ is a map, then we define its kernel as the relation $\operatorname{Ker} f = \{(x, y) \in A^2 \mid f(x) = f(y)\}$. If f is a homomorphism of algebras, then $\operatorname{Ker} f \in \operatorname{Con} A$.

3. Free trees

Let k be a positive integer and X a set. For a map $\Phi : [\Omega]^{k-1} \to [\Omega]^{<\omega}$ we say that a k-element set $B \subseteq \Omega$ is *free with respect to* Φ if $b \notin \Phi(B \setminus \{b\})$ for all $b \in B$.

The following statement of infinite combinatorics is a one direction of a theorem due to K. Kuratowski [4]

Theorem 3.1. *Let Ω be a set of cardinality at least $\aleph_{k-1}$. Then for every map $\Phi : [\Omega]^{k-1} \to [\Omega]^{<\omega}$ there is a k-element set free set $B \subseteq \Omega$.*

The special case of this principle (for $k = 3$) has been used in several papers ([11], [7], [6] and others) to prove negative results concerning the representability of distributive algebraic lattices as congruence lattices of algebras. The general Kuratowski's theorem played an important role in recent solution of Congruence

Lattice Problem by Wehrung ([12]). For our purpose we need a modification of this principle, recently discovered by Růžička ([8]).

Let m, n, k be natural numbers with $k > 0$, $m \leq n$ and let $g : \{m, \dots, n-1\} \to k$ be a map. We denote

$$T_{n,k}(g) = \{f : n \to k \mid f \text{ extends } g\}. \tag{2}$$

If $0 < m$ and $i \in \{0, \dots, k-1\}$ then we also use

$$T_{n,k}(g, i) = \{f \in T_{n,k}(g) \mid f(m-1) = i\}, \tag{3}$$

$$T_{n,k}(g, \neg i) = \{f \in T_{n,k}(g) \mid f(m-1) \neq i\}, \tag{4}$$

Definition 3.2. ([8]) *Let Ω be a set and let $\Phi : [\Omega]^{<\omega} \to [\Omega]^{<\omega}$ be a map. Let k and n be natural numbers with $k > 0$. We say that a family $\mathcal{T} = (\alpha(f) \mid f : n \to k)$ with all $\alpha(f) \in \Omega$ is a* free k-tree of height n with respect to Φ *if*

$$\{\alpha(f) \mid f \in T_{n,k}(g, i)\} \cap \Phi(\{\alpha(f) \mid f \in T_{n,k}(g, \neg i)\}) = \emptyset, \tag{5}$$

for every $0 < m \leq n$, every $g : \{m, \dots, n-1\} \to k$ and every $i \in k$.

Theorem 3.3. ([8]) *Let Ω be a set of cardinality at least $\aleph_{k-1}$. Then for every map $\Phi : [\Omega]^{<\omega} \to [\Omega]^{<\omega}$ and every natural n there is a free k-tree of height n with respect to Φ.*

Notice that free k-trees of height 1 are just the free sets, so 3.3 generalizes 3.1

4. Main result

Throughout this section we fix a positive integer m. Let F be the free algebra (lattice) in the variety $\mathcal{M}_3^{01}$ with X as the set of free generators. We assume that $|X| \geq \aleph_2$ and choose a special denotation for the elements of X, depending on m, namely $X = \{x_k^\xi \mid k \in \{1, ..2m\},\ \xi \in \Omega\}$, where Ω is some set of cardinality at least $\aleph_2$. Formally we set $x_0^\xi = 0$, $x_{2m+1}^\xi = 1$ for every $\xi \in \Omega$.

Now we define the congruences $\boldsymbol{a}_i^\xi \in \operatorname{Con}_c F$ $(i = 0, 1)$ as

$$\boldsymbol{a}_0^\xi = \theta(x_0^\xi, x_1^\xi) \vee \theta(x_2^\xi, x_3^\xi) \vee \dots \vee \theta(x_{2m}^\xi, x_{2m+1}^\xi);$$

$$\boldsymbol{a}_1^\xi = \theta(x_1^\xi, x_2^\xi) \vee \theta(x_3^\xi, x_4^\xi) \vee \dots \vee \theta(x_{2m-1}^\xi, x_{2m}^\xi).$$

Clearly, $\boldsymbol{a}_0^\xi \vee \boldsymbol{a}_1^\xi = 1$ for every ξ, since this congruence collapses 0 and 1.

For $Y \subseteq \Omega$ let $F(Y)$ denote the subalgebra of F generated by all elements x_k^ξ, $\xi \in Y$, $k \in \{0, \dots, 2m+1\}$.

Every $\psi \in \operatorname{Con}_c F$ is generated by a finite subset of F^2 and every element of F belongs to $F(Z)$ for some finite set $Z \subseteq \Omega$. Hence, for every $\psi \in \operatorname{Con}_c F$ there exist a finite set $Y \subseteq \Omega$ such that ψ is generated by $\psi \restriction F(Y)$. We pick such a set for every ψ, call it the support, and denote it by $\operatorname{supp}(\psi)$. (We do not require any kind of mimimality, just the finiteness.) The importance of the support lies in the following, rather trivial, observation, which will be frequently used in the sequel.

Lemma 4.1. *Let $\psi \in \operatorname{Con}_c F$, $\varphi \in \operatorname{Con} F$, $\operatorname{supp}(\psi) \subseteq Y \subseteq \Omega$. Then $\psi \subseteq \varphi$ if and only if $\psi \restriction F(Y) \subseteq \varphi \restriction F(Y)$.*

Further, for every $n = 0, \ldots, 2m$, let $f_n : F \to \{0, 1\}$ be the unique bounded lattice homomorphism determined by the condition

$$f_n(x_i^\xi) = \begin{cases} 0 & \text{if } i \in \{1, \ldots, 2m - n\} \\ 1 & \text{if } i \in \{2m - n + 1, \ldots, 2m\} \end{cases}$$

We have the following easy assertion.

Lemma 4.2. *Let $\xi \in \Omega$, $n \in \{0, \ldots, 2m\}$, $i \in \{0, 1\}$. Suppose that p is a homomorphism $F \to \{0, 1\}$ which coincides with f_n on the set $\{x_k^\xi \mid k = 1, \ldots, 2m\}$. Then $\boldsymbol{a}_i^\xi \subseteq \operatorname{Ker} p$ iff $i + n$ is odd.*

Proof. If n is odd then $p(x_{2k}^\xi) = p(x_{2k+1}^\xi)$ for every $k = 0, \ldots, m$, hence $(x_{2k}^\xi, x_{2k+1}^\xi)$ belongs to $\operatorname{Ker} p$ and therefore $\boldsymbol{a}_0^\xi \subseteq \operatorname{Ker} p$. On the other hand, $p(x_{2m-n}^\xi) \neq p(x_{2m-n+1}^\xi)$, hence $(x_{2m-n}^\xi, x_{2m-n+1}^\xi) \in \boldsymbol{a}_1^\xi \setminus \operatorname{Ker} p$, showing $\boldsymbol{a}_1^\xi \not\subseteq \operatorname{Ker} p$. The proof for n even is similar. $\square$

The next assertion contains the substantial part of our proof. Let ε be the parity function, i.e. $\varepsilon(k) = 0$ for k even and $\varepsilon(k) = 1$ for j odd.

Lemma 4.3. *Let A be an algebra which admits a congruence preserving semilattice operation $\vee$ (which induces the order relation $\leq$). Suppose that $\mu : \operatorname{Con}_c A \to \operatorname{Con}_c F$ is an isomorphism. Let $u, v \in A$ be such that for every $\xi \in \Omega$ there are elements $u = z_m^\xi \leq z_{m-1}^\xi \leq \cdots \leq z_0^\xi = v$ such that*

$$\mu\theta(z_k^\xi, z_{k+1}^\xi) \subseteq \boldsymbol{a}_{\varepsilon(k)}^\xi \tag{6}$$

for every $k = 0, \ldots, m - 1$. Then

$$\mu\theta(v, u) \subseteq \operatorname{Ker} f_m. \tag{7}$$

Proof. If Y is a finite subset of Ω and $k \in \{0, \ldots, m - 1\}$, then clearly

$$\theta\left(\bigvee_{\xi \in Y} z_k^\xi, \bigvee_{\xi \in Y} z_{k+1}^\xi\right) \subseteq \bigvee_{\xi \in Y} \theta(z_k^\xi, z_{k+1}^\xi). \tag{8}$$

From (6) we obtain

$$\mu\theta\left(\bigvee_{\xi \in Y} z_k^\xi, \bigvee_{\xi \in Y} z_{k+1}^\xi\right) \subseteq \bigvee_{\xi \in Y} \boldsymbol{a}_{\varepsilon(k)}^\xi \tag{9}$$

for every $k = 0, \ldots, m - 1$.

For $Y \subseteq \Omega$ let $S(Y)$ be the $\vee$-subsemilattice of A generated by all elements z_k^ξ with $\xi \in Y$, $0 \leq k \leq m$. Notice that $S(Y)$ is finite whenever Y is finite.

Now we define a map $\Phi : [\Omega]^{<\omega} \to [\Omega]^{<\omega}$ by

$$\Phi(Y) = Y \cup \bigcup \{\operatorname{supp}(\mu(\theta(x_1, y_1) \vee \cdots \vee \theta(x_l, y_l))) \mid l \in \omega,\ x_i, y_i \in S(Y)\}. \tag{10}$$

Hence, we take all compact congruences on A generated by elements of $S(Y)$ (there are finitely many of them) and unite the supports of their μ-images with Y.

Since $|\Omega| \geq \aleph_2$, by 3.3 there exists a free 3-tree $\mathcal{T} = \{\alpha(f) \mid f : m \to 3\}$ of height m with respect to Φ.

Note that the inclusion $Y \subseteq \Phi(Y)$ ensures the injectivity of α. Indeed, let $f_1, f_2 : m \to 3$ be different. Let $j = \max\{k \in m \mid f_1(k) \neq f_2(k)\}$ and let g be the restriction of f_1 (and f_2) to $\{j+1,\dots,m\}$. Then (5) for this g implies that $\alpha(f_1)\neq\alpha(f_2)$.

Our proof will be completed by the following claim:

Claim. *Let $j\in\{0,\dots,m\}$, $g:\ \{j,\dots,m-1\}\to\{0,1\}$ and $n\in\{j,j+1,\dots,2m-j\}$. Let $t:\ F\to\{0,1\}$ be a homomorphism such that $t(x_k^{\alpha(f)})=f_n(x_k^{\alpha(f)})$ for every $f\in T_{m,3}(g)$ and every $k=1,\dots,2m$. Then*

$$\mu\theta\left(v,\bigvee\{z_j^{\alpha(f)}\mid f\in T_{m,2}(g)\}\right)\subseteq \operatorname{Ker} t. \tag{11}$$

Indeed, for $j=m$ (which means that g is the empty map), $n=m$ and $t=f_m$ we have $z_m^{\alpha(f)}=u$ for every f, so the above claim says that $\mu\theta(v,u)\subseteq\operatorname{Ker} f_m$.

It remains to prove the Claim. We proceed by induction on j. The statement is trivial for $j=0$, since $z_0^{\xi}=v$ for every $\xi\in\Omega$ and $\theta(v,v)=0$.

Suppose now that j,g,n,t satisfy the assumptions of the claim, $0<j$. Denote

$$x_0=\bigvee\{z_j^{\alpha(f)}\mid f\in T_{m,2}(g,0)\}; \tag{12}$$

$$x_1=\bigvee\{z_j^{\alpha(f)}\mid f\in T_{m,2}(g,1)\}. \tag{13}$$

Choose any $h\in T_{m,3}(g,2)$ and define elements $\boldsymbol{u}_0,\boldsymbol{u}_1\in\operatorname{Con}_c F$ as follows.

$$\boldsymbol{u}_0=\mu\left(\bigvee\left\{\theta(x_0\vee z_l^{\alpha(h)},x_0\vee z_{l+1}^{\alpha(h)})\mid 0\le l<m,\ l\text{ is even}\right\}\right); \tag{14}$$

$$\boldsymbol{u}_1=\mu\left(\bigvee\left\{\theta(x_1\vee z_l^{\alpha(h)},x_1\vee z_{l+1}^{\alpha(h)})\mid 0< l<m,\ l\text{ is odd}\right\}\right). \tag{15}$$

The construction of $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ comes from the "Erosion Lemma" of [12], which plays a central role in Wehrung's proof. In the next few lines we repeat (for the convenience of the reader) Wehrung's arguments showing the essential facts about $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$.

Since $x_0\vee z_k^{\alpha(h)}\in S(Y)$, where $Y=\{\alpha(f)\mid f\in T_{m,3}(g,\neg 1)\}$, we obtain that $\operatorname{supp}(\boldsymbol{u}_0)\subseteq\Phi(\{\alpha(f)\mid f\in T_{m,3}(g,\neg 1)\})$. The freeness of the tree $\mathcal{T}$ implies that

$$\operatorname{supp}(\boldsymbol{u}_0)\cap\{\alpha(f)\mid f\in T_{m,3}(g,1)\}=\emptyset. \tag{16}$$

For similar reasons,

$$\operatorname{supp}(\boldsymbol{u}_1)\cap\{\alpha(f)\mid f\in T_{m,3}(g,0)\}=\emptyset. \tag{17}$$

Further, $\mu\theta(z_k^{\alpha(h)},z_{k+1}^{\alpha(h)})\subseteq\boldsymbol{a}_0^{\alpha(h)}$ for every even k, which implies that $\mu\theta(x_0\vee z_k^{\alpha(h)},x_0\vee z_{k+1}^{\alpha(h)})\subseteq\boldsymbol{a}_0^{\alpha(h)}$. Consequently,

$$\boldsymbol{u}_0\subseteq\boldsymbol{a}_0^{\alpha(h)},\ \text{and similarly, }\boldsymbol{u}_1\subseteq\boldsymbol{a}_1^{\alpha(h)}. \tag{18}$$

The compatibility of $\vee$ implies that

$$\theta(x_0\vee z_k^{\alpha(h)},x_0\vee z_{k+1}^{\alpha(h)})\supseteq\theta(x_0\vee x_1\vee z_k^{\alpha(h)},x_0\vee x_1\vee z_{k+1}^{\alpha(h)}) \tag{19}$$

and also

$$\theta(x_1\vee z_k^{\alpha(h)},x_1\vee z_{k+1}^{\alpha(h)})\supseteq\theta(x_0\vee x_1\vee z_k^{\alpha(h)},x_0\vee x_1\vee z_{k+1}^{\alpha(h)}), \tag{20}$$

for every k. From the definition of $\boldsymbol{u}_0$, $\boldsymbol{u}_1$ we obtain that

$$(21)\quad \boldsymbol{u}_0 \vee \boldsymbol{u}_1 \supseteq \bigvee_{k=0}^{m-1} \mu\theta(x_0 \vee x_1 \vee z_k^{\alpha(h)}, x_0 \vee x_1 \vee z_{k+1}^{\alpha(h)}) \supseteq$$
$$\supseteq \mu\theta(x_0 \vee x_1 \vee z_0^{\alpha(h)}, x_0 \vee x_1 \vee z_m^{\alpha(h)}) = \mu\theta(v, x_0 \vee x_1).$$

Now we define a special homomorphism $r : F \to M_3$. Since F is free, it is sufficient to give its value on the free generators of F. The rules are as follows.

(R1) $r(x_{2m-n+1}^{\alpha(f)}) = a$ for every $f \in T_{m,3}(g, 0)$;
(R2) $r(x_{2m-n}^{\alpha(f)}) = b$ for every $f \in T_{m,3}(g, 1)$;
(R3) $r(x_{2m-n}^{\alpha(h)}) = c$ if j is even and $r(x_{2m-n+1}^{\alpha(h)}) = c$ if j is odd;
(R4) $r(x_k^{\xi}) = t(x_k^{\xi})$ in all other cases.

The values of t, r and the functions $r_1, \dots, r_4$ (which will appear later) are displayed on the following table.

	$T_{m,3}(g,0)$	$T_{m,3}(g,1)$	h for even j	h for odd j
t	$0\ldots 00\overbrace{11\ldots 1}^{n}$	$0\ldots 00\overbrace{11\ldots 1}^{n}$	$0\ldots 00\overbrace{11\ldots 1}^{n}$	$0\ldots 00\overbrace{11\ldots 1}^{n}$
r	$0\ldots 00a1\ldots 1$	$0\ldots 0b11\ldots 1$	$0\ldots 0c11\ldots 1$	$0\ldots 00c1\ldots 1$
r_1	$0\ldots 0001\ldots 1$	any	$0\ldots 0111\ldots 1$	$0\ldots 0011\ldots 1$
r_2	$0\ldots 0011\ldots 1$	any	$0\ldots 0011\ldots 1$	$0\ldots 0001\ldots 1$
r_3	any	$0\ldots 0011\ldots 1$	$0\ldots 0111\ldots 1$	$0\ldots 0011\ldots 1$
r_4	any	$0\ldots 0111\ldots 1$	$0\ldots 0011\ldots 1$	$0\ldots 0001\ldots 1$

Each entry represents the values of a particular function in $x_1^{\xi}, \dots, x_{2m}^{\xi}$, where ξ is specified in the first line. (For instance, $T_{m,3}(g, 0)$ means "every $\xi = \alpha(f)$ with $f \in T_{m,3}(g, 0)$.")

Let $p : \{a, b, 0, 1\} \to \{0, 1\}$ be the lattice homomorphism defined on a sublattice of M_3 by $p(0) = p(b) = 0$, $p(a) = p(1) = 1$. It is easy to check that

$$(22)\qquad pr \restriction F(\Omega \setminus \{\alpha(h)\}) = t \restriction F(\Omega \setminus \{\alpha(h)\}).$$

This implies that

$$(23)\qquad \operatorname{Ker} t \restriction F(\Omega \setminus \{\alpha(h)\}) \supseteq \operatorname{Ker} r \restriction F(\Omega \setminus \{\alpha(h)\}).$$

We wish to prove that $\mu\theta(x_0 \vee x_1, v) \subseteq \operatorname{Ker} t$. Assume for contradiction that $\mu\theta(x_0 \vee x_1, v) \not\subseteq \operatorname{Ker} t$. Since $x_0, x_1, v \in S(\{\alpha(f) \mid f \in T_{m,3}(g, \neg 2)\})$, we have $\operatorname{supp} \mu\theta(x_0 \vee x_1, v) \subseteq \Phi(\{\alpha(f) \mid f \in T_{m,3}(g, \neg 2)\}$. The freeness of $\mathcal{T}$ implies that $\alpha(h) \notin \operatorname{supp}(\mu\theta(x_0 \vee x_1, v)$, by Lemma 4.1 we obtain that

$$(24)\qquad \mu\theta(x_0 \vee x_1, v) \restriction F(\Omega \setminus \{\alpha(h)\}) \not\subseteq \operatorname{Ker} t \restriction F(\Omega \setminus \{\alpha(h)\}).$$

Then also

$$(25)\qquad \mu\theta(x_0 \vee x_1, v) \restriction F(\Omega \setminus \{\alpha(h)\}) \not\subseteq \operatorname{Ker} r \restriction F(\Omega \setminus \{\alpha(h)\}),$$

hence $\mu\theta(x_0 \vee x_1, v) \not\subseteq \operatorname{Ker} r$, which implies by (21) that $\boldsymbol{u}_0 \vee \boldsymbol{u}_1 \not\subseteq \operatorname{Ker} r$. Thus, $\boldsymbol{u}_0 \not\subseteq \operatorname{Ker} r$ or $\boldsymbol{u}_1 \not\subseteq \operatorname{Ker} r$. We show that both cases lead to a contradiction.

Case A. Let $\boldsymbol{u}_0 \not\subseteq \operatorname{Ker} r$. Denote $U = \Omega \setminus \{\alpha(f) \mid f \in T_{m,3}(g, 1)\}$. By (16), $\operatorname{supp}(\boldsymbol{u}_0) \subseteq U$, hence

$$(26)\qquad \boldsymbol{u}_0 \restriction F(U) \not\subseteq \operatorname{Ker} r \restriction F(U).$$

The restriction $r \restriction F(U)$ is a homomorphism $F(U) \to \{0,1,a,c\} \subseteq M_3$. There are projections $p_1, p_2 : \{0,1,a,c\} \to \{0,1\}$ defined by $p_1(a) = p_1(0) = 0$, $p_1(c) = p_1(1) = 1$ and $p_2(a) = p_2(1) = 1$, $p_2(c) = p_2(0) = 0$. Let $r_i : F \to \{0,1\}$ $(i = 1,2)$ be any homomorphism satisfying $r_i(x_k^\xi) = p_i(r(x_k^\xi))$ for every k and every $\xi \in U$. (Choose $r_i(x_k^\xi)$ for $\xi \notin U$ arbitrarily.) Then $r_i \restriction F(U) = p_i r \restriction F(U)$ $(i = 1,2)$. Since $\operatorname{Ker} p_1 \cap \operatorname{Ker} p_2 = 0$ (the smallest congruence of $\{0,1,a,c\}$), we obtain that

$$\operatorname{Ker} r \restriction F(U) = \operatorname{Ker} r_1 \restriction F(U) \cap \operatorname{Ker} r_2 \restriction F(U). \tag{27}$$

This leads to two subcases: $\boldsymbol{u}_0 \nsubseteq \operatorname{Ker} r_1$ or $\boldsymbol{u}_0 \nsubseteq \operatorname{Ker} r_2$. The subcases are very similar, so we deal with them simultaneously.

Let g^+ be the extension of g to $\{j-1, \dots, m-1\}$ by setting $g^+(j-1) = 0$. Denote $A_0 = T_{m,2}(g^+) = T_{m,2}(g,0)$. Since t coincides with f_n on the set $\{x_k^{\alpha(f)} \mid f \in T_{m,3}(g,0) = T_{m,3}(g^+)\}$, the homomorphism r_1 coincides with f_{n-1} on this set. The induction hypothesis for $j-1$, g^+, $n-1$ and r_1 yields that

$$\mu\theta\left(v, \bigvee_{f\in A_0} z_{j-1}^{\alpha(f)}\right) \subseteq \operatorname{Ker} r_1. \tag{28}$$

Similarly, r_2 coincides on the same set with f_n, hence

$$\mu\theta\left(v, \bigvee_{f\in A_0} z_{j-1}^{\alpha(f)}\right) \subseteq \operatorname{Ker} r_2. \tag{29}$$

Further, $x_0 \le x_0 \vee z_k^{\alpha(h)} \le v$ for every k implies that $\boldsymbol{u}_0 \subseteq \mu\theta(v, x_0)$. Then

$$\boldsymbol{u}_0 \subseteq \mu\theta\left(\bigvee_{f\in A_0} z_{j-1}^{\alpha(f)}, \bigvee_{f\in A_0} z_j^{\alpha(f)}\right) \vee \mu\theta\left(v, \bigvee_{f\in A_0} z_{j-1}^{\alpha(f)}\right). \tag{30}$$

From (9) we obtain

$$\boldsymbol{u}_0 \subseteq \bigvee_{f\in A_0} \boldsymbol{a}_{\varepsilon(j-1)}^{\alpha(f)} \vee \mu\theta\left(v, \bigvee_{f\in A_0} z_{j-1}^{\alpha(f)}\right). \tag{31}$$

Now (28) and (29) together with (31) imply that

$$\text{if} \quad \boldsymbol{u}_0 \nsubseteq \operatorname{Ker} r_i \quad \text{then} \quad \boldsymbol{a}_{\varepsilon(j-1)}^{\alpha(f)} \nsubseteq \operatorname{Ker} r_i \tag{32}$$

for some $f \in A_0$ $(i \in \{1,2\})$.

On the other hand, $\boldsymbol{u}_0 \subseteq \boldsymbol{a}_0^{\alpha(h)}$ implies that, for $i \in \{1,2\}$,

$$\text{if} \quad \boldsymbol{u}_0 \nsubseteq \operatorname{Ker} r_i \quad \text{then} \quad \boldsymbol{a}_0^{\alpha(h)} \nsubseteq \operatorname{Ker} r_i \tag{33}$$

Now we argue that (32) and (33) are incompatible. By Lemma 4.2, the satisfaction of $\boldsymbol{a}_{\varepsilon(j-1)}^{\alpha(f)} \nsubseteq \operatorname{Ker} r_i$ and $\boldsymbol{a}_0^{\alpha(h)} \nsubseteq \operatorname{Ker} r_i$ can be easily checked by looking at the values $r_i(x_k^{\alpha(f)})$ and $r_i(x_k^{\alpha(h)})$ $(k = 1, \dots, 2m)$. The results of this checking depend on the parity of j and n. The function r_1 coincides with f_{n-1} on $\{z_k^{\alpha(f)} \mid k = 1, \dots, 2m\}$ (for any $f \in A_0$) and with $f_{n+\varepsilon(j-1)}$ on $\{z_k^{\alpha(h)} \mid k = 1, \dots, 2m\}$. In the case that $\boldsymbol{u}_0 \nsubseteq \operatorname{Ker} r_1$ we obtain that (32) holds iff $j-1+n-1$ is odd, while (33) holds iff $0+n+j-1$ is odd. Clearly, these requirements are incompatible. Similarly, the function r_2 coincides with f_n on $\{z_k^{\alpha(f)} \mid k = 1, \dots, 2m\}$ (for any $f \in A_0$) and with $f_{n-\varepsilon(j)}$ on $\{z_k^{\alpha(h)} \mid k = 1, \dots, 2m\}$. Hence, in the case $\boldsymbol{u}_0 \nsubseteq \operatorname{Ker} r_2$ we have that (32) holds iff $j-1+n$ is odd, while (33) requires $0+n-j$ odd, a contradiction.

Case B. Let $\boldsymbol{u}_1 \nsubseteq \operatorname{Ker} r$. Denote $U = \Omega \setminus \{\alpha(f) \mid f \in T_{m,3}(g,0)\}$. By (17), $\operatorname{supp}(\boldsymbol{u}_1) \subseteq U$, hence

$$\boldsymbol{u}_1 \restriction F(U) \nsubseteq \operatorname{Ker} r \restriction F(U). \tag{34}$$

The restriction $r \restriction F(U)$ is a homomorphism $F(U) \to \{0,1,b,c\} \subseteq M_3$. We consider the projections $p_3, p_4 : \{0,1,b,c\} \to \{0,1\}$, defined by $p_3(b) = p_3(0) = 0$, $p_3(c) = p_3(1) = 1$ and $p_4(b) = p_4(1) = 1$, $p_4(c) = p_4(0) = 0$. Let $r_i : F \to \{0,1\}$ ($i = 3,4$) be any homomorphism satisfying $r_i(x_k^\xi) = p_i(r(x_k^\xi))$ for every k and every $\xi \in U$. (Choose $r_i(x_k^\xi)$ for $\xi \notin U$ arbitrarily.) Then $r_i \restriction F(U) = p_i r \restriction F(U)$ ($i = 3,4$). Since $\operatorname{Ker} p_3 \cap \operatorname{Ker} p_4 = 0$ (the smallest congruence of $\{0,1,b,c\}$), we obtain that

$$\operatorname{Ker} r \restriction F(U) = \operatorname{Ker} r_3 \restriction F(U) \cap \operatorname{Ker} r_4 \restriction F(U). \tag{35}$$

This leads to two subcases: $\boldsymbol{u}_1 \nsubseteq \operatorname{Ker} r_3$ or $\boldsymbol{u}_1 \nsubseteq \operatorname{Ker} r_4$. Their discussion is similar to the Case A. We extend g to g^+ by setting $g^+(j-1) = 1$ and denote $A_1 = T_{m,2}(g^+) = T_{m,2}(g,1)$. Instead of (31) we have (by the same argument)

$$\boldsymbol{u}_1 \subseteq \bigvee_{f \in A_1} \boldsymbol{a}_{\varepsilon(j-1)}^{\alpha(f)} \vee \mu\theta \left(v, \bigvee_{f \in A_1} z_{j-1}^{\alpha(f)} \right). \tag{36}$$

On the set $\{x_k^{\alpha(f)} \mid f \in T_{m,3}(g,1) = T_{m,3}(g^+)\}$ the homomorphism r_3 coincides with f_n and r_4 with f_{n+1}. The induction hypothesis yields that

$$\mu\theta \left(v, \bigvee_{f \in A_1} z_{j-1}^{\alpha(f)} \right) \subseteq \operatorname{Ker} r_i \tag{37}$$

(for $i \in \{3,4\}$). Now (37) together with (36) imply that

$$\text{if} \quad \boldsymbol{u}_1 \nsubseteq \operatorname{Ker} r_i \quad \text{then} \quad \boldsymbol{a}_{\varepsilon(j-1)}^{\alpha(f)} \nsubseteq \operatorname{Ker} r_i \tag{38}$$

for some $f \in A_1$ ($i \in \{3,4\}$).

On the other hand, $\boldsymbol{u}_1 \subseteq \boldsymbol{a}_1^{\alpha(h)}$ implies that, for $i \in \{3,4\}$,

$$\text{if} \quad \boldsymbol{u}_1 \nsubseteq \operatorname{Ker} r_i \quad \text{then} \quad \boldsymbol{a}_1^{\alpha(h)} \nsubseteq \operatorname{Ker} r_i \tag{39}$$

Again we claim that (38) and (39) are incompatible. On the set $\{z_k^{\alpha(h)} \mid k = 1, \dots, 2m\}$ the function r_3 coincides with $f_{n+\varepsilon(j-1)}$ and r_4 with $f_{n-\varepsilon(j)}$. In the case $\boldsymbol{u}_1 \nsubseteq \operatorname{Ker} r_3$ the condition (38) implies that $j-1+n$ is odd, while (39) requires $1+n+j-1$ odd, which is impossible. Finally, if $\boldsymbol{u}_1 \nsubseteq \operatorname{Ker} r_4$, then (38) needs $j-1+n+1$ odd, while (39) needs $1+n-j$ odd. This contradiction completes the proof. □

Now we can deduce our main results.

Theorem 4.4. *Let L be an algebra with m-permutable congruences, which admits a compatible lattice structure. Then $\operatorname{Con}_c L$ is not isomorphic to $\operatorname{Con}_c F$. (And consequently, $\operatorname{Con} L$ is not isomorphic to $\operatorname{Con} F$.)*

Proof. For contradiction, suppose that $\mu : \operatorname{Con}_c L \to \operatorname{Con}_c F$ is an isomorphism. Since $\operatorname{Con}_c F$ has a largest element, $\operatorname{Con}_c L$ must have a largest element too, which means that the largest congruence of L is compact. Then there are $u, v \in L$, $u \le v$, such that $\theta(u, v) = 1 \in \operatorname{Con} L$.

For every $\xi \in \Omega$ we have $\mu^{-1}(\boldsymbol{a}_0^\xi) \vee \mu^{-1}(\boldsymbol{a}_1^\xi) = 1$. Since L is m-permutable, by 2.2 there are $u = z_m^\xi \le z_{m-1}^\xi \le \cdots \le z_0^\xi = v$ such that $(z_k^\xi, z_{k+1}^\xi) \in \mu^{-1}(\boldsymbol{a}_{\varepsilon(k)}^\xi)$ for every k. This clearly implies (6), so by 4.3, $\mu\theta(v, u) \subseteq \operatorname{Ker} f_m$. This is a contradiction, because $\operatorname{Ker} f_m \ne 1$. □

The above theorem applies to lattices, lattice ordered algebras, and other algebras that admit a compatible lattice structure. However, it is easy to observe that the meet operation in 2.2 has only been used to ensure that $z_k \le v$. If there is a largest element 1 with respect to the order induced by the join operation, it can play the role of v and one semilattice operation is sufficient. We say that an algebra A admits a compatible $(\vee, 1)$-semilattice structure if there is $1 \in A$ and a compatible binary operation $\vee$ on A such that $(A, \vee)$ is a semilattice with the largest element 1.

Theorem 4.5. *Let A be a an algebra with m-permutable congruences, which admits a compatible $(\vee, 1)$-semilattice structure. Then* $\operatorname{Con} A$ *is not isomorphic to* $\operatorname{Con} F$.

Proof. Suppose that $\mu : \operatorname{Con}_c A \to \operatorname{Con}_c F$ is an isomorphism. Again, $\operatorname{Con}_c A$ must have a largest element, which means that there are $u_i, v_i \in A$, $i = 1, \dots, n$, such that $\bigvee_{i=1}^n \theta(u_i, v_i) = 1 \in \operatorname{Con} A$. We denote $X = \{u_1, v_1, \dots, u_n, v_n\}$. Then clearly $\bigvee\{\theta(1, x) \mid x \in X\} = 1$.

Now we modify the proof of 2.2. For every $x \in X$ and every $\xi \in \Omega$ we have $(1, x) \in \mu^{-1}(\boldsymbol{a}_0^\xi) \vee \mu^{-1}(\boldsymbol{a}_1^\xi) = 1$. Since A is m-permutable, there are $w_0 = 1, w_1, \dots, w_m = x$ such that $(w_k, w_{k+1}) \in \mu^{-1}(\boldsymbol{a}_{\varepsilon(k)}^\xi)$ for every k. We set $z_k = (w_k \vee w_{k+1} \vee \cdots \vee w_m)$ $(k = 0, \dots, m)$. Now $(w_k, w_{k+1}) \in \mu^{-1}(\boldsymbol{a}_{\varepsilon(k)}^\xi)$ implies that $(z_k, z_{k+1}) = (\bigvee_{i=k+1}^m w_i \vee w_k, \bigvee_{i=k+1}^m w_i \vee w_{k+1}) \in \mu^{-1}(\boldsymbol{a}_{\varepsilon(k)}^\xi)$. From 4.3 we obtain that $\mu\theta(1, x) \subseteq \operatorname{Ker} f_m$. Then also $\mu(1) = \bigvee\{\mu\theta(1, x) \mid x \in X\} \subseteq \operatorname{Ker} f_m$, a contradiction. □

This result raises the following question.

Problem 4.6. *Which algebras admit a compatible $(\vee, 1)$-semilattice structure?*

Another observation is that instead of F one can take the (sufficiently large) free algebra in any variety containing $\mathcal{M}_3^{01}$. Indeed, the existence of suitable homomorphisms $F \to M_3$ and $F \to \{0, 1\}$ was the only property of F needed for the proof. Hence, F can be chosen locally finite. (As the variety $\mathcal{M}_3^{01}$ is finitely generated.)

Finally, let us remark that instead of M_3 one can use the 5-element nonmodular lattice $N_5 = \{0, 1, a, b, c\}$ with $0 < b < a < 1$ and c the common complement of a and b. (In this denotation, exactly the same proof works.) Hence, F can also be taken as the free bounded lattice in the variety generated by N_5 (with at least $\aleph_2$ generators).

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