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Local separation in distributive semilattices

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Abstract. We introduce the Local Separation Property (LSP) for distributive semilattices. We show that LSP holds in many semilattices of the form $\mathrm{Con}_c\ A$, where A is a lattice. On the other hand, we construct an abstract example of a distributive lattice without LSP. Our research is connected with the well known open problem whether every distributive algebraic lattice is isomorphic to the congruence lattice of some lattice.

1. Introduction

By a distributive semilattice we mean a join-semilattice having the smallest element 0 and satisfying the following condition:

if $x \leq y \vee z$, then there exist $y' \leq y$, $z' \leq z$ such that $x = y' \vee z'$.

The ideals of a distributive semilattice S form a distributive algebraic lattice $\mathrm{Id}(S)$ and S is isomorphic to the semilattice of compact elements of $\mathrm{Id}(S)$ (under the assignment $x \mapsto \downarrow x = \{y \in S \mid y \leq x\}$). Also conversely, the compact elements of every distributive algebraic lattice L form a distributive semilattice L_c and L is isomorphic to $\mathrm{Id}(L_c)$.

For a lattice L let $\mathrm{M}(L) = \{x \in L \mid x < \bigwedge\{y \in L \mid y > x\}\}$ denote the set of all completely meet-irreducible elements. It is well known that in any algebraic lattice every element is a meet of completely meet-irreducible elements.

For an algebra A let $\mathrm{Con}\ A$ denote the lattice of all congruences of A under inclusion. This lattice is always algebraic and its compact elements (i.e. compact, or finitely generated congruences) form a join-semilattice $\mathrm{Con}_c\ A$.

If A is a lattice, then $\mathrm{Con}\ A$ is distributive. It is not known if every distributive algebraic lattice is isomorphic to $\mathrm{Con}\ A$ for some lattice A. This is a longstanding open problem known as the Congruence Lattice Problem (CLP in short). The equivalent semilattice formulation of CLP asks whether every distributive semilattice is isomorphic to $\mathrm{Con}_c\ A$ for some lattice A. We refer to [1] (appendix C) and [8] for the survey on this problem.

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One of the important directions in investigating CLP is connected with uniform refinement properties. They are usually considered as additional requirements, which distributive semilattices should satisfy in order to be isomorphic to $\mathrm{Con}_c A$ for a (particular kind of) lattice A. So far, none of them was able to solve the problem, although they led to many interesting results. (See [8].)

In this paper we introduce a new uniform refinement property for distributive semilattices, called the Local Separation Property (LSP). This property seems to be considerably weaker than all other uniform refinement properties. Nevertheless, we were able to construct a distributive semilattice without LSP. On the other hand, we can prove that LSP holds in $\mathrm{Con}_c A$ for a quite large class of lattices A. In fact, we do not know whether there is a lattice A, for which $\mathrm{Con}_c A$ does not satisfy LSP.

2. Refinable decomposition systems

Definition 2.1. Let S be a distributive semilattice and $e \in S$. Let $\mathcal{F} = \{(a_i, b_i) \mid i{\in}I\}$ be a system of pairs of elements of S. We say that $\mathcal{F}$ is a decomposition system at e if $a_i \vee b_i = e$ for every $i \in I$. Such a decomposition system is called *uniformly refinable* if there are compact elements c_{ij} for $i, j \in I$, $i \neq j$ such that

(i) $c_{ij} \leq a_i, b_j$ for every $i, j \in I,\ i \neq j$;
(ii) $c_{ij} \vee a_j \vee b_i = e$ for every $i, j \in I,\ i \neq j$;
(iii) $c_{ik} \leq c_{ij} \vee c_{jk}$ for every $i, j, k \in I$, $i \neq j \neq k \neq i$.

Following [7], we say that a distributive semilattice S has the *weak uniform refinement property* (WURP) if every decomposition system in S (at every $e \in S$) is uniformly refinable.

If S is a lattice (i.e. meets exist in S), then S has WURP. Indeed, it is easy to see that the elements $c_{ij} = a_i \wedge b_j$ have the required properties.

If the cardinality of S is at most $\aleph_1$, then S has WURP. This is a difficult result, which can be deduced from the results in [10] and [8].)

Now we present another important example of a uniformly refinable decomposition system. For any elements x, y of a lattice A let $\theta(x, y)$ denote the congruence on A generated by the pair (x, y).

Theorem 2.2. *Let $S = \mathrm{Con}_c A$ for some lattice A. Let $x, y \in A$ and let J denote the interval $[x \wedge y, x \vee y]$. For every $z \in J$ let $a_z = \theta(x \wedge y, z)$, $b_z = \theta(z, x \vee y)$. Then*

$$\mathcal{F}(x, y) = \{(a_z, b_z) \mid z \in J\}$$

is a uniformly refinable decomposition system at $e = \theta(x, y)$.

Proof. It is clear that $a_z, b_z \in S$ and $a_z \vee b_z = e$ for every $z \in J$. For every $u, v \in J$ let $c_{uv} = \theta(u \wedge v, u) = \theta(v, u \vee v)$. This is a compact congruence. (Notice that c_{uv} is the least congruence of A for which the congruence classes of u and v satisfy $[u] \leq [v]$.) The condition 2.1(i) is obviously satisfied. Further, $(x \wedge y, v) \in a_v$ implies $(x \wedge y, u \wedge v) \in a_v$. Since $(u \wedge v, u) \in c_{uv}$, $(u, x \vee y) \in b_u$, we obtain that $(x \wedge y, x \vee y) \in c_{uv} \vee a_v \vee b_u$, hence $c_{uv} \vee a_v \vee b_u \geq \theta(x, y)$ and 2.1(ii) holds.

Now, let u, v, w be different elements of J. From $(v \wedge w, v) \in c_{vw}$ we obtain that $(u \wedge v, u \wedge v \wedge w) \in c_{vw}$. Since $(u \wedge v, u) \in c_{uv}$, we have $(u \wedge v \wedge w, u) \in c_{uv} \vee c_{vw}$, which implies $(u \wedge w, u) \in c_{uv} \vee c_{vw}$. Hence, $c_{uw} \leq c_{uv} \vee c_{vw}$ and 2.1(iii) holds. □

The above construction could be generalized to the case when e is not principal, due to the following result.

Theorem 2.3. *Let S be a distributive semilattice, $e_1, \ldots, e_n \in S$. Let $\mathcal{F}_k = \{(a_i, b_i) \mid i \in I_k\}$ be a uniformly refinable decomposition system at e_k, $k = 1, \ldots, n$. Then*

$$\mathcal{F}_1 \times \cdots \times \mathcal{F}_n = \{(a_{i_1} \vee \cdots \vee a_{i_n}, b_{i_1} \vee \cdots \vee b_{i_n}) \mid (i_1, \ldots, i_n) \in I_1 \times \cdots \times I_n\}$$

is a uniformly refinable decomposition system at $e = e_1 \vee \cdots \vee e_n$.

Proof. Obviously, $\mathcal{F}_1 \times \cdots \times \mathcal{F}_n$ is a decomposition system at e. For every $i, j \in I_1 \times \cdots \times I_n$, $i = (i_1, \ldots, i_n)$, $j = (j_1, \ldots, j_n)$ we denote $a_i = a_{i_1} \vee \cdots \vee a_{i_n}$, $b_j = b_{j_1} \vee \cdots \vee b_{j_n}$ and set $c_{ij} = c_{i_1 j_1} \vee \cdots \vee c_{i_n j_n}$. It is easy to check that the conditions of Definition 2.1 are satisfied. □

Hence, semilattices of compact congruences of lattices contain large uniformly refinable decomposition systems. On the other hand, it is not easy to construct a decomposition system which is not uniformly refinable. However, such systems do exist. The first such example has been constructed by F. Wehrung in [9]. A more direct construction has been presented in [6]. Another important example is the following result.

Theorem 2.4. (See [7].) *Let F be the free lattice in any non-distributive variety of lattices with at least $\aleph_2$ generators. Then* $\mathrm{Con}_c\, F$ *does not satisfy WURP.*

Hence, non-refinable decomposition systems can occur in the semilattices of compact congruences of lattices. This in particular means, that WURP cannot solve the CLP. That is why we introduce the following essential weakening of WURP. Instead of requiring the refinability for every decomposition system, we will just demand the existence of "rich" decomposition systems. Recall that, for a distributive semilattice S, $\mathrm{M}(\mathrm{Id}(S))$ denotes the set of all completely meet-irreducible elements of the algebraic lattice $\mathrm{Id}(S)$, i.e. the set of all completely meet-irreducible ideals of S.

Definition 2.5. Let S be a distributive semilattice, $e \in S$, $P \in \mathrm{M}(\mathrm{Id}(S))$. We say that S has the local separation property (LSP) at (e, P) if there are uniformly refinable decomposition systems $\mathcal{F}_1, \dots, \mathcal{F}_n$ at e such that for every $Q \in \mathrm{M}(\mathrm{Id}(S))$ with $e \in P \vee Q$ (the join in $\mathrm{Id}(S)$) there exists $(a, b) \in \bigcup_{i=1}^{n} \mathcal{F}_i$ with $a \in P$, $b \in Q$. We say that S has the LSP if it has the LSP at every (e, P).

We use the term "separation", because LSP is connected with the separating the points of the space $M(\mathrm{Id}(S))$ with a natural topology. (See [3].)

Lemma 2.6. *Every distributive semilattice with WURP has LSP.*

Proof. Let $e \in S$, $P \in \mathrm{M}(\mathrm{Id}(S))$. We set $\mathcal{F} = \{(a, b) \in S^2 \mid a \vee b = e\}$. By WURP, $\mathcal{F}$ is uniformly refinable. For every $Q \in \mathrm{M}(\mathrm{Id}(S))$, $e \in P \vee Q$ implies that $e \leq a_0 \vee b_0$ for some $a_0 \in P$, $b_0 \in Q$. Since S is distributive, $e = a \vee b$ for some $a \leq a_0$, $b \leq b_0$, hence $(a, b) \in \mathcal{F}$, $a \in P$, $b \in Q$. $\square$

Lemma 2.7. *Let S be a distributive semilattice, $e_1, \dots, e_n \in S$, $P \in \mathrm{M}(\mathrm{Id}(S))$. If S has LSP at every (e_i, P), then it has LSP at $(e_1 \vee \cdots \vee e_n, P)$.*

Proof. Let $e = e_1 \vee \cdots \vee e_n$ and assume that S has LSP at every (e_i, P). Thus, for every i we have suitable uniformly refinable decomposition systems $\mathcal{F}_1^i, \dots \mathcal{F}_{k_i}^i$ at e_i. For every n-tuple $s = (s_1, \dots s_n)$ with $1 \leq s_i \leq k_i$ we have the uniformly refinable decomposition system $\mathcal{F}_s = \mathcal{F}_{s_1}^1 \times \cdots \times \mathcal{F}_{k_n}^n$ at e, by Theorem 2.3. We claim that the collection of all $\mathcal{F}_s$ has the required property. Let $Q \in \mathrm{M}(\mathrm{Id}(S))$, $e \in P \vee Q$. Then $e_i \in P \vee Q$ for every i, so $a_i \in P$, $b_i \in Q$ for some $(a_i, b_i) \in \mathcal{F}_{s_i}^i$. Let $s = (s_1, \dots, s_n)$, $a = a_1 \vee \cdots \vee a_n$, $b = b_1 \vee \cdots \vee b_n$. Clearly, $(a, b) \in \mathcal{F}_s$, $a \in P$, $b \in Q$. $\square$

3. The variety $\mathcal{M}_3$

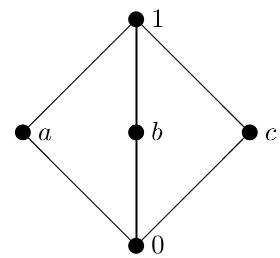
In this section we prove that LSP is strictly weaker than WURP. We show that LSP holds in the semilattice $\mathrm{Con}_c\ A$ for every algebra A in $\mathcal{M}_3$ (the variety generated by the lattice M_3) depicted in Figure 1, which, by Theorem 2.4, is not the case for WURP.

We start our proof with a general observation.

Lemma 3.1. *Let A be any algebra. The completely meet-irreducible ideals in $\mathrm{Con}_c\ A$ are exactly the sets of the form*

$$P_\alpha = \{\beta \in \mathrm{Con}_c\ A \mid \beta \subseteq \alpha\},$$

where α is a congruence on A such that the quotient algebra A/α is subdirectly irreducible.



M_3

FIGURE 1

Proof. The ideal lattice of $\mathrm{Con}_c\ A$ is isomorphic to Con A, so every ideal in $\mathrm{Con}_c\ A$ is equal to P_α for some $\alpha \in \mathrm{Con}\ A$. And obviously, $\alpha \in \mathrm{M}(\mathrm{Con}\ A)$ if and only if A/α is subdirectly irreducible. □

By the Jónsson Lemma (see also [1], Corollary V.10), every subdirectly irreducible algebra in $\mathcal{M}_3$ is a homomorphic image of a subalgebra of M_3. It is therefore easy to check that the variety $\mathcal{M}_3$ has (up to isomorphism) only two subdirectly irreducible members: the lattice M_3 and the 2-element chain $C_2 = \{0, 1\}$.

Theorem 3.2. *For every $A \in \mathcal{M}_3$, the distributive semilattice $S = \mathrm{Con}_c\ A$ has LSP.*

Proof. Let $e \in S$ and $P \in \mathrm{M}(\mathrm{Id}S)$. We need to show that S has LSP at (e, P). By Lemma 2.7 we can assume that $e = \theta(x, y)$ (the smallest congruence containing the pair (x, y)) for some $x, y \in A$. As $\theta(x, y) = \theta(x \wedge y, x \vee y)$, we can assume that $x \leq y$.

By Lemma 3.1, $P = P_\alpha$ for some $\alpha \in \mathrm{Con}\ A$ such that A/α is isomorphic to C_2 or M_3. The restriction of α to the interval $[x, y]$, denoted by α', is a congruence on the lattice $[x, y]$ and the quotient $[x, y]/\alpha'$ is a convex sublattice of C_2 or M_3. Hence, it can be a 1-element lattice, C_2, or M_3. We discuss all three cases.

If $[x, y]/\alpha'$ is a 1-element lattice, then $e \subseteq \alpha$, hence $e \in P$ and the decomposition system $\mathcal{F} = \{(e, 0)\}$ has the required property.

Let $[x, y]/\alpha'$ be isomorphic to M_3. Then there exist $z \in [x, y]$ such that both $[x, z]/\alpha''$ and $[z, y]/\alpha'''$ (where α'' and α''' are the restrictions of α to $[x, z]$ and $[z, y]$ respectively) are isomorphic to C_2. Let $e_1 = \theta(x, z)$, $e_2 = \theta(z, y)$. Then $e = e_1 \vee e_2$ and by Lemma 2.7 it suffices to show that S has the LSP at (e_1, P) and (e_2, P), which reduces our case to the following one.

Let $[x,y]/\alpha'$ be isomorphic to C_2. For every $z \in [x,y]$ let $a_z = \theta(x,z)$, $b_z = \theta(z,y)$. We claim that the systems

$$\mathcal{F}_1 = \{(a_z, b_z) \mid z \in [x,y]\}, \quad \mathcal{F}_2 = \{(b_z, a_z) \mid z \in [x,y]\}$$

have the required properties. By Theorem 2.2, both $\mathcal{F}_1$ and $\mathcal{F}_2$ are uniformly refinable decomposition systems at e.

Let $Q \in \mathrm{M}(\mathrm{Id}(S))$, $e \in P \vee Q$. We have $Q = P_\beta$ for some $\beta \in \mathrm{Con}\, A$ such that A/β is isomorphic to C_2 or M_3. Let β' be the restriction of β to the interval, $[x,y]$. Similarly as above, we have three possibilities for $[x,y]/\beta'$.

I. If $[x,y]/\beta'$ is a 1-element lattice, then $e \in Q$, $0 \in P$ and $(0,e) = (a_x, b_x) \in \mathcal{F}_1$.

II. Let $[x,y]/\beta'$ be isomorphic to M_3. Then β' is the kernel of a surjective homomorphism $g \colon [x,y] \to M_3$. Similarly, α' is the kernel of a surjective homomorphism $f \colon [x,y] \to C_2$. Let us choose $u, v, w \in [x,y]$ such that $g(u) = a$, $g(v) = b$, $g(w) = c$. The homomorphism f must collapse at least two of u, v, w. Without loss of generality, $f(u) = f(v)$.

If $f(u) = 0$, then consider $z = u \vee v$. We have $f(z) = 0 = f(x)$, hence $(x,z) \in \alpha$, and $g(z) = 1 = g(y)$, hence $(z,y) \in \beta$. Consequently, $a_z \in P$, $b_z \in Q$ and $(a_z, b_z) \in \mathcal{F}_1$.

If $f(u) = 1$, then we consider $z = u \wedge v$. We have $f(z) = 1 = f(y)$, $g(z) = 0 = g(x)$, hence $(x,z) \in \beta$, $(z,y) \in \alpha$, which implies that $b_z \in P$, $a_z \in Q$, and clearly $(b_z, a_z) \in \mathcal{F}_2$.

III. Finally, let $[x,y]/\beta'$ be isomorphic to C_2. From $e \in P \vee Q$ we obtain that $e \subseteq \alpha \vee \beta$, hence $(x,y) \in \alpha \vee \beta$. By the definition of the join of equivalence relations, there exist elements $z_0 = x, z_1, \dots, z_k = y$ such that $(z_i, z_{i+1}) \in \alpha \cup \beta$ for every $i = 0, \dots, k-1$. Let $t_i = y \wedge \bigvee_{j=0}^{i} z_j$. Then

$$x = t_0 \leq t_1 \leq \cdots \leq t_k = y.$$

and it is easy to show by induction that $(t_i, t_{i+1}) \in \alpha \cup \beta$ for every i. We can also assume that the sequence $t_0, \dots, t_k$ does not contain redundant elements, i.e., (t_i, t_{i+1}) belongs to exactly one of the congruences α, β. Now, if $(x, t_1) \in \alpha$, then $(x, t_1) \notin \beta$, and so $(t_1, y) \in \beta$, and consequently $a_{t_1} \in P$, $b_{t_1} \in Q$, $(a_{t_1}, b_{t_1}) \in \mathcal{F}_1$. Similarly, if $(x, t_1) \in \beta$, we get $b_{t_1} \in P$, $a_{t_1} \in Q$ and $(b_{t_1}, a_{t_1}) \in \mathcal{F}_2$. This completes the proof. □

The same proof would work for any lattice in $\mathcal{M}_\infty$, the variety generated by all lattices of length 2. For lattices outside this variety, the decomposition systems considered in the above proof are, in general, not rich enough. Nevertheless, it is still possible that LSP holds in $\mathrm{Con}_c\, A$ for every lattice A.

Problem 3.3. Does LSP hold in $\mathrm{Con}_c\, A$ for every lattice A?

In the next Section we construct an example of a distributive semilattice without LSP. Thus, the positive answer to 3.3 would imply the negative solution of CLP.

4. A distributive semilattice without LSP

In this Section we construct a distributive semilattice which does not have LSP. We use the topological representation mentioned in the Section 2 and define our semilattice as the semilattice of all compact open subsets of a suitable topological space.

For any function f let $\mathrm{dom}(f)$ and $\mathrm{rng}(f)$ denote its domain and range, respectively.

Let M denote the 5-element set $\{0,1,a,b,c\}$. Let X be any set. Let $T_X = \{f \in M^X \mid f(X) \subseteq \{0,1\}$ or $\{a,b,c\} \subseteq f(X)\}$.

For every $u, v \in \{a,b,c\}$, $u \neq v$ we define functions

$$p_0^{\{u,v\}}, p_1^{\{u,v\}} \colon \{0,1,u,v\} \to \{0,1\}$$

(abbreviated by p_0^{uv}, p_1^{uv}) as follows:

$$\begin{aligned}
p_0^{uv}(0) = p_1^{uv}(0) &= 0, & p_0^{uv}(1) = p_1^{uv}(1) &= 1 \quad \text{for every } u, v;\\
p_0^{ab}(a) = p_0^{ab}(b) &= 0, & p_1^{ab}(a) = p_1^{ab}(b) &= 1;\\
p_0^{bc}(b) = p_0^{bc}(c) &= 0, & p_1^{bc}(b) = p_1^{bc}(c) &= 1;\\
p_0^{ac}(a) = p_1^{ac}(c) &= 0, & p_0^{ac}(c) = p_1^{ac}(a) &= 1.
\end{aligned}$$

Further we denote

$$S_0 = \{r \colon X_0 \to M \mid X_0 \subseteq X \text{ is finite, } \mathrm{rng}(r) \subseteq \{0,1\}\};$$

$$S_1 = \{r \colon X_0 \to M \mid X_0 \subseteq X \text{ is finite, } \{a,b,c\} \subseteq \mathrm{rng}(r)\}.$$

For every $r \in S_0$ let

$$\begin{aligned}
K_r = \big\{ f \in M^X \;\big|\; & f(\mathrm{dom}(r)) \subseteq \{0,1,u,v\} \text{ for some } u, v \in \{a,b,c\}\\
& \text{and } \big(r = p_0^{uv} \cdot (f \restriction \mathrm{dom}(r)) \text{ or } r = p_1^{uv} \cdot (f \restriction \mathrm{dom}(r))\big)\big\}.
\end{aligned}$$

It is easy to see that this definition is unambiguous even if $f(\mathrm{dom}(r)) \subseteq \{0,1,u\}$. (In this case there are two possible choices for v.) If $f(\mathrm{dom}(r)) \subseteq \{0,1\}$ then $f \in K_r$ iff $r = f \restriction \mathrm{dom}(r)$.

For every $r \in S_1$ let

$$K_r = \big\{ f \in M^X \;\big|\; r = f \restriction \mathrm{dom}(r) \big\}.$$

Finally, for every $r \in S_0 \cup S_1$ let $G_r = K_r \cap T_X$ and let $\mathcal{G} = \{G_r \mid r \in S_0 \cup S_1\}$.

Lemma 4.1. *Let* $r \in S_0 \cup S_1$, $f, g, g_0, g_1 \in M^X$.

(i) *If* $f \restriction \mathrm{dom}(r) = g \restriction \mathrm{dom}(r)$ *then* $f \in K_r$ *iff* $g \in K_r$.

(ii) *If* $f(\mathrm{dom}(r)) \subseteq \{0,1,u,v\}$ *for some* $u,v \in \{a,b,c\}$ *and* $g_n \restriction \mathrm{dom}(r) = p_n^{uv} \cdot f \restriction \mathrm{dom}(r)$ *(*$n = 0,1$*), then* $f \in K_r$ *iff* $g_0 \in K_r$ *or* $g_1 \in K_r$*.*

Proof. Obvious. □

Lemma 4.2. *For any* $G_r, G_s \in \mathcal{G}$*, the set* $G_r \cap G_s$ *is a union of some sets from* $\mathcal{G}$*. Hence,* $\mathcal{G}$ *is a base of some topology on* T_X*.*

Proof. For every $f \in G_r \cap G_s$ we will find $G_t \in \mathcal{G}$ with $f \in G_t \subseteq G_r \cap G_s$. (If $G_r \cap G_s = \emptyset$, we have nothing to prove.)

Suppose first that $\mathrm{rng}(f) \subseteq \{0,1\}$. This is only possible if $r,s \in S_0$, $r = f \restriction \mathrm{dom}(r)$, $s = f \restriction \mathrm{dom}(s)$. Choose a finite set $X_0 \subseteq X$ with $X_0 \supseteq \mathrm{dom}(r) \cup \mathrm{dom}(s)$ and let $t = f \restriction X_0$. Clearly, $f \in G_t \subseteq G_r \cap G_s$.

Now let $\mathrm{rng}(f) \nsubseteq \{0,1\}$. Choose a finite set $X_0 \subseteq X$ such that $X_0 \supseteq \mathrm{dom}(r) \cup \mathrm{dom}(s)$ and $f(X_0) = \mathrm{rng}(f)$. Let $t = f \restriction X_0$. Then $t \in S_1$ and $f \in G_t$. For every $g \in G_t$ we have $g \restriction X_0 = f \restriction X_0$, which by Lemma 4.1 implies that $g \in G_r \cap G_s$. □

Lemma 4.3. *Every* $G_r \in \mathcal{G}$ *is compact.*

Proof. First notice that in the usual product topology of the space M^X the sets K_r are open and compact. Suppose that $G_r \subseteq \bigcup_{i \in I} G_{r_i}$. We claim that the set K_r is covered by K_{r_i}. Let $f \in K_r$. If $f \in T_X$ then clearly $f \in G_{r_i} \subseteq K_{r_i}$ for some i. Let $f \in M^X \setminus T_X$. Then necessarily $r \in S_0$ and $\mathrm{rng}(f)$ contains at most two of a, b, c. Let us assume that $\mathrm{rng}(f) \subseteq \{0,1,a,b\}$. (The other two cases are similar.) Then $r = p_n^{ab} \cdot f \restriction \mathrm{dom}(r)$, where $n \in \{0,1\}$. Let $g = p_n^{ab} \cdot f$. Then $g \restriction \mathrm{dom}(r) = r$, hence $g \in K_r$. Since $\mathrm{rng}(g) \subseteq \{0,1\}$, we have $g \in G_r$. Thus, $g \in G_{r_i} \subseteq K_{r_i}$ for some i. By Lemma 4.1, $f \in K_{r_i}$.

Thus, in the product topology, the compact set K_r is covered by open sets K_{r_i}. Hence, there is a finite set $I_0 \subseteq I$ such that $K_r \subseteq \bigcup\{K_{r_i} \mid i \in I_0\}$. Then $G_r = K_r \cap T_X \subseteq \bigcup\{G_{r_i} \mid i \in I_0\}$, which shows that G_r is compact. □

The above proof works also in the case when r is an empty map, so the whole space $T_X = G_\emptyset$ is compact too.

Thus, T_X has a basis of compact open sets. Let L_X be the family of all open subsets of T_X ordered by the set inclusion. It is clear that L_X is a distributive algebraic lattice. Compact elements of L_X are exactly the (topologically) compact open subsets of T_X and they form a distributive semilattice S_X. The whole space T_X is compact, so it is the largest element of S_X. We have the following description of elements of S_X.

Lemma 4.4. *A set* G *belongs to* S_X *if and only if* $G = G_1 \cup \cdots \cup G_n$ *for some sets* $G_1, \ldots, G_n \in \mathcal{G}$*.*

Proof. A finite union of compact sets is compact. Conversely, every open set is a union of some sets from $\mathcal{G}$. If this set is compact, the union can be chosen finite. $\square$

For every $G \in S_X$ we fix its representation in the form $G = G_{r_1} \cup \dots \cup G_{r_n}$ for some $r_1, \dots, r_n \in S_0 \cup S_1$ and define $\operatorname{dom}(G) = \operatorname{dom}(r_1) \cup \dots \cup \operatorname{dom}(r_n)$. Hence, for every compact open set G we have a finite set $\operatorname{dom}(G) \subseteq X$.

Lemma 4.5. *Let G be compact open. Let $f, g, g_0, g_1 \in T_X$.*

(i) *If $f \restriction \operatorname{dom}(G) = g \restriction \operatorname{dom}(G)$ then $f \in G$ iff $g \in G$.*
(ii) *If $f(\operatorname{dom}(G)) \subseteq \{0, 1, u, v\}$ for some $u, v \in \{a, b, c\}$ and $g_n \restriction \operatorname{dom}(G) = p_n^{uv} \cdot f \restriction \operatorname{dom}(G)$ $(n = 0, 1)$, then $f \in G$ iff $g_0 \in G$ or $g_1 \in G$.*

Proof. This is a direct consequence of Lemmas 4.1 and 4.4. $\square$

Lemma 4.6. *Let $f, g \in T_X$, $f \neq g$. Then there exists an open set G such that $f \in G$, $g \notin G$.*

Proof. There exists a finite set $X_0 \subseteq X$ such that $f(X_0) = f(X)$, $g(X_0) = g(X)$ and $f \restriction X_0 \neq g \restriction X_0$. Let us set $s = f \restriction X_0$. Then clearly $s \in S_0 \cup S_1$ and $f \in G_s$. Further, $f \restriction X_0 \neq g \restriction X_0$ implies $g \notin G_s$. (Notice that the case $g(X_0) \subseteq \{0, 1, u, v\}$ for some $u, v \in \{a, b, c\}$ is only possible when $g(X_0) \subseteq \{0, 1\}$.) $\square$

The proof that our semilattice does not have LSP is based on the following statement of infinite combinatorics, see C. Kuratowski [2]. Recall that $[X]^2$ denotes the family of all 2-element subsets of X and $[X]^{<\omega}$ stands for the family of all finite subsets of X.

Lemma 4.7. *Let X be a set with the cardinality at least $\aleph_2$. Then for every map $\Phi\colon [X]^2 \to [X]^{<\omega}$ there exists a three-element set M such that $x \notin \Phi(M \setminus \{x\})$ for every $x \in M$.*

For every $f \in T_X$ the set $T_X \setminus \{f\}$ is a maximal open (by Lemma 4.6) subset of T_X. Consequently,

$$I_f = \{U \in S_X \mid f \notin U\}$$

is a maximal (and hence completely meet-irreducible) ideal of S_X.

Theorem 4.8. *Let $g \in T_X$ be defined by $g(x) = 0$ for every $x \in X$. If $|X| \geq \aleph_2$ then S_X does not have LSP at (T_X, I_g).*

Proof. Since I_g is a maximal ideal, we have $T_X \in I_g \vee Q$ for every $Q \in \mathrm{M}(\mathrm{Id}(S_X))$ with $Q \not\subseteq I_g$. For contradiction, suppose that there are uniformly refinable decomposition systems $\mathcal{F}_1, \dots, \mathcal{F}_n$ at T_X such that for every $Q \not\subseteq I_g$ there is $(A, B) \in \bigcup_{i=1}^n \mathcal{F}_i$ with $A \in I_g$, $B \in Q$.

For every $x \in X$ let $f_x \in T_X$ be the characteristic function of the set $\{x\}$, i.e. $f_x(x) = 1$ and $f_x(y) = 0$ for $y \neq x$. Every ideal I_{f_x} is maximal, so $I_{f_x} \not\subseteq I_g$. For $k = 1, \dots, n$ let

$$X_k = \{x \in X \mid A \in I_g \text{ and } B \in I_{f_x} \text{ for some } (A, B) \in \mathcal{F}_k\}.$$

Equivalently, $X_k = \{x \in X \mid g \notin A,\ f_x \notin B \text{ for some } (A, B) \in \mathcal{F}_k\}$. By our assumption, $X_1 \cup \dots \cup X_n = X$, so some of the sets X_k must have the cardinality at least $\aleph_2$. We can assume that $|X_1| \geq \aleph_2$.

Let us denote the elements of $\mathcal{F}_1$ by (A_i, B_i), $i \in I$. The uniform refinability of $\mathcal{F}_1$ means that there are sets $C_{ij} \in S_X$ $(i, j \in I,\ i \neq j)$ such that

(1) $C_{ij} \subseteq A_i \cap B_j$ for every different i, j;
(2) $C_{ij} \cup A_j \cup B_i = T_X$ for every different i, j;
(3) $C_{ik} \subseteq C_{ij} \cup C_{jk}$ for every different i, j, k.

For every $x \in X_1$ we fix $i(x) \in I$ with $g \notin A_{i(x)}$, $f_x \notin B_{i(x)}$. Let us set $x \sim y$ iff $i(x) = i(y)$. From Lemma 4.5 we deduce that $x \in \mathrm{dom}(B_{i(x)})$. (Otherwise $g \restriction \mathrm{dom}(B_{i(x)}) = f_x \restriction \mathrm{dom}(B_{i(x)})$, hence $g \notin B_{i(x)}$, which is impossible since $A_{i(x)} \cup B_{i(x)} = T_X$.) Since $\mathrm{dom}(B_{i(x)})$ is finite, the equivalence $\sim$ has finite equivalence classes. It is therefore possible to choose a set $Y \subseteq X_1$ such that $|Y| \geq \aleph_2$ and $i(x) \neq i(y)$ for every $x, y \in Y$, $x \neq y$. For every $x, y \in Y$, $x \neq y$ we define

$$\begin{aligned} \Phi(\{x, y\}) = \quad & Y \cap \big(\mathrm{dom}(A_{i(x)}) \cup \mathrm{dom}(B_{i(x)}) \cup \mathrm{dom}(A_{i(y)}) \\ & \qquad \cup \mathrm{dom}(B_{i(y)}) \cup \mathrm{dom}(C_{i(x)i(y)}) \cup \mathrm{dom}(C_{i(y)i(x)})\big). \end{aligned}$$

Hence, Φ is a function $[Y]^2 \to [Y]^{<\omega}$. By Lemma 4.7, there are elements $x_1, x_2, x_3 \in Y$ such that $x_1 \notin \Phi(\{x_2, x_3\})$, $x_2 \notin \Phi(\{x_1, x_3\})$, $x_3 \notin \Phi(\{x_1, x_2\})$. Let us write f_m, A_m, B_m, C_{mn} instead of f_{x_m}, $A_{i(x_m)}$, $B_{i(x_m)}$, $C_{i(x_m)i(x_n)}$. Let us define a function $h\colon X \to M$ by

$$h(x) = \begin{cases} a & \text{if } x = x_1; \\ b & \text{if } x = x_2; \\ c & \text{if } x = x_3; \\ 0 & \text{otherwise.} \end{cases}$$

Since $x_1 \notin \mathrm{dom}(A_3)$, we have $f_1 \restriction \mathrm{dom}(A_3) = g \restriction \mathrm{dom}(A_3)$, so $g \notin A_3$ implies $f_1 \notin A_3$ by Lemma 4.5. Also, $f_1 \notin B_1$, so $f_1 \in C_{13}$ by (2). Further, $f_1 \restriction \mathrm{dom}(C_{13}) = p_1^{ac} \cdot (h \restriction \mathrm{dom}(C_{13}))$, so $h \in C_{13}$ by Lemma 4.5.

Now we claim that $h \notin C_{12}$. Let $f_{12} \in T_X$ be defined by $f_{12}(x_1) = f_{12}(x_2) = 1$ and $f_{12}(y) = 0$ for every $y \notin \{x_1, x_2\}$. Since $x_1 \notin \mathrm{dom}(B_2)$, we have $f_{12} \restriction \mathrm{dom}(B_2) = f_2 \restriction \mathrm{dom}(B_2)$, so $f_2 \notin B_2$ implies that $f_{12} \notin B_2$, hence $f_{12} \notin C_{12} \subseteq B_2$.

Since $x_3 \notin \mathrm{dom}(C_{12})$, we have $h(\mathrm{dom}(C_{12})) \subseteq \{0, 1, a, b\}$. It is easy to see that $g \restriction \mathrm{dom}(C_{12}) = p_0^{ab} \cdot (h \restriction \mathrm{dom}(C_{12}))$ and $f_{12} \restriction \mathrm{dom}(C_{12}) = p_1^{ab} \cdot (h \restriction \mathrm{dom}(C_{12}))$. Since $g \notin C_{12} \subseteq A_1$ and $f_{12} \notin C_{12}$, we obtain that $h \notin C_{12}$ by Lemma 4.5.

Finally, we claim that $h \notin C_{23}$. We consider $f_{23} \in T_X$ defined by $f_{23}(x_2) = f_{23}(x_3) = 1$ and $f_{23}(y) = 0$ otherwise. Similarly as above, $h(\mathrm{dom}(C_{23})) \subseteq \{0, 1, b, c\}$, $g \restriction \mathrm{dom}(C_{23}) = p_0^{ab} \cdot (h \restriction \mathrm{dom}(C_{23}))$ and $f_{23} \restriction \mathrm{dom}(C_{23}) = p_1^{ab} \cdot (h \restriction \mathrm{dom}(C_{23}))$, $g \notin C_{23} \subseteq A_2$ and $f_{23} \notin C_{23}$ hence $h \notin C_{23}$ by Lemma 4.5.

Thus, $h \in C_{13} \setminus (C_{12} \cup C_{23})$, which contradicts (3). □

5. The case $|X| \leq \aleph_1$

In contrast to the previous section, we will prove that L_X is isomorphic to Con A for some $A \in \mathcal{M}_3$ whenever $|X| \leq \aleph_1$. This is very nontrivial. Fortunately, most of the work has been done in [4].

Lemma 5.1. (See [4], Theorem 4.2.) *Let L be a distributive algebraic lattice containing at most $\aleph_1$ compact elements. The following conditions are equivalent:*

(1) *L is isomorphic to* Con A *for some bounded lattice $A \in \mathcal{M}_3$;*

(2) *L is isomorphic to the open sets lattice of some topological space Z satisfying the following conditions:*
   - (i) *Z is compact and has a base of compact open sets;*
   - (ii) *Z is a disjoint union of two Hausdorff zerodimensional spaces Z_0 and Z_1;*
   - (iii) *Z_0 is a closed subspace of Z;*
   - (iv) *if $x \in Z_1$, $y \in Z \setminus \{x\}$ then there exists a clopen set G with $x \in G \subseteq Z_1$, $y \notin G$;*
   - (v) *if $x, y, z \in Z$ are mutually different then there exist open sets A, B, C such that $x \in A$, $y \in B$, $z \in C$ and $A \cap B \cap C = \emptyset$.*

We want to apply this characterization to the space T_X.

Lemma 5.2. *For every $r \in S_1$, the set G_r is closed (and hence clopen).*

Proof. Let $f \in T_X \setminus G_r$. Choose a finite set $X_0 \subseteq X$ such that $\mathrm{dom}(r) \subseteq X_0$ and $f(X) = f(X_0)$. Let $s = f \restriction X_0$. Then $s \in S_0 \cup S_1$, $f \in G_s$ and we claim that $G_r \cap G_s = \emptyset$. Let $g \in G_r$. If $s \in S_0$, then $g \notin G_s$ because $g(\mathrm{dom}(s)) \supseteq g(\mathrm{dom}(r)) \supseteq \{a, b, c\}$. If $s \in S_1$, then $g \restriction \mathrm{dom}(r) = r \neq f \restriction \mathrm{dom}(r)$, hence $g \restriction \mathrm{dom}(s) \neq f \restriction \mathrm{dom}(s) = s$, so $g \notin G_s$. □

Lemma 5.3. *Let $W \subseteq T_X$ be open, $f \in W$,* $\mathrm{rng}(f) \supseteq \{a, b, c\}$. *Then there is $r \in S_1$ such that $f \in G_r \subseteq W$.*

Proof. Since W is open, there is $s \in S_0 \cup S_1$ with $f \in G_s \subseteq W$. There is a finite set $Y \subseteq X$ such that $\mathrm{dom}(s) \subseteq Y$ and $f(Y) \supseteq \{a, b, c\}$. Let $r = f \restriction Y$. Clearly, $f \in G_r \subseteq G_s$. □

Let us denote $Z_0 = \{f \in T_X \mid f(X) \subseteq \{0, 1\}\}$, $Z_1 = T_X \setminus Z_0$.

Lemma 5.4. *If $f, g, h \in T_X$ are mutually different then there exist open sets $A, B, C \subseteq T_X$ such that $f \in A$, $g \in B$, $h \in C$ and $A \cap B \cap C = \emptyset$.*

Proof. By Lemma 4.6, all finite sets are closed. If any of f, g, h belongs to Z_1 (say, $f \in Z_1$), then by Lemmas 5.3 and 5.2 there exists a clopen set G with $f \in G$, $g, h \notin G$. We can set $A = G$, $B = C = T_X \setminus G$.

Assume now that $f, g, h \in Z_0$. Since they are different, there is a finite set (in fact, a 3-element set) X_0 such that $r = f \restriction X_0$, $s = g \restriction X_0$ and $t = h \restriction X_0$ are different. We set $A = G_r$, $B = G_s$, $C = G_t$. Then $f \in A$, $g \in B$, $h \in C$. For contradiction suppose that $k \in A \cap B \cap C$. Then there are $u, v \in \{a, b, c\}$ such that each of r, s, t is equal to $p_0^{uv} \cdot k \restriction X_0$ or to $p_1^{uv} \cdot k \restriction X_0$. Since r, s, t are different, this is impossible. □

Theorem 5.5. *If $|X| \leq \aleph_1$ then there is a lattice $M \in \mathcal{M}_3$ such that L_X is isomorphic to* Con M.

Proof. If $|X| \leq \aleph_1$ then $|\mathcal{G}| \leq \aleph_1$. By Lemma 4.4, T_X has at most $\aleph_1$ compact open sets and hence L_X has at most $\aleph_1$ compact elements. It suffices to show that the space T_X satisfies (i)–(v) of Lemma 5.1.

By 4.2 and 4.3, T_X has a basis of compact open sets. For the compactness of T_X see the remark after 4.3. From the definition it is easy to see that Z_0 inherits the topology from $\{0, 1, a, b, c\}^X$, a power of a discrete space. Thus, Z_0 is Hausdorff zerodimensional. The same is true for Z_1 by Lemmas 5.2 and 5.3. Further, (iii) follows from the fact that every $f \in Z_1$ belongs to some G_r with $r \in S_1$. Finally (iv) and (v) were proved in Lemma 5.3 and 5.4. □

References

[1] G. Grätzer, *General Lattice Theory* (Second edition with new appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, F. Wehrung and R. Wille), Birkhäuser Verlag, Basel, 1998.

[2] C. Kuratowski, *Sur une caractérisation des alephs,* Fund. Math. **38** (1951), 14–17.

[3] M. Ploščica, *Separation properties in congruence lattices of lattices,* Colloquium Mathematicum **83**(2000), 71–84.

[4] M. Ploščica, *Dual spaces of some congruence lattices,* Topology Appl. **131** (2003), 1–14.

[5] M. Ploščica, *Separation in distributive congruence lattices,* Algebra Universalis **49** (2003), 1–12.

[6] M. Ploščica and J. Tůma, *Uniform refinements in distributive semilattices,* in: Contributions to General Algebra **10** (Proc. conf. Klagenfurt '97), Verlag Johannes Heyn, 1998, pp. 251–262.

[7] M. Ploščica, J. Tůma and F. Wehrung, *Congruence lattices of free lattices in non-distributive varieties,* Colloquium Math. **76** (1998), 269–278.

[8] J. Tůma and F. Wehrung, *A survey of recent results on congruence lattices of lattices,* Algebra Universalis **48** (2002), 439–471.

[9] F. Wehrung, *Non-measurability properties of interpolation vector spaces,* Israel J. Math. **103** (1998), 197–206.
[10] F. Wehrung, *Forcing extensions of partial lattices,* Journal of Algebra **262** (2003), 127–193.

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