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IDEAL LATTICES OF LOCALLY MATRICIAL ALGEBRAS

MIROSLAV PLOŠČICA

ABSTRACT. We give another proof of Růžička's result that every infinite algebraic distributive lattice whose compact elements form a lattice is isomorphic to the lattice of all two-sided ideals of some locally matricial algebra. Our construction is more elementary and explicit.

1. Introduction

This paper is a contribution to the following problem.

PROBLEM 1.1. *Which lattices are isomorphic to the ideal lattices of locally matricial algebras?*

The motivation for this research is the theory of Γ-invariants of strongly uniform modules over a regular ring. Namely, if a lattice is isomorphic to the ideal lattice of a locally matricial k-algebra S, then it is also isomorphic to the submodule lattice of some right module over the matricial k-algebra $S \otimes_k S^{\mathrm{op}}$. (See [2] and [5] for more details.)

An equivalent formulation of (1.1) is to characterize $(\vee, 0)$-semilattices, which are isomorphic to the semilattices of finitely generated ideals of locally matricial algebras. It is well known that such semilattices must be distributive in the sense of [4]. Conversely, G. M. Bergman in [1] proved that every *countable* distributive $(\vee, 0)$-semilattice has such a representation. However, for uncountable semilattices the distributivity is not sufficient and the problem (1.1) remains open. (See [6].)

In [5] P. Růžička developed a method of representation of distributive semilattices as semilattices of finitely generated ideals of locally matricial algebras. Especially, he proved that every distributive *lattice* with 0 admits such a representation.

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In this paper we give another proof of Růžička's result. Our construction is based on the same ideas, but it is more elementary and explicit. Especially, we avoid the use of direct limits and several auxiliary categories. We hope that this approach could be helpful for a further progress in the problem (1.1).

Now we recall some basic concepts. Let k be a field, p a nonnegative integer. Let $\mathbb{M}_p(k)$ denote the k-algebra of all matrices $p \times p$ over k. A matricial k-algebra is an k-algebra of the form

$$\mathbb{M}_{p(1)}(k) \times \cdots \times \mathbb{M}_{p(n)}(k),$$

for some natural numbers $n, p(1), \dots, p(n)$. (See [3].) A k-algebra is *locally matricial* if it is a direct limit of matricial k-algebras. The ideal lattice of a k-algebra R will be denoted by $\mathrm{Id}(R)$. It is well known that if R is locally matricial then $\mathrm{Id}(R)$ is distributive. The finitely generated (or *compact*) ideals of R form a $\vee$-subsemilattice $\mathrm{Id}^c(R)$ of $\mathrm{Id}(R)$, which in the locally matricial case is also distributive. (But in general, it is not a lattice.)

We refer to [4] as the basic reference for the lattice theory concepts. Especially, we use some elementary results about prime ideals and filters (dual ideals) in distributive lattices ([4], sections II.1 and II.5).

For a partially ordered set $(P, \leq)$ and $x \in P$, $X \subseteq P$ we denote

$$\downarrow x = \{y \in P \colon y \leq x\}, \quad \downarrow X = \{y \in P \colon y \leq x \text{ for some } x \in X\}.$$

Sometimes we work simultaneously with several orders $\leq_\alpha$, $\leq_\beta$, ... on the same underlying set. Then we use the denotations like $\downarrow_\alpha x$, $\downarrow_\beta Y$, etc.

2. Technical tools

For any bounded distributive lattice L we denote by $\mathcal{F}(L)$ the family of all finite $0,1$-sublattices of L. Further, let $\mathcal{P}(L)$ be the set of all (proper) prime filters of L. Let $\mathcal{S}(L)$ denote the family of all finite subsets of L containing 0 and 1.

Let B_L denote the set of all triples $\alpha = (P_\alpha, \leq_\alpha, \varphi_\alpha)$, where

(1) $P_\alpha \in \mathcal{P}(L)$;

(2) $\leq_\alpha$ is a linear order on the set P_α;

(3) φ_α is a function $\mathcal{S}(L) \to L$ such that $\varphi_\alpha(Z) \in Z \cap P_\alpha$ for every $Z \in \mathcal{S}(L)$;

(4) for every $x \in P_\alpha$ such that $P_\alpha \setminus \downarrow_\alpha x \in \mathcal{P}(L)$, there exists $Z \in \mathcal{S}(L)$ with $\varphi_\alpha(Z) \in \downarrow_\alpha x$.

Here $\downarrow_\alpha x$ denotes the set $\{y \in L\colon y \leq_\alpha x\}$. For any $\alpha \in B_L$ and any $0,1$-sublattice X of L we denote

$$X_\alpha = \bigl\{x \in X \cap P_\alpha\colon X \cap P_\alpha \setminus \downarrow_\alpha x \in \mathcal{P}(X) \\ \text{and } \varphi_\alpha(Z) \notin \downarrow_\alpha x \text{ for every } Z \in \mathcal{S}(X)\bigr\}.$$

So, (4) above can be reformulated as $L_\alpha = \emptyset$. For every $\alpha \in B_L$ and $X \in \mathcal{F}(L)$ we define the restriction $\alpha\restriction X = (P_{\alpha\restriction X}, \leq_{\alpha\restriction X}, \varphi_{\alpha\restriction X})$ as follows:

(5) $P_{\alpha\restriction X} = X \cap P_\alpha \setminus \downarrow_\alpha X_\alpha$;

(6) $x \leq_{\alpha\restriction X} y$ iff $x, y \in P_{\alpha\restriction X}$ and $x \leq_\alpha y$;

(7) $\varphi_{\alpha\restriction X} = \varphi_\alpha\restriction\mathcal{S}(X)$.

Notice that (5) implies

(8) $\varphi_\alpha(Z) \in Z \cap P_{\alpha\restriction X}$ for every $Z \in \mathcal{S}(X)$.

Lemma 2.1. *For every $\alpha \in B_L$ and $X \in \mathcal{F}(L)$, $\alpha\restriction X$ belongs to B_X.*

Proof. Denote $\gamma = \alpha\restriction X$. If $X_\alpha = \emptyset$ then $P_\gamma = P_\alpha \cap X$. If $X_\alpha \neq \emptyset$ then there exists $y = \max_\alpha X_\alpha$ (the maximum with respect to the order $\leq_\alpha$) and $P_\gamma = X \cap P_\alpha \setminus \downarrow_\alpha y$. In both cases, $P_\gamma \in \mathcal{P}(X)$. Thus, (1) holds. Since the restriction of a linear order is again a linear order, we have (2). For $Z \in \mathcal{S}(X)$ we have $\varphi_\gamma(Z) = \varphi_\alpha(Z)$, so (3) follows from (8).

Finally, suppose that $x \in P_\gamma$ is such that $P_\gamma \setminus \downarrow_\gamma x \in \mathcal{P}(X)$. From $x \in P_\gamma$ we obtain that $x \notin X_\alpha$ and $X \cap P_\alpha \setminus \downarrow_\alpha x = P_\gamma \setminus \downarrow_\gamma x$. (If $z \in X \cap P_\alpha \setminus \downarrow_\alpha x$, then $x <_\alpha z$, so $x \notin \downarrow_\alpha X_\alpha$ implies $z \notin \downarrow_\alpha X_\alpha$ and hence $z \in P_\gamma$.) Thus, $\varphi_\gamma(Z) = \varphi_\alpha(Z) \in \downarrow_\alpha x$ for some $Z \in \mathcal{S}(X)$. We have already proved that $\varphi_\gamma(Z) \in Z \cap P_\gamma$, so $\varphi_\gamma(Z) \in (Z \cap P_\gamma) \cap \downarrow_\alpha x \subseteq \downarrow_\gamma x$. Thus, (4) holds. $\square$

Thus, we have defined the restrictions of elements of B_L to the elements of B_X. Further, we need the following technical tool. Let $\alpha, \beta \in B_L$, $X \in \mathcal{F}(L)$. We say that α and β are X-equivalent ($\alpha \sim_X \beta$) iff the following conditions are satisfied.

(9) $P_\alpha = P_\beta$ and $P_{\alpha\restriction X} = P_{\beta\restriction X}$;

(10) $\varphi_\alpha(Z) = \varphi_\beta(Z)$ for every $Z \in \mathcal{S}(L) \setminus \mathcal{S}(X)$;

(11) there exists an isomorphism of ordered sets $t\colon (P_\alpha, \leq_\alpha) \to (P_\beta, \leq_\beta)$ such that $t(x) = x$ for every $x \in P_\alpha \setminus P_{\alpha\restriction X}$.

It is easy to see that $\sim_X$ is indeed an equivalence relation on B_L.

Lemma 2.2. *For every $\alpha \in B_L$, $X \in \mathcal{F}(L)$ and $\delta \in B_X$ with $P_\delta = P_{\alpha\restriction X}$, there exists unique $\alpha^\delta \in B_L$ such that*

(i) *$\alpha^\delta\restriction X = \delta$;*

(ii) *$\alpha^\delta \sim_X \alpha$.*

Moreover, if $\beta \in B_L$ *is such that* $\alpha \sim_X \beta$, *then* $\alpha^\delta = \beta^\delta$.

Proof. The ordered sets $(P_\delta, \le_\delta)$ and $(P_\delta, \le_{\alpha\restriction X})$ are finite chains of the same cardinality, so there exists an order isomorphism

$$t_0\colon (P_\delta, \le_\delta) \to (P_\delta, \le_{\alpha\restriction X}).$$

Extend it to a map $P_\alpha \to P_\alpha$ by putting $t(y) = y$ for every $y \in P_\alpha \setminus P_\delta$. Define the order relation $\le_\varepsilon$ on P_α by $x \le_\varepsilon y$ iff $t(x) \le_\alpha t(y)$. Since t is a bijection, $\le_\varepsilon$ is a linear order. Further, we set $\varphi_\varepsilon = \varphi_\delta \cup \bigl(\varphi_\alpha\restriction(\mathcal{S}(L) \setminus \mathcal{S}(X))\bigr)$. We claim that the triple $\alpha^\delta = (P_\alpha, \le_\varepsilon, \varphi_\varepsilon)$ has the required properties.

Since $\le_\alpha$ and $\le_\varepsilon$ coincide on $P_\alpha \setminus P_{\alpha\restriction X}$, we obtain that

(12) $\downarrow_\varepsilon x = \downarrow_\alpha x$ for every $x \in P_\alpha$ such that $\downarrow_\alpha x \cap P_{\alpha\restriction X} = \emptyset$ (especially, for every $x \in X_\alpha$).

Claim 1. $\alpha^\delta \in B_L$.

Proof of Claim. (1) and (2) are clear. Let $Z \in \mathcal{S}(L)$. If $Z \nsubseteq X$ then $\varphi_\varepsilon(Z) = \varphi_\alpha(Z) \in Z \cap P_\alpha$ because $\alpha \in B_L$. If $Z \subseteq X$ then $\varphi_\varepsilon(Z) = \varphi_\delta(Z) \in Z \cap P_\delta = Z \cap P_{\alpha\restriction X} \subseteq Z \cap P_\alpha$. Thus, (3) holds. To prove (4), suppose now that $x \in P_\alpha$ and $P_\alpha \setminus \downarrow_\varepsilon x \in \mathcal{P}(L)$. We distinguish two cases.

(I) Suppose that $\downarrow_\alpha x \cap P_{\alpha\restriction X} = \emptyset$. By (12) we have $P_\alpha \setminus \downarrow_\alpha x \in \mathcal{P}(L)$. Since $\alpha \in B_L$, there exists $Z \in \mathcal{S}(L)$ with $\varphi_\alpha(Z) \in \downarrow_\alpha x = \downarrow_\varepsilon x$. If $Z \subseteq X$, then $\varphi_\alpha(Z) = \varphi_{\alpha\restriction X}(Z) \in P_{\alpha\restriction X}$, which is impossible because $\downarrow_\alpha x \cap P_{\alpha\restriction X} = \emptyset$. Hence, $Z \nsubseteq X$ and $\varphi_\varepsilon(Z) = \varphi_\alpha(Z) \in \downarrow_\alpha x = \downarrow_\varepsilon x$.

(II) Let $\downarrow_\alpha x \cap P_{\alpha\restriction X} = \downarrow_\alpha x \cap P_\delta \neq \emptyset$. Then also $Y = \downarrow_\varepsilon x \cap P_\delta \neq \emptyset$. Indeed, if $x \notin P_\delta$ and $x >_\alpha z \in P_\delta$, then $x >_\varepsilon t^{-1}(z) \in P_\delta$. Hence, there exists $v = \max_\varepsilon Y$ (the maximum with respect to $\le_\varepsilon$). Since $\le_\varepsilon$ and $\le_\delta$ coincide on P_δ, we have $P_\delta \setminus \downarrow_\varepsilon x = P_\delta \setminus \downarrow_\varepsilon v = P_\delta \setminus \downarrow_\delta v$. For every $y \in X_\alpha$ we have $y <_\alpha x$. Since $\downarrow_\alpha y = \downarrow_\varepsilon y$, we obtain that $y <_\varepsilon x$. Now we compute $P_\delta \setminus \downarrow_\delta v = P_{\alpha\restriction X} \setminus \downarrow_\varepsilon x = (X \cap P_\alpha) \setminus \downarrow_\alpha X_\alpha \setminus \downarrow_\varepsilon x = (X \cap P_\alpha) \setminus \downarrow_\varepsilon X_\alpha \setminus \downarrow_\varepsilon x = (X \cap P_\alpha) \setminus \downarrow_\varepsilon x = X \cap (P_\alpha \setminus \downarrow_\varepsilon x)$. By our assumption, $P_\alpha \setminus \downarrow_\varepsilon x \in \mathcal{P}(L)$, hence $P_\delta \setminus \downarrow_\delta v \in \mathcal{P}(X)$. The condition (4) for $\delta \in B_X$ yields that there exists $Z \in \mathcal{S}(X)$ with $\varphi_\varepsilon(Z) = \varphi_\delta(Z) \in \downarrow_\delta v \subseteq \downarrow_\varepsilon v \subseteq \downarrow_\varepsilon x$. This completes the proof of Claim 1.

Claim 2. $X_{\alpha^\delta} = X_\alpha$.

Proof of Claim. Suppose first that $y \in X_\alpha$. Then $\downarrow_\varepsilon y = \downarrow_\alpha y$, so $X \cap (P_\alpha \setminus \downarrow_\varepsilon y) \in \mathcal{P}(X)$. Further, $\delta \in B_X$ implies that, for $Z \in \mathcal{S}(X)$, $\varphi_\varepsilon(Z) = \varphi_\delta(Z) \in P_\delta$. Since $\downarrow_\varepsilon y \cap X = \downarrow_\alpha y \cap X$ is disjoint to $P_{\alpha\restriction X} = P_\delta$, we have $\varphi_\varepsilon(Z) \notin \downarrow_\varepsilon y$, so $y \in X_{\alpha^\delta}$.

Conversely, suppose that $y \in X_{\alpha^\delta}$. If $y \in P_{\alpha\restriction X}$ then $z <_\varepsilon y$ for every $z \in X_\alpha$. (Indeed, $z = t(z) <_\alpha t(y) \in P_{\alpha\restriction X}$.) Consequently, $P_{\alpha\restriction X} \setminus \downarrow_\varepsilon y = X \cap$

$(P_\alpha \setminus {\downarrow_\varepsilon} y) \in \mathcal{P}(X)$. Since $\leq_\varepsilon$ and $\leq_\delta$ coincide on $P_{\alpha\restriction X} = P_\delta$, we have $P_\delta \setminus {\downarrow_\delta} y = P_{\alpha\restriction X} \setminus {\downarrow_\varepsilon} y \in \mathcal{P}(X)$. Further, for every $Z \in \mathcal{S}(X)$ we have $\varphi_\delta(Z) = \varphi_\varepsilon(Z) \notin {\downarrow_\varepsilon} y \supseteq {\downarrow_\delta} y$, which contradicts the assumption $\delta \in B_X$ (property (4)). Thus, $y \in P_\alpha \cap X \setminus P_{\alpha\restriction X}$, hence ${\downarrow_\alpha} y = {\downarrow_\varepsilon} y$, so $X \cap (P_\alpha \setminus {\downarrow_\alpha} y) = X \cap (P_\alpha \setminus {\downarrow_\varepsilon} y) \in \mathcal{P}(X)$. Let $Z \in \mathcal{S}(X)$. We have $y \leq_\alpha z$ for some $z \in X_\alpha$ and therefore $\varphi_\alpha(Z) \notin {\downarrow_\alpha} z \supseteq {\downarrow_\alpha} y$. This shows that $y \in X_\alpha$. The proof of Claim is complete.

The equality $X_\alpha = X_{\alpha^\delta}$ implies that $P_\delta = P_{\alpha\restriction X} = P_{\alpha^\delta\restriction X}$. Since the order $\leq_\varepsilon$ restricted to $P_{\alpha\restriction X}$ equals $\leq_\delta$, and $\varphi_\delta = \varphi_\varepsilon \restriction \mathcal{S}(X)$, we have proved (i).

The statement (ii) is clear. We have already proved (9). Conditions (10) and (11) follow directly from the definition of α^δ.

To prove the uniqueness, suppose that $\gamma \in B_L$, $\alpha \sim_X \gamma$ and $\gamma\restriction X = \delta$. Then $P_\gamma = P_\alpha = P_{\alpha^\delta}$ and $\varphi_\gamma = \varphi_\delta \cup \varphi_\alpha \restriction \big(\mathcal{S}(L) \setminus \mathcal{S}(X)\big)$. It remains to show that the orders $\leq_\varepsilon$ and $\leq_\gamma$ are equal. Since $\alpha^\delta \sim_X \alpha \sim_X \gamma$, we have order isomorphisms $t \colon (P_\alpha, \leq_\alpha) \to (P_\alpha, \leq_\varepsilon)$ and $s \colon (P_\alpha, \leq_\gamma) \to (P_\alpha, \leq_\alpha)$ such that $t(x) = s(x) = x$ for all $x \in P_\alpha \setminus P_\delta$ (as $P_\delta = P_{\alpha^\delta\restriction X} = P_{\gamma\restriction X}$). Then $ts \colon (P_\alpha, \leq_\gamma) \to (P_\alpha, \leq_\varepsilon)$ is also an isomorphism. Let $P_\delta = \{x_1, \dots, x_m\}$ with $x_1 <_\gamma \cdots <_\gamma x_m$. Since $\gamma\restriction X = \delta$ we have $x_1 <_\delta \cdots <_\delta x_m$. Since $\alpha^\delta\restriction X = \delta$ we have $x_1 <_\varepsilon \cdots <_\varepsilon x_m$. Since ts maps the set P_δ isomorphically onto P_δ, we obtain that $ts(x_i) = x_i$ for every $i = 1, \dots, m$. Hence, ts is an identity and the orders $\leq_\gamma$ and $\leq_\varepsilon$ coincide. Thus, $\gamma = \alpha^\delta$.

Finally, if $\alpha \sim_X \beta$ then, by (ii), $\alpha \sim_X \beta \sim_X \beta^\delta$ and, by (i), $\beta^\delta\restriction X = \delta$. The uniqueness property of α^δ ensures that $\beta^\delta = \alpha^\delta$. ☐

Lemma 2.3. *Let $\alpha, \beta \in B_L$, $X, Y \in \mathcal{F}(L)$ with $X \subseteq Y$. Then*

- (i) $(\alpha\restriction Y)\restriction X = \alpha\restriction X$*;*
- (ii) $\alpha \sim_X \beta$ *if and only if* $\alpha \sim_Y \beta$ *and* $\alpha\restriction Y \sim_X \beta\restriction Y$*;*

Proof. For every $x \in X \cap P_{\alpha\restriction Y}$ and $y \in Y_\alpha$ we have $y <_\alpha x$, so

(13) $X \cap (P_{\alpha\restriction Y} \setminus {\downarrow_{\alpha\restriction Y}} x) = X \cap (P_{\alpha\restriction Y} \setminus {\downarrow_\alpha} x) = X \cap (P_\alpha \setminus {\downarrow_\alpha} Y_\alpha \setminus {\downarrow_\alpha} x) = X \cap (P_\alpha \setminus {\downarrow_\alpha} x)$.

Claim. $X_{\alpha\restriction Y} = X_\alpha \cap P_{\alpha\restriction Y}$.

Proof of Claim. Let $x \in X_{\alpha\restriction Y}$. By (13) we have $X \cap (P_\alpha \setminus {\downarrow_\alpha} x) \in \mathcal{P}(X)$. For any $Z \subseteq X \subseteq Y$, (8) yields $\varphi_\alpha(Z) \in P_{\alpha\restriction Y}$. Now, $\varphi_\alpha(Z) = \varphi_{\alpha\restriction Y}(Z) \notin {\downarrow_{\alpha\restriction Y}} x$ implies $\varphi_\alpha(Z) \notin {\downarrow_\alpha} x$, which shows that $x \in X_\alpha$.

Conversely, let $x \in X_\alpha \cap P_{\alpha\restriction Y}$. By (13), $X \cap (P_{\alpha\restriction Y} \setminus {\downarrow_{\alpha\restriction Y}} x) \in \mathcal{P}(X)$. For every $Z \subseteq X$, $\varphi_\alpha(Z) \notin {\downarrow_\alpha} x$, which clearly implies that $\varphi_{\alpha\restriction Y}(Z) \notin {\downarrow_{\alpha\restriction Y}} x$, hence $x \in X_{\alpha\restriction Y}$ and the Claim is proved.

By the above claim we have ${\downarrow_\alpha} x \cap P_{\alpha\restriction Y} = \emptyset$ for every $x \in X_\alpha \setminus P_{\alpha\restriction Y}$, hence

(14) $P_{(\alpha\restriction Y)\restriction X} = X \cap P_{\alpha\restriction Y} \setminus {\downarrow_{\alpha\restriction Y}}(X_\alpha \cap P_{\alpha\restriction Y}) = X \cap (P_{\alpha\restriction Y} \setminus {\downarrow_{\alpha\restriction Y}} X_\alpha) = X \cap (P_{\alpha\restriction Y} \setminus {\downarrow_\alpha} X_\alpha) = X \cap (P_\alpha \setminus {\downarrow_\alpha} Y_\alpha \setminus {\downarrow_\alpha} X_\alpha)$.

Since $Y_\alpha \subseteq X_\alpha$, we obtain that $P_{(\alpha\restriction Y)\restriction X} = P_{\alpha\restriction X}$. Further, both $\leq_{(\alpha\restriction Y)\restriction X}$ and $\leq_{\alpha\restriction X}$ are restrictions of $\leq_\alpha$, so they must coincide. Similarly, $\varphi_{(\alpha\restriction Y)\restriction X}$ and $\varphi_{\alpha\restriction X}$ are restrictions of φ_α, so they coincide too. This completes the proof of (i).

To prove (ii) we need to show that (9), (10) and (11) are equivalent to the following six conditions.

(9a) $P_\alpha = P_\beta$ and $P_{\alpha\restriction Y} = P_{\beta\restriction Y}$;

(9b) $P_{\alpha\restriction Y} = P_{\beta\restriction Y}$ and $P_{(\alpha\restriction Y)\restriction X} = P_{(\beta\restriction Y)\restriction X}$;

(10a) $\varphi_\alpha(Z) = \varphi_\beta(Z)$ for every $Z \in \mathcal{S}(L) \setminus \mathcal{S}(Y)$;

(10b) $\varphi_{\alpha\restriction Y}(Z) = \varphi_{\beta\restriction Y}(Z)$ for every $Z \in \mathcal{S}(Y) \setminus \mathcal{S}(X)$;

(11a) there is an order isomorphism $t_1\colon (P_\alpha, \leq_\alpha) \to (P_\beta, \leq_\beta)$ such that $t_1(x) = x$ for every $x \in P_\alpha \setminus P_{\alpha\restriction Y}$;

(11b) there is an order isomorphism $t_2\colon (P_{\alpha\restriction Y}, \leq_{\alpha\restriction Y}) \to (P_{\beta\restriction Y}, \leq_{\beta\restriction Y})$ such that $t_2(x) = x$ for every $x \in P_{\alpha\restriction Y} \setminus P_{(\alpha\restriction Y)\restriction X}$.

Assume first that (9), (10) and (11) are satisfied. Thus, $P_\alpha = P_\beta$. We claim that $Y_\alpha = Y_\beta$. Let $y \in Y_\alpha$. The condition (11) implies that $\downarrow_\alpha x = \downarrow_\beta x$ for every $x \in P_\alpha \setminus P_{\alpha\restriction X}$, especially $\downarrow_\alpha y = \downarrow_\beta y$ (since $Y_\alpha \subseteq P_\alpha \setminus P_{\alpha\restriction Y} \subseteq P_\alpha \setminus P_{\alpha\restriction X}$). Hence, $Y \cap (P_\beta \setminus \downarrow_\beta y) = Y \cap (P_\alpha \setminus \downarrow_\alpha y) \in \mathcal{P}(Y)$. Further, let $Z \subseteq Y$. If $Z \nsubseteq X$, then by (10), $\varphi_\beta(Z) = \varphi_\alpha(Z) \notin \downarrow_\alpha y = \downarrow_\beta y$. If $Z \subseteq X$ then $\varphi_\beta(Z) = \varphi_{\beta\restriction X}(Z) \in P_{\beta\restriction X} = P_{\alpha\restriction X} \subseteq P_\alpha \setminus \downarrow_\alpha y = P_\alpha \setminus \downarrow_\beta y$, so $\varphi_\beta(Z) \notin \downarrow_\beta y$. Hence, $y \in Y_\beta$, so we have $Y_\alpha \subseteq Y_\beta$. Because of the symmetry, $Y_\alpha = Y_\beta$. Consequently, $P_{\alpha\restriction Y} = Y \cap (P_\alpha \setminus \downarrow_\alpha Y_\alpha) = Y \cap (P_\beta \setminus \downarrow_\beta Y_\beta) = P_{\beta\restriction Y}$, which shows (9a).

In (i) we have proved that $P_{(\alpha\restriction Y)\restriction X} = P_{\alpha\restriction X}$, $P_{(\beta\restriction Y)\restriction X} = P_{\beta\restriction X}$, so (9b) is proved too.

Next, $\varphi_{\alpha\restriction ?Y}(Z) = \varphi_\alpha(Z)$, $\varphi_{\beta\restriction Y}(Z) = \varphi_\beta(Z)$, for every $Z \subseteq Y$, so (10a) and (10b) are (together) obviously equivalent to (10).

Further, (11a) follows easily from (11), since $P_{\alpha\restriction Y} \supseteq P_{\alpha\restriction X}$. The isomorphism $t\colon (P_\alpha, \leq_\alpha) \to (P_\beta, \leq_\beta)$, which exists by (11), is identical on $P_\alpha \setminus P_{\alpha\restriction Y} \subseteq P_\alpha \setminus P_{\alpha\restriction X}$. Hence, it maps $P_{\alpha\restriction Y}$ onto $P_{\beta\restriction Y}$ $(= P_{\alpha\restriction Y})$ and we can consider the restriction $t\restriction P_{\alpha\restriction Y}$, which shows (11b).

Conversely, suppose that (9a)–(11b) are satisfied. Since $P_{(\alpha\restriction Y)\restriction X} = P_{\alpha\restriction X}$, $P_{(\beta\restriction Y)\restriction X} = P_{\beta\restriction X}$, (9a) and (9b) imply (9). Next, (10) obviously follows from (10a) and (10b). Since t_1 maps $P_{\alpha\restriction Y}$ onto $P_{\beta\restriction Y}$, the restriction

$$t_1 \restriction P_{\alpha\restriction Y}\colon (P_{\alpha\restriction Y}, \leq_{\alpha\restriction Y}) \to (P_{\beta\restriction Y}, \leq_{\beta\restriction Y})$$

is also an isomorphism. However, $(P_{\alpha\restriction Y}, \leq_{\alpha\restriction Y})$ is a finite chain, so the isomorphisms $t_1 \restriction P_{\alpha\restriction Y}$ and t_2 must coincide. Hence, $t_1(x) = x$ for every $x \in P_{\alpha\restriction Y} \setminus P_{\alpha\restriction X}$, so t_1 satisfies the requirements of (11). $\square$

Lemma 2.4. *Let $X \in \mathcal{F}(L)$, $X \neq L$. Let $P \in \mathcal{P}(X)$, $Q \in \mathcal{P}(L)$ with $P \subseteq Q$. Then there exists $\delta \in B_L$ such that $P_\delta = Q$, $P_{\delta\restriction X} = P$.*

Proof. Put $P_\delta = Q$. Choose a linear order $\leq_\delta$ on P_δ in such a way that $x <_\delta 1 \leq_\delta y$ for every $x \in X \cap Q \setminus P$ and $y \in Q \setminus (X \cap Q \setminus P)$. Let $X \cap Q \setminus P = \{x_1, \dots, x_m\}$ with $x_1 <_\delta \dots <_\delta x_m$. (The case $m = 0$ is possible.) Choose $y \in L \setminus X$ arbitrarily and for any $Z \in \mathcal{S}(Y)$ define

$$\varphi_\delta(Z) = \begin{cases} x_1 & \text{if } m > 0 \text{ and } Z = \{0, 1, y, x_1\}\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

We claim that $\delta = (P_\delta, \leq_\delta, \varphi_\delta) \in B_L$. Clearly, (1), (2) and (3) are satisfied. Let $x \in P_\delta$. If $x \notin X \cap Q \setminus P$, then $1 \leq_\delta x$, hence $(P_\delta \setminus {\downarrow}_\delta x) \notin \mathcal{P}(Y)$. If $x \in X \cap Q \setminus P$, then $x_1 \in {\downarrow}_\delta x$, so we have $\varphi_\delta(Z) \in {\downarrow}_\delta x$ for $Z = \{0, 1, y, x_1\}$ and (4) holds too.

It remains to show that $P_{\delta\restriction X} = P$. It is clear that $X_\delta \subseteq \{x_1, \dots, x_m\}$. (If $x \in P_\delta \setminus \{x_1, \dots, x_m\}$ then $1 \notin X \cap (P_\delta \setminus {\downarrow}_\delta x) \notin \mathcal{P}(X)$.) Thus, if $X \cap Q \setminus P = \emptyset$, then $X_\delta = \emptyset$ and $P_{\delta\restriction X} = P_\delta \cap X = Q \cap X = P$. Let $X \cap Q \setminus P \neq \emptyset$. We claim that $x_m \in X_\delta$. Clearly, $X \cap (P_\delta \setminus {\downarrow}_\delta x_m) = X \cap \big(Q \setminus (X \cap Q \setminus P)\big) = P \in \mathcal{P}(X)$. For every $Z \in \mathcal{S}(X)$ we have $\varphi_\delta(Z) = 1$ (as $\{0, 1, y, x_1\} \notin \mathcal{S}(X)$), hence $\varphi_\delta(Z) \notin {\downarrow}_\delta x_m$. Thus, $x_m \in X_\delta$. Clearly, $x_m = \max_\delta X_\delta$, hence $P_{\delta\restriction X} = X \cap (P_\delta \setminus {\downarrow}_\delta x_m) = P$. ◻

3. Construction

Let F be a field. For any set S let F_S denote the set of all functions $f \colon S \times S \to F$ (i.e., possibly infinite matrices) with the property that for every $i \in S$ the sets $\{j \in S \colon f(i,j) \neq 0\}$ and $\{j \in S \colon f(j,i) \neq 0\}$ are finite. We define the F-algebra operations similarly as for usual matrices. For $f, g \in F_S$, $r \in F$, we set $(f+g)(i,j) = f(i,j) + g(i,j)$, $(rf)(i,j) = r \cdot f(i,j)$ and $(fg)(i,j) = \sum\limits_{k \in S} f(i,k)g(k,j)$. It is not difficult to check that the operations are well defined and F_S is an F-algebra.

Let K be an infinite distributive lattice with 0. Let L be the lattice which arises from K by adding a new greatest element 1. For every $X \in \mathcal{F}(L)$ and every $\sigma, \tau \in B_X$ with $P_\sigma = P_\tau$ we define a map $f_{\sigma\tau} \colon B_L \times B_L \to F$ by

$$f_{\sigma\tau}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha\restriction X = \sigma, \beta\restriction X = \tau \text{ and } \alpha \sim_X \beta\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

It is easy to see that $f_{\sigma\tau} \in F_{B_L}$. Let A be the subalgebra of F_{B_L} generated by all $f_{\sigma\tau}$ such that $P_\sigma \neq \{1\}$. We claim that A is locally matricial and the lattice of its two-sided finitely generated ideals is isomorphic to K. The proof will take the rest of this paper.

Lemma 3.1. *$X \in \mathcal{F}(L)$, $\rho, \pi, \sigma, \tau \in B_X$ and $P_\pi = P_\rho$, $P_\sigma = P_\tau$.*

(i) *If $\pi = \sigma$ then $f_{\rho\pi} f_{\sigma\tau} = f_{\rho\tau}$.*

(ii) *If $\pi \neq \sigma$ then $f_{\rho\pi} f_{\sigma\tau} = 0$.*

Proof. (i) By the definition, $f_{\rho\pi} f_{\sigma\tau}(\alpha, \beta) = \sum_{\gamma \in B_L} f_{\rho\sigma}(\alpha, \gamma) f_{\sigma\tau}(\gamma, \beta)$. If $\alpha \restriction X \neq \rho$ or $\beta \restriction X \neq \tau$ or not$(\alpha \sim_X \beta)$ then clearly $f_{\rho\sigma} f_{\sigma\tau}(\alpha, \beta) = 0 = f_{\rho\tau}(\alpha, \beta)$. Suppose now that $\alpha \restriction X = \rho$, $\beta \restriction X = \tau$ and $\alpha \sim_X \beta$. Then $f_{\rho\tau}(\alpha, \beta) = 1$. Further, $f_{\rho\sigma}(\alpha, \gamma) = 1$ iff $\gamma \restriction X = \sigma$ and $\alpha \sim_X \gamma$. By 2.2, this is true iff $\gamma = \alpha^\sigma$. Similarly, $f_{\sigma\tau}(\gamma, \beta) = 1$ iff $\gamma = \beta^\sigma$. By 2.2 we have $\alpha^\sigma = \beta^\sigma$, so $f_{\rho\sigma} f_{\sigma\tau}(\alpha, \beta) = f_{\rho\sigma}(\alpha, \alpha^\sigma) f_{\sigma\tau}(\beta^\sigma, \beta) = 1$. Thus, the functions $f_{\rho\sigma} f_{\sigma\tau}$ and $f_{\rho\tau}$ coincide for every $\alpha, \beta \in B_L$.

(ii) Let $\pi \neq \sigma$. Let $\alpha, \beta \in B_L$. For every $\gamma \in B_L$ we have $\gamma \restriction X \neq \pi$ or $\gamma \restriction X \neq \sigma$. Hence, $f_{\rho\pi}(\alpha, \gamma) f_{\sigma\tau}(\gamma, \beta) = 0$. □

Lemma 3.2. *Let $X, Y \in \mathcal{F}(L)$, $X \subseteq Y$, $\sigma, \tau \in B_X$, $P_\sigma = P_\tau$. Then*

$$f_{\sigma\tau} = \sum \{ f_{\gamma\delta} \colon (\gamma, \delta) \in \mathcal{U} \},$$

where $\mathcal{U} = \{ (\gamma, \delta) \in B_Y \times B_Y \colon \gamma \restriction X = \sigma,\ \delta \restriction X = \tau,\ \gamma \sim_X \delta \}$.

Proof. Let $\alpha, \beta \in B_L$. Suppose first that $f_{\sigma\tau}(\alpha, \beta) = 1$. We define $\pi = \alpha \restriction Y$, $\rho = \beta \restriction Y$. By 2.3 we have $\pi \restriction X = \sigma$, $\rho \restriction X = \tau$, $\pi \sim_X \rho$ and $\alpha \sim_Y \beta$. Hence, $(\pi, \rho) \in \mathcal{U}$ and $f_{\pi\rho}(\alpha, \beta) = 1$. For any $(\gamma, \delta) \in \mathcal{U} \setminus \{ (\pi, \rho) \}$ we have $\gamma \neq \alpha \restriction Y$ or $\delta \neq \beta \restriction Y$, hence $f_{\gamma\delta}(\alpha, \beta) = 0$. Thus, $\sum \{ f_{\gamma\delta}(\alpha, \beta) \colon (\gamma, \delta) \in \mathcal{U} \} = 1$.

Conversely, suppose that $f_{\pi\rho}(\alpha, \beta) = 1$ for some $(\pi, \rho) \in \mathcal{U}$. We have $\alpha \restriction X = (\alpha \restriction Y) \restriction X = \pi \restriction X = \sigma$ and similarly $\beta \restriction X = \tau$. Further, $f_{\pi\rho}(\alpha, \beta) = 1$ implies $\alpha \sim_Y \beta$ and $(\pi, \rho) \in \mathcal{U}$ implies $\alpha \restriction Y = \pi \sim_X \rho = \beta \restriction Y$. By 2.3 we have $\alpha \sim_X \beta$. Thus, $f_{\sigma\tau}(\alpha, \beta) = 1$. □

The above two lemmas lead to the following description of elements of A.

Lemma 3.3. *Every element of A can be expressed in the form $\sum_{i=1}^{n} q_i f_{\gamma_i \delta_i}$, where $q_i \in F$ and all γ_i and δ_i belong to the same B_X (for some $X \in \mathcal{F}(L)$) and $P_{\gamma_i} \neq \{1\}$.*

Proof. The set of all elements of this form contains all generators of A and is a subalgebra of F_{B_L}. Indeed, let $x = \sum q_i f_{\gamma_i \delta_i}$, $y = \sum r_j f_{\sigma_j \tau_j}$. Because of 3.2 we can assume that all γ_i, δ_i, σ_j, τ_j belong to the same B_X. (If $\gamma_i \in B_{X_i}$, $\sigma_j \in B_{Y_j}$, then choose $X \in \mathcal{F}(L)$ with $X_i \subseteq X$, $Y_j \subseteq X$, for every i, j.) Then, by 3.1, $xy = \sum \{ q_i r_j f_{\gamma_i \tau_j} : \delta_i = \sigma_j \}$. □

For $X \in \mathcal{F}(L)$, $\sigma \in B_X$, the set P_σ is a prime filter in the finite distributive lattice X, hence it has a smallest element $\operatorname{val}(\sigma)$. This element is join-irreducible in X and will be called the value of σ.

Lemma 3.4. *If* $\sigma \in B_Y$, $X \subseteq Y$, *then* $\operatorname{val}(\sigma\restriction X) \geq \operatorname{val}(\sigma)$.

Proof. The claim follows from $P_{\sigma\restriction X} \subseteq P_\sigma$. $\square$

For $x \in A$ let $[x]$ denote the two-sided ideal generated by x.

Lemma 3.5. *If* $X \in \mathcal{F}(L)$, $\pi, \rho, \sigma, \tau \in B_X$, $P_\pi = P_\rho = P_\sigma = P_\tau \neq \{1\}$, *then* $[f_{\pi\rho}] = [f_{\sigma\tau}]$.

Proof.
By 3.1, $f_{\pi\rho} = f_{\pi\sigma} f_{\sigma\tau} f_{\tau\rho}$. $\square$

Lemma 3.6. *Suppose that* $X, Y \in \mathcal{F}(L)$, $\sigma, \tau \in B_X$, $\pi, \rho \in B_Y$, $P_\sigma = P_\tau$, $P_\pi = P_\rho$. *If* $1 \neq \operatorname{val}(\sigma) \geq \operatorname{val}(\pi)$ *then* $f_{\pi\rho} \in [f_{\sigma\tau}]$.

Proof. Suppose first that $X \subsetneq Y$. The inequality $\operatorname{val}(\sigma) \geq \operatorname{val}(\pi)$ implies that $P_\sigma \subseteq P_\pi$. By 2.4 there exists $\delta \in B_Y$ with $P_\delta = P_\pi$, $P_{\delta\restriction X} = P_\sigma$. Denote $\varepsilon = \delta\restriction X$. By 3.2, $f_{\varepsilon\varepsilon} = \sum\{f_{\alpha\beta} : (\alpha, \beta) \in \mathcal{U}\}$, where

$$\mathcal{U} = \bigl\{(\alpha, \beta) \in B_Y \times B_Y : \alpha\restriction X = \beta\restriction X = \varepsilon,\ \alpha \sim_X \beta\bigr\}.$$

Clearly, $(\delta, \delta) \in \mathcal{U}$. For any $(\alpha, \beta) \in \mathcal{U} \setminus \bigl\{(\delta, \delta)\bigr\}$ we have $\alpha \neq \delta$ or $\beta \neq \delta$, so $f_{\delta\delta} f_{\alpha\beta} f_{\delta\delta} = 0$. By the distributivity and 3.1, $f_{\delta\delta} f_{\varepsilon\varepsilon} f_{\delta\delta} = f_{\delta\delta} f_{\delta\delta} f_{\delta\delta} = f_{\delta\delta}$, which shows that $f_{\delta\delta} \in [f_{\varepsilon\varepsilon}]$, or equivalently, $[f_{\delta\delta}] \subseteq [f_{\varepsilon\varepsilon}]$. By 3.5 we have $[f_{\pi\rho}] = [f_{\delta\delta}] \subseteq [f_{\varepsilon\varepsilon}] = [f_{\sigma\tau}]$, so $f_{\pi\rho} \in [f_{\sigma\tau}]$.

Now the general case. Let X, Y be arbitrary. Since L is infinite, it is possible to choose $Z \in \mathcal{F}(L)$ with $Y \subseteq Z$ and $X \subsetneq Z$. By 3.2, $f_{\pi\rho} = \sum\{f_{\alpha\beta} : (\alpha, \beta) \in \mathcal{V}\}$, where $\mathcal{V} = \bigl\{(\alpha, \beta) \in B_Z \times B_Z : \alpha\restriction Y = \pi,\ \beta\restriction Y = \rho,\ \alpha \sim_Y \beta\bigr\}$. For every $(\alpha, \beta) \in \mathcal{V}$ we have $P_\alpha \cap Y \supseteq P_\pi$, hence $\min P_\pi \geq \min P_\alpha$, so $\operatorname{val}(\alpha) \leq \operatorname{val}(\pi) \leq \operatorname{val}(\sigma)$. By the first part of this proof, $f_{\alpha\beta} \in [f_{\sigma\tau}]$ for every $(\alpha, \beta) \in \mathcal{V}$. Then also $f_{\pi\rho} \in [f_{\sigma\tau}]$. $\square$

For every $r \in K$, $r > 0$, let I_r denote the set of all $x \in A$ which can be, for some n, expressed in the form $x = \sum_{i=1}^{n} q_i f_{\delta_i\varepsilon_i}$, where all $f_{\delta_i\varepsilon_i}$ are some generators of A satisfying $\operatorname{val}(\delta_i) = \operatorname{val}(\varepsilon_i) \leq r$. For $r = 0$ we set $I_r = \{0\}$.

Lemma 3.7. *For every* $r \in K$, $I_r \in \operatorname{Id}^c(A)$.

Proof. The case $r = 0$ is trivial, let $r > 0$. The set I_r is certainly closed under subtraction and constant multiplication. Suppose now that $x = \sum q_i f_{\delta_i\varepsilon_i} \in I_r$ and $y = \sum r_j f_{\alpha_j\beta_j} \in A$. Because of 3.2, we can assume that all $\delta_i, \varepsilon_i, \alpha_j, \beta_j$ belong to the same B_X for some $X \in \mathcal{F}(L)$. Then, by 3.1,

$$xy = \sum\{q_i r_j f_{\delta_i\beta_j} : \varepsilon_i = \alpha_j\},$$

which is an element of I_r (and similarly for yx). Thus, I_r is a two-sided ideal.

It remains to prove that I_r is finitely generated. Let $X = \{0, r, 1\} \in \mathcal{F}(L)$ and $P = \{r, 1\}$. Then P is a prime filter in X and, by 2.4, there is $\alpha \in B_X$ with $P_\alpha = P$. Clearly, $\operatorname{val}(\alpha) = r$, so $f_{\alpha\alpha} \in I_r$. For every $x \in I_r$ we have $x \in [f_{\alpha\alpha}]$ by 3.6. Thus, $I_r = [f_{\alpha\alpha}]$. □

Lemma 3.8. *$I_r \subseteq I_s$ if and only if $r \leq s$.*

Proof. If $r \leq s$ then obviously $I_r \subseteq I_s$. Let $r \not\leq s$. There exist a prime filter $P \in \mathcal{P}(L)$ such that $r \in P$, $s \notin P$. Consider $X = \{0, r, 1\} \in \mathcal{F}(L)$. By 2.4, there exists $\alpha \in B_L$ such that $P_\alpha = P$, $P_{\alpha \restriction X} = \{r, 1\}$. Denote $\sigma = \alpha \restriction X$. Then $\operatorname{val}(\sigma) = r$, so $f_{\sigma\sigma} \in I_r$. We claim that $f_{\sigma\sigma} \notin I_s$. For contradiction, suppose that $f_{\sigma\sigma} = \sum_{i=1}^{n} q_i f_{\delta_i \varepsilon_i}$ with $\delta_i, \varepsilon_i \in B_{Y_i}$, $\operatorname{val}(\delta_i) \leq s$ for every i. Then $\operatorname{val}(\delta_i) \notin P$. Since $P_{\alpha \restriction Y_i} \subseteq P$, we have $\delta_i \neq \alpha \restriction Y_i$, hence $f_{\delta_i \varepsilon_i}(\alpha, \alpha) = 0$. On the other hand, $f_{\sigma\sigma}(\alpha, \alpha) = 1$, a contradiction. □

Lemma 3.9. *Let $x = \sum_{i=1}^{n} q_i f_{\delta_i \varepsilon_i}$, where $\delta_i, \varepsilon_i \in B_X$ for some fixed X and all i, $\operatorname{val}(\delta_i) \neq 1$. Suppose that all q_i are nonzero and all pairs $(\delta_i, \varepsilon_i)$ are different. Then $[x] = \bigvee_{i=1}^{n} [f_{\delta_i \varepsilon_i}]$.*

Proof. Clearly, $[x] \subseteq \bigvee [f_{\delta_i \varepsilon_i}]$. Conversely, for every j we have $f_{\delta_j \delta_j} x f_{\varepsilon_j \varepsilon_j} = \sum_{i=1}^{n} q_i f_{\delta_j \delta_j} f_{\delta_i \varepsilon_i} f_{\varepsilon_j \varepsilon_j} = q_j f_{\delta_j \varepsilon_j}$, so $f_{\delta_j \varepsilon_j} \in [x]$, hence $[f_{\delta_j \varepsilon_j}] \subseteq [x]$. □

Lemma 3.10. *$\operatorname{Id}^c(A)$ is isomorphic to K.*

Proof. After 3.8 it suffices to show that every element of $\operatorname{Id}^c(A)$ is of the form I_r for some $r \in K$. Let J be a two-sided ideal of A generated by some elements $x^1, \dots, x^m$. Because of 3.2 and 3.9 we can assume that there is $Y \in \mathcal{F}(L)$ such that $x^i = f_{\delta_i \varepsilon_i}$ for some $\delta_i, \varepsilon_i \in B_Y$, $\operatorname{val}(\delta_i) \neq 1$. Let $r = \bigvee_{i=1}^{m} \operatorname{val}(\delta_i)$. Clearly, $r \in K$. If $r = 0$ then clearly $J = \{0\} = I_0$. Let $r > 0$. Clearly, $x^i \in I_r$ for every i, so $J \subseteq I_r$. Conversely, by 2.4, there is $\sigma \in B_X$ with $\operatorname{val}(\sigma) = r$ and $X = \{0, r, 1\} \subseteq Y$. By 3.2 we have $f_{\sigma\sigma} = \sum \{ f_{\alpha\beta} \colon (\alpha, \beta) \in \mathcal{U} \}$, where

$$\mathcal{U} = \bigl\{ (\alpha, \beta) \in B_Y \times B_Y \colon \alpha \restriction X = \beta \restriction X = \sigma,\ \alpha \sim_X \beta \bigr\}.$$

For every $(\alpha, \beta) \in \mathcal{U}$ we have $\operatorname{val}(\alpha) \leq \operatorname{val}(\sigma) = r = \bigvee \operatorname{val}(\delta_i)$. Since $\operatorname{val}(\alpha)$ is join-irreducible in Y, we obtain that $\operatorname{val}(\alpha) \leq \operatorname{val}(\delta_i)$ for some i. By 3.5, $f_{\alpha\beta} \in [f_{\delta_i \varepsilon_i}] \subseteq J$. Consequently, $f_{\sigma\sigma} \in J$. Since $f_{\sigma\sigma}$ generates I_r, we obtain that $I_r \subseteq J$. □

Lemma 3.11. *Let* $X \in \mathcal{F}(L)$. *If* $\delta, \varepsilon \in B_X$ *and* $\operatorname{val}(\delta) = \operatorname{val}(\varepsilon) \neq 1$ *then* $f_{\delta\varepsilon} \neq 0$.

Proof. There exists $P \in \mathcal{P}(L)$ with $P \cap X = P_\delta$. By 2.4, there is $\alpha \in B_L$ with $P_\alpha = P$, $P_{\alpha\restriction X} = P_\delta$. Let $\beta = \alpha^\delta$, $\gamma = \alpha^\varepsilon$. By 2.2 we have $\beta \sim_X \gamma$, $\beta\restriction X = \delta$, $\gamma\restriction X = \varepsilon$, hence $f_{\delta\varepsilon}(\beta,\gamma) = 1$. □

Lemma 3.12. *A is locally matricial.*

Proof. For every $X \in \mathcal{F}(L)$ let A_X be the set of all $x \in A$ of the form

$$x = \sum\bigl\{q_{ij} f_{\delta_i\delta_j} \colon \delta_i, \delta_j \in B_X,\ P_{\delta_i} = P_{\delta_j} \neq \{1\}\bigr\}$$

for some $q_{ij} \in F$. By 3.1, A_X is a subalgebra of A. Clearly, 3.2 implies that $A_X \subseteq A_Y$ whenever $X \leq Y$. Thus, A is a directed union of its subalgebras A_X, $X \in \mathcal{F}(L)$.

For every $\vee$-irreducible $r \in X$ let $B_r = \bigl\{\beta \in B_X \colon \operatorname{val}(\beta) = r\bigr\}$. Thus, B_X is a disjoint union of the sets B_r. Let R be the direct product of all F_{B_r}. We can view R as the set of all functions $h \colon B_X \times B_X \to F$ such that $h(\beta,\gamma) = 0$ whenever β and γ belong to different B_r. Since X is finite, R is a matricial F-algebra. We claim that R is isomorphic to A_X. We define a map $G \colon R \to A_X$ naturally by

$$G(h) = \sum\bigl\{h(\beta,\gamma) f_{\beta\gamma} \colon \beta, \gamma \in B_X,\ \operatorname{val}(\beta) = \operatorname{val}(\gamma) \neq 1\bigr\}.$$

It is clear that G is surjective and preserves the addition and the constant multiplication. By 3.1, it also preserves the multiplication. For the injectivity of G we need to show that its kernel is trivial. Let $\sum h(\beta,\gamma) f_{\beta\gamma} = 0$. Let $\delta, \varepsilon \in B_X$. Multiplying by $f_{\delta\delta}$ from the left and by $f_{\varepsilon\varepsilon}$ from the right we obtain that $0 = h(\delta,\varepsilon) f_{\delta\varepsilon}$. By 3.1, $h(\delta,\varepsilon) = 0$. The proof is complete. □

Now we have completed the proof of our main result.

Theorem 3.13. *Every infinite distributive lattice with* 0 *is isomorphic to* $\operatorname{Id}^c(A)$ *for some locally matricial algebra* A. (*Consequently, every infinite algebraic distributive lattice whose compact elements form a lattice is isomorphic to* $\operatorname{Id}(A)$.)

Our construction does not work for finite lattices. (The infiniteness was needed in 3.6.) However, the theorem remains true in the finite case. In fact, G. M. Bergman proved ([1]) that every countable, distributive semilattice is representable as $\operatorname{Id}^c(A)$ for some locally matricial algebra A. (See also [5].) This result cannot be extended to the uncountable case. A counterexample of size $\aleph_1$ has been constructed by F. Wehrung in [7].

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Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK–040 01 Košice
SLOVAKIA

E-mail: ploscica@saske.sk