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FINITE CONGRUENCE LATTICES IN CONGRUENCE DISTRIBUTIVE VARIETIES

MIROSLAV PLOŠČICA

Abstract. We present a general theorem characterizing finite congruence lattices of algebras belonging to a congruence distributive variety. We apply the result to some small varieties of lattices.

1. Introduction

This paper is a contribution to the following general problem: *Characterize lattices isomorphic to congruence lattices of algebras in a given variety* $\mathcal{V}$. This problem has proved to be very hard and there are very few varieties for which we have a satisfactory answer. For instance, there is a longstanding problem whether every algebraic distributive lattice is isomorphic to the congruence lattice of some lattice.

In this paper we restrict our attention to congruence distributive varieties (CD varieties), i.e. varieties in which every member has a distributive congruence lattice. The most common examples of such varieties are the varieties of lattices and lattice-ordered structures (l-groups, MV-algebras, etc.). We consider the finite case of the above problem and present a general theorem characterizing finite lattices isomorphic to congruence lattices of algebras in a given CD variety $\mathcal{V}$. As we have already mentioned, the analogous question for infinite lattices is much more complicated. Some attempts to investigate the infinite case are contained in [5].

Our result is based on the observation that an algebra with a prescribed congruence lattice can often be constructed as the limit of a suitable commutative diagram of subdirectly irreducible algebras. Thus, our result is useful mainly for those varieties, where we have a good knowledge of subdirectly irreducible algebras.

Our basic reference books are [1],[2] and [4]. All the unexplained notions and facts can be found there. We use the following special denotations. If P is a partially ordered set and $x \in P$ then $\uparrow x = \{y \in P \mid y \geq x\}$. If $f: A \to B$ is a mapping then the binary relation $\mathrm{Ker}(f) = \{(x, y) \in A^2 \mid f(x) = f(y)\}$ is called the kernel of f. The symbol Δ_A denotes the diagonal relation $\{(x, x) \mid x \in A\}$ on A. If α is an equivalence relation on a set A, then A/α denotes the set of equivalence classes of α and x/α is the equivalence class containing $x \in A$. The Cartesian product of sets A_i

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$(i \in I)$ is denoted by $\Pi_{i\in I} A_i$. We use the notation $a = (a_i)_{i\in I}$ for elements of such product and speak about a_i as the i-th coordinate of a.

2. The general result

First we present a purely set-theoretical construction. An *ordered diagram of sets* is a triple $(P, \mathcal{A}, \mathcal{F})$, where P is a partially ordered set, $\mathcal{A} = \{A_p \mid p \in P\}$ is a family of sets indexed by P and $\mathcal{F} = \{f_{pq} \mid p, q \in P,\ p \leq q\}$ is a family of functions $f_{pq} :\ A_p \to A_q$ such that

(1) f_{pp} is the identity map for every $p \in P$;
(2) $f_{qr} f_{pq} = f_{pr}$ for every $p, q, r \in P$, $p \leq q \leq r$.

For any ordered diagram of sets we define its *limit* as

$$\lim(P, \mathcal{A}, \mathcal{F}) = \{a \in \Pi_{p\in P} A_p \mid a_q = f_{pq}(a_p) \text{ for every } p, q \in P,\ p \leq q\}.$$

Thus, our limit is the limit in the sense of the category theory (applied to the category of sets). In the universal algebra, this construction is often called *the inverse limit* (see [2]). Now we introduce the crucial concept of this paper.

Definition 2.1. *An ordered diagram of sets $(P, \mathcal{A}, \mathcal{F})$ is called admissible if the following conditions are satisfied:*

(i) *for every $p \in P$ and every $u \in A_p$ there exists $a \in \lim(P, \mathcal{A}, \mathcal{F})$ such that $a_p = u$;*
(ii) *for every $p, q \in P$, $p \not\leq q$ there exist $a, b \in \lim(P, \mathcal{A}, \mathcal{F})$ such that $a_p = b_p$ and $a_q \neq b_q$.*

Admissible ordered diagrams arise naturally from systems of equivalences on a given set, as shown by the following assertion.

Lemma 2.2. *Let P be some set of equivalences on a set A, ordered by the set inclusion. For every $\alpha \in P$ let $A_\alpha = A/\alpha$. For $\alpha \subseteq \beta$ let $f_{\alpha\beta} :\ A/\alpha \to A/\beta$ be the natural projection map, i.e. $f_{\alpha\beta}(x/\alpha) = x/\beta$. Then $(P, \mathcal{A}, \mathcal{F})$ is an admissible ordered diagram of sets.*

Proof. It is clear that $(P, \mathcal{A}, \mathcal{F})$ is an ordered diagram of sets. For every $x \in A$ the element $\overline{x} = (x/\alpha)_{\alpha\in P}$ belongs to $\lim(P, \mathcal{A}, \mathcal{F})$, so 2.1(i) is satisfied. Further, let $\alpha, \beta \in P$, $\alpha \nsubseteq \beta$. Then there are $x, y \in A$ with $(x, y) \in \alpha$, $(x, y) \notin \beta$. We have $\overline{x}, \overline{y} \in \lim(P, \mathcal{A}, \mathcal{F})$, $\overline{x}_\alpha = x/\alpha = y/\alpha = \overline{y}_\alpha$ and $\overline{x}_\beta = x/\beta \neq y/\beta = \overline{y}_\beta$, which proves 2.1(ii). □

On the other hand, it is not difficult to construct ordered diagrams which are not admissible.

Now we turn our attention to congruence lattices by recalling some fundamental facts from the lattice theory and the universal algebra. An element $x \neq 1$ of a lattice L is called meet-irreducible if $x = y \wedge z$ implies $x \in \{y, z\}$. An element x is strictly (or completely) meet-irreducible if $x = \inf X$ $(X \subseteq L)$ implies $x \in X$. Equivalently, x is strictly meet-irreducible if the set $\{y \in L \mid y > x\}$ has a smallest element. Of course, if L is finite then every meet-irreducible element is strictly meet-irreducible. If x is meet-irreducible and L is distributive then, for every $x_1, \ldots, x_n \in L$, $x_1 \wedge \cdots \wedge x_n \leq x$

implies that $x_i \leq x$ for some i. Let $M(L)$ denote the set of all strictly meet-irreducible elements of L.

We do not distinguish between an algebra and its underlying set. Let $\operatorname{Con} A$ denote the congruence lattice of an algebra A. It is well known that an algebra A is subdirectly irreducible iff Δ_A (the smallest element of $\operatorname{Con} A$) belongs to $M(\operatorname{Con} A)$. More generally, $\theta \in M(\operatorname{Con} A)$ iff the factor algebra A/θ is subdirectly irreducible. This is a direct consequence of the fact that $\operatorname{Con} A/\theta$ is isomorphic to $\uparrow \theta$ (a subset of $\operatorname{Con} A$).

If $\mathcal{V}$ is a variety (a class of algebras closed under direct products, subalgebras and homomorphic images) then $SI(\mathcal{V})$ denotes the class of all subdirectly irreducible members of $\mathcal{V}$. The facts recalled above have the following consequence.

Lemma 2.3. *Let $\mathcal{V}$ be a variety and $L = \operatorname{Con} A$ for some $A \in \mathcal{V}$. Then for every $x \in M(L)$, the lattice $\uparrow x$ is isomorphic to $\operatorname{Con} T$ for some $T \in SI(\mathcal{V})$.*

This lemma provides a basic information about congruence lattices of algebras in $\mathcal{V}$. It is especially effective in the case of a CD variety $\mathcal{V}$ and a finite lattice L, because finite distributive lattices are determined uniquely by the ordered sets of their meet-irreducible elements. However, 2.3 does not provide a complete characterization of finite congruence lattices for CD varieties.

Theorem 2.4. *Let $\mathcal{V}$ be a CD variety. Let L be a finite distributive lattice. Let $P = M(L)$. For every $p \in P$ let $A_p \in \mathcal{V}$ and for every $p \leq q$ let $f_{pq} : A_p \to A_q$ be a homomorphism such that $(P, \mathcal{A}, \mathcal{F})$ is an admissible ordered diagram of sets (with $\mathcal{A} = \{A_p \mid p \in P\}$ and $\mathcal{F} = \{f_{pq} \mid p \leq q\}$) and, moreover,*

(*) *for all $p \in P$, the sets $\{\operatorname{Ker}(f_{pq}) \mid q \geq p\}$ and $M(\operatorname{Con} A_p)$ coincide.*

Then $\lim(P, \mathcal{A}, \mathcal{F})$ is an algebra whose congruence lattice is isomorphic to L.

Proof. Let $A = \lim(P, \mathcal{A}, \mathcal{F})$. It is easy to see that A is a subalgebra of the algebra $\Pi_{p \in P} A_p$. Hence, $A \in \mathcal{V}$, so $\operatorname{Con} A$ is distributive. Since every finite distributive lattice is determined by its ordered set of meet-irreducible elements, it suffices to prove that the ordered sets P and $M(\operatorname{Con} A)$ are isomorphic. (Strictly speaking, we have not yet proved that $\operatorname{Con} A$ is finite. However, $\operatorname{Con} A$ is algebraic and every algebraic lattice with finitely many completely meet-irreducible elements is finite.)

Condition (*) implies that $\operatorname{Ker}(f_{pp}) = \Delta_{A_p}$ belongs to $M(\operatorname{Con} A_p)$, which means that A_p is subdirectly irreducible. The condition 2.1(i) ensures that the natural projections $h_p : A \to A_p$ ($h_p(x) = x_p$) are surjective. Hence, $A/\operatorname{Ker}(h_p)$ is isomorphic to A_p and therefore $\operatorname{Ker}(h_p) \in M(\operatorname{Con} A)$. Thus, we can define a map $\varphi : P \to M(\operatorname{Con} A)$ by $\varphi(p) = \operatorname{Ker}(h_p)$. We claim that this is the required isomorphism.

Let $p, q \in P$, $p \leq q$. If $(x, y) \in \operatorname{Ker}(h_p)$ then $x_p = y_p$, hence $x_q = f_{pq}(x_p) = f_{pq}(y_p) = y_q$ and $(x, y) \in \operatorname{Ker}(h_q)$. Thus, $\varphi(p) \leq \varphi(q)$.

Let $p, q \in P$, $p \nleq q$. By 2.1(ii), there are $x, y \in A$ such $x_p = y_p$ and $x_q \neq y_q$, hence $(x, y) \in \operatorname{Ker}(h_p) \setminus \operatorname{Ker}(h_q)$. Thus $\varphi(p) \nleq \varphi(q)$. Especially, φ is injective.

It remains to prove the surjectivity of φ. Let $\beta \in M(\operatorname{Con} A)$. Clearly, $\bigwedge_{p \in P} \operatorname{Ker}(h_p) = \Delta_A \leq \beta$. Since P is finite and β is meet-irreducible, we obtain that $\beta \geq \operatorname{Ker}(h_p)$ for some $p \in P$. Consider the natural map $k : A_p \to A/\beta$ defined by $k(x_p) = x/\beta$. It is easy to check that k is well defined ($x_p = y_p$ implies $x/\beta = y/\beta$) and that it is a surjective homomorphism. Since A/β is subdirectly irreducible and isomorphic to $A_p/\operatorname{Ker}(k)$, we have $\operatorname{Ker}(k) \in M(\operatorname{Con} A_p)$. By our assumption (*) we have $\operatorname{Ker}(k) = \operatorname{Ker}(f_{pq})$ for some $q \geq p$. For any $x, y \in A$ we have $(x, y) \in \beta$ iff $k(x_p) = k(y_p)$ iff $(x_p, y_p) \in \operatorname{Ker}(k) = \operatorname{Ker}(f_{pq})$ iff $x_q = f_{pq}(x_p) = f_{pq}(y_p) = y_q$ iff $(x, y) \in \operatorname{Ker}(h_q)$. Hence, $\beta = \varphi(q)$. $\square$

Theorem 2.5. *Let $\mathcal{V}$ be a CD variety. Let L be a finite distributive lattice. Let $P = M(L)$. The following conditions are equivalent.*

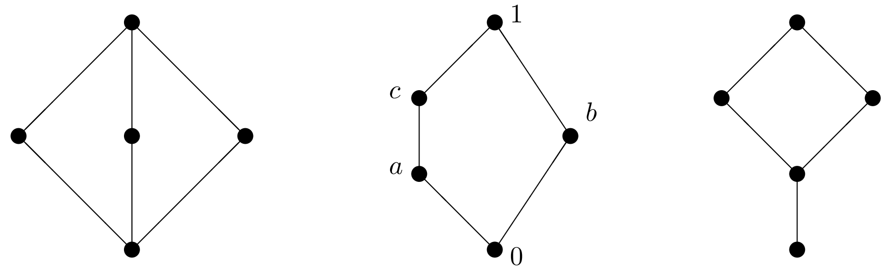
(1) *There is $A \in \mathcal{V}$ such that $\operatorname{Con} A$ is isomorphic to L.*
(2) *There are algebras $A_p \in \mathcal{V}$ (for $p \in P$) and homomorphisms $f_{pq} : A_p \to A_q$ such that $(P, \mathcal{A}, \mathcal{F})$ is an admissible ordered diagram of sets (with $\mathcal{A} = \{A_p \mid p \in P\}$ and $\mathcal{F} = \{f_{pq} \mid p \leq q\}$) and, moreover,*
(*) *for all $p \in P$, the sets $\{\operatorname{Ker}(f_{pq}) \mid q \geq p\}$ and $M(\operatorname{Con} A_p)$ coincide.*

Proof. We have just proved the implication (2)$\Longrightarrow$(1). Conversely, let (1) hold. Then $M(\operatorname{Con} A)$ is isomorphic to P, we can assume that $M(\operatorname{Con} A) = P$. For every $\alpha \in M(\operatorname{Con} A)$ let $A_\alpha = A/\alpha$. For every $\alpha, \beta \in M(\operatorname{Con} A)$ with $\alpha \leq \beta$ we have a natural homomorphism $f_{\alpha\beta} : A_\alpha \to A_\beta$. By 2.2, we have constructed an admissible ordered diagram of sets. By the well known isomorphism theorem, $\operatorname{Con} A/\alpha = \{\beta/\alpha \mid \beta \in \operatorname{Con} A,\ \beta \geq \alpha\}$ and therefore $M(\operatorname{Con} A_\alpha) = M(\operatorname{Con} A/\alpha) = \{\beta/\alpha \mid \beta \in M(\operatorname{Con} A),\ \beta \geq \alpha\} = \{\operatorname{Ker}(f_{\alpha\beta}) \mid \beta \in M(\operatorname{Con} A),\ \beta \geq \alpha\}$, which shows (*). $\square$

We admit that our theorem looks rather complicated. It might be possible to find simpler equivalent conditions, especially under some additional assumptions on the variety $\mathcal{V}$. We believe that this paper is a good starting point for the research in this direction.

Nevertheless, our result is applicable to concrete varieties. If $\mathcal{V}$ contains only finitely many subdirectly irreducible algebras, then we have only finitely many ways to construct an ordered diagram with a prescribed P satisfying 2.5(2). (Notice that (*) implies subdirect irreducibility of all A_p.) We investigate some varieties in the next section.

3. Small varieties of lattices



M_3 N_5 Con N_5

The application of 2.5 to a variety $\mathcal{V}$ requires a good knowledge of subdirectly irreducible algebras in $\mathcal{V}$. Such knowledge is available in the case of small varieties of lattices. We refer to [3] as a good source of information. We often use the well known fact that if a CD variety $\mathcal{V}$ is generated by a single finite algebra A then all subdirectly irreducible members of $\mathcal{V}$ are homomorphic images of subalgebras of A.

The smallest nontrivial variety of lattices is $\mathcal{D}$, the variety of distributive lattices. The description of $\operatorname{Con} A$ for $A \in \mathcal{D}$ is well known, even in the infinite case. (See [1], II.3 and II.4.) A finite lattice D is isomorphic to $\operatorname{Con} A$ for some $A \in \mathcal{D}$ if and only if D is Boolean.

The variety $\mathcal{D}$ is covered (in the lattice of varieties) by the varieties $\mathcal{M}_3$ and $\mathcal{N}_5$ generated by the lattices M_3 and N_5 depicted above.

The variety $\mathcal{M}_3$ has two subdirectly irreducible members: M_3 and the 2-element chain $C_2 = \{0, 1\}$. Both M_3 and C_2 are simple, their congruence lattice is a 2-element chain. If $A \in \mathcal{M}_3$ then by 2.3 every element of $M(\operatorname{Con} A)$ is a coatom of A. Hence, we obtain the same result as for $\mathcal{D}$: a finite lattice D is isomorphic to $\operatorname{Con} A$ for some $A \in \mathcal{M}_3$ iff D is Boolean. The congruence lattices of infinite $A \in \mathcal{M}_3$ have a much more complicated structure and they differ from the congruence lattices of distributive lattices ([5]).

The variety $\mathcal{N}_5$ has two subdirectly irreducible members: N_5 and C_2. The lattice $\operatorname{Con} N_5$ is depicted above. It has 3 meet-irreducible elements, namely the zero congruence Δ_{N_5} and the kernels of the two surjective homomorphisms $f_1,\ f_2 :\ N_5 \to C_2$ given by $f_1(0) = f_1(a) = f_1(c) = 0$, $f_1(b) = f_1(1) = 1$, $f_2(0) = f_2(b) = 0$, $f_2(a) = f_2(c) = f_2(1) = 1$.

Theorem 3.1. *For a finite distributive lattice D, the following conditions are equivalent:*

(1) $D \cong \operatorname{Con} A$ *for some* $A \in \mathcal{N}_5$*;*
(2) $M(D)$ *is a union of two disjoint antichains* M_1 *and* M_2 *such that for every* $x \in M_2$ *there are exactly two* $y \in M_1$ *with* $x < y$*.*

Proof. The implication (1)$\Longrightarrow$(2) follows from 2.3. If $D = \operatorname{Con} A$, $A \in \mathcal{N}_5$ then $M_1 = \{\alpha \in M(D) \mid A/\alpha \cong C_2\}$, $M_2 = \{\alpha \in M(D) \mid A/\alpha \cong N_5\}$.

Suppose now that (2) is satisfied. Let $P = M(D)$. For every $p \in M_1$ let $A_p = C_2$. For every $p \in M_2$ let $A_p = N_5$. For every $p \in M_2$ we have exactly two elements of M_1 greater than p; we denote them by p' and p''. (Choose arbitrarily which one is p' and which is p''.) Further, define $f_{pp'} = f_1$, $f_{pp''} = f_2$ (with f_1, f_2 defined above). It is easy to see that $(P, \mathcal{A}, \mathcal{F})$ is an ordered diagram of sets and 2.4(*) is satisfied. (As usual, we set $\mathcal{A} = \{A_p \mid p \in P\}$, $\mathcal{F} = \{f_{pq} \mid p \leq q\}$.) It remains to prove that $(P, \mathcal{A}, \mathcal{F})$ is admissible. For every $x \in \Pi_{p \in M_1} A_p$ we define $x^* \in \Pi_{p \in P} A_p$ as follows. If $p \in M_1$ then $x^*_p = x_p$ and if $p \in M_2$ then

$$x^*_p = \begin{cases} 0 & \text{if } x_{p'} = x_{p''} = 0; \\ a & \text{if } x_{p'} = 0,\ x_{p''} = 1; \\ b & \text{if } x_{p'} = 1,\ x_{p''} = 0; \\ 1 & \text{if } x_{p'} = x_{p''} = 1. \end{cases}$$

Further, let $x^{**} \in \Pi_{p \in P}$ be such that $x^{**}_p = c$ whenever $x^*_p = a$ and $x^{**}_p = x^*_p$ otherwise. It is easy to check that $x^*, x^{**} \in \lim(P, \mathcal{A}, \mathcal{F})$. Since x can be arbitrary, x^*_p and x^{**}_p can take any possible value for any $p \in P$. Hence, 2.1(i) holds. To prove 2.1(ii) let $p \not\leq q$. We distinguish several cases.

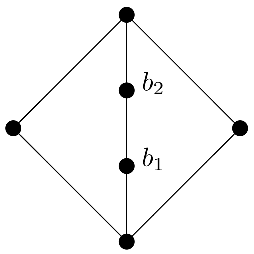
Let $q \in M_2$. Choose $x \in \Pi_{x \in M_1} A_p$ such that $x_{q'} = 0$, $x_{q''} = 1$. Then $x^*_q = a$. Define $y \in \Pi_{p \in P} A_p$ by $y_q = c$ and $y_r = x^*_r$ for every $r \neq q$. Then also $y \in \lim(P, \mathcal{A}, \mathcal{F})$ and we have $x^*_p = y_p$ and $x^*_q \neq y_q$.

Let $p, q \in M_1$. Choose any $x, y \in \Pi_{p \in M_1} A_p$ with $x_p = y_p$, $x_q \neq y_q$. Then $x^*_p = y^*_p$ and $x^*_q \neq y^*_q$.

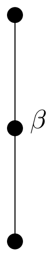
Finally, let $p \in M_2$, $q \in M_1$. Since $p \nleq q$, we have $q \notin \{p', p''\}$. Choose $x, y \in \Pi_{p \in M_1} A_p$ with $x_{p'} = y_{p'}$, $x_{p''} = y_{p''}$ and $x_q \neq y_q$. Clearly, $x^*_p = y^*_p$ and $x^*_q \neq y^*_q$.

Thus, the diagram $(P, \mathcal{A}, \mathcal{F})$ is admissible. By 2.4, $\lim(P, \mathcal{A}, \mathcal{F})$ has the congruence lattice isomorphic to D. $\square$

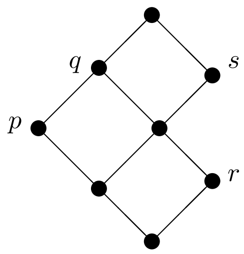
Finally, let us present one example. Let $\mathcal{K}$ be the variety generated by the lattice K depicted below.



K



$\mathrm{Con}\, K$



D

The lattice K is subdirectly irreducible. Its only nontrivial congruence β collapses the elements b_1 and b_2. Thus, K/β is isomorphic to M_3. It is not difficult to check that all sudirectly irreducible algebras in $\mathcal{K}$ are (up to isomorphism) K, M_3, N_5, C_2.

Let us consider the finite distributive lattice D depicted above. We are looking for $A \in \mathcal{K}$ such that $\mathrm{Con}\, A \cong D$. The lattice D satisfies the necessary condition in 2.3. However, we claim that there is no such algebra A.

The ordered set $M(D)$ consists of elements p, q, r, s. (See the picture.) Suppose for contradiction that there are subdirectly irreducible $A_p, A_q, A_r, A_s \in \mathcal{K}$ together with corresponding homomorphisms satisfying 2.5(2). By 2.3, the only possibility is $A_p = K$, $A_r = N_5$. By (*), $\mathrm{Ker}(f_{pq})$ must be equal to β and hence A_q must be M_3. However, A_q is also a homomorphic image of A_r, which is impossible.

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Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 04001 Košice, Slovakia