Congruence-preserving functions on Stone and Kleene algebras

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Abstract. We investigate local polynomial functions on Stone algebras and on Kleene algebras. We find a generating set for the clone of all local polynomial functions. We also represent local polynomial functions on a given algebra by polynomial functions of some canonical extension of this algebra.

1. Introduction

Let f be an n-ary function on a set A. Let ρ be a k-ary relation on a set A. We say that f preserves ρ if $(f(a_{11},\dots,a_{1n}),\dots,f(a_{k1},\dots,a_{kn}))\in\rho$ whenever $(a_{11},\dots,a_{k1})\in\rho,\dots,(a_{1n},\dots,a_{kn})\in\rho$.

An n-ary function f on an algebra $\mathbf{A}$ is called *compatible* (or *congruence-preserving*) if it preserves all congruences of $\mathbf{A}$.

It is clear that all polynomial functions on $\mathbf{A}$ are compatible. The algebra $\mathbf{A}$ is called *affine complete* if, conversely, every compatible function is polynomial. Affine completeness has been investigated for various kinds of algebras. (See [7] for a survey.) In general, however, there are non-polynomial compatible functions and our aim is to investigate this phenomenon and characterize the compatible functions. As a first step in achieving this goal we concentrate on an important subclass of compatible functions. An n-ary function f on an algebra $\mathbf{A}$ is said to be a *local polynomial function* of $\mathbf{A}$ if it can be interpolated by a polynomial of $\mathbf{A}$ on every finite subset of A^n. It is well-known (see [10]) that an n-ary function on an algebra $\mathbf{A}$ is a local polynomial iff it preserves all diagonal subalgebras of finite powers of $\mathbf{A}$. (A subalgebra $\mathbf{B}$ of $\mathbf{A}^k$ is diagonal if $(x,x,\dots,x)\in B$ for every $x\in A$.) It is also easy to find a local polynomial function which is not polynomial.

It is clear that compatible (local polynomial) functions form a clone. So, we have the following problem.

Problem 1.1. *Given an algebra* $\mathbf{A}$*, find a nice generating set for the clone of all compatible (local polynomial) functions.*

Another approach to the description of compatible functions (suggested in [3] for the case of distributive lattices) is via special extensions of algebras. An algebra $\mathbf{B}$

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is called *a CEP-extension* of $\mathbf{A}$, if $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and every congruence of $\mathbf{A}$ is a restriction of some congruence of $\mathbf{B}$. In such a case, every polynomial of $\mathbf{B}$, which preserves $\mathbf{A}$, restricts to a compatible function on $\mathbf{A}$. So, there is a possibility to describe compatible functions on $\mathbf{A}$ as polynomials of a suitable extension of $\mathbf{A}$.

Problem 1.2. *Given an algebra* $\mathbf{A}$*, construct (if possible) its CEP-extension* $\mathbf{B}$ *such that every compatible (local polynomial) function on* $\mathbf{A}$ *is a restriction of a polynomial of* $\mathbf{B}$.

The above problems have been solved for Boolean algebras in [2] (every Boolean algebra is affine complete) and partially (for local polynomial functions) for distributive lattices in [9]. In the present paper we consider the varieties of Stone and Kleene algebras. The affine completeness of these algebras has been characterized in [5], [4], and [6]. We generalize these theorems in the sense that their most difficult part follows easily from our results. (See Section 2 for the details.) Our proof covers both the case of Stone algebras and Kleene algebras. Moreover, we solve (the local polynomial versions of) Problems 1.1 and 1.2. This has not been known either for Stone algebras, or for Kleene algebras. We hope that the approach developed here could also be applied to other kinds of algebras.

Since the results for Stone and Kleene algebras are identical, we find it convenient to work in the variety $\mathcal{K}\vee\mathcal{S}$ generated by the class of all Kleene and Stone algebras. Some parts of our construction work even in a larger class of Ockham algebras.

An *Ockham algebra* is an algebra $\langle L; \vee, \wedge, ^*, 0, 1\rangle$, where $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and * is a unary operation such that $0^* = 1$, $1^* = 0$ and for all $x, y \in L$,

$$(x \wedge y)^* = x^* \vee y^* \,, \tag{1}$$
$$(x \vee y)^* = x^* \wedge y^* \,. \tag{2}$$

We refer to [1] as the basic source of information about Ockham algebras. The variety $\mathcal{K}\vee\mathcal{S}$ is defined by the following additional identities:

$$x \le x^{**}\,; \tag{3}$$
$$x \ge x^* \wedge x^{**}\,; \tag{4}$$
$$x \wedge x^* \le y \vee y^*\,; \tag{5}$$
$$x \vee y^* \vee y^{**} \ge x^{**}\,. \tag{6}$$

Equivalently, $\mathcal{K}\vee\mathcal{S}$ is the variety generated by the 3-element Kleene algebra $\mathbf{K_3} = \{0, a, 1\}$ and 3-element Stone algebra $\mathbf{S_3} = \{0, b, 1\}$. The variety $\mathcal{K}$ of Kleene algebras is characterized in $\mathcal{K}\vee\mathcal{S}$ by the identity

$$x = x^{**} \tag{7}$$

and the variety $\mathcal{S}$ of Stone algebras is characterized in $\mathcal{K}\vee\mathcal{S}$ by the identity

$$x \wedge x^* = 0\,. \tag{8}$$

Now we recall some basic denotations and results for $\mathcal{K}\vee\mathcal{S}$-algebras.

Lemma 1.3. *Every algebra in $\mathcal{K} \vee \mathcal{S}$ satisfies the following identities.*

$$x = x^{**} \wedge (x \vee x^*); \tag{9}$$
$$x \wedge y^* \wedge y^{**} \leq x^{**}. \tag{10}$$

Proof. It is easy to check that the identities hold in both $\mathbf{S_3}$ and $\mathbf{K_3}$. □

For every Ockham algebra $\mathbf{A}$ we define

$$\mathbf{A}^{\vee} = \{x \vee x^* \mid x \in A\} = \{x \in A \mid x \geq x^*\},$$
$$\mathbf{A}^{\wedge} = \{x \wedge x^* \mid x \in A\} = \{x \in A \mid x \leq x^*\}.$$

If $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$, then $\mathbf{A}^{\vee}$ is a filter and $\mathbf{A}^{\wedge}$ is an ideal of the lattice $\mathbf{A}$. Further, for any $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ we define

$$\mathbf{A}^{**} = \{x^{**} \mid x \in A\} = \{x \in A \mid x = x^{**}\}.$$

Since it easily follows from (3) that $x^* = x^{***}$, the second equality is immediate. Note that the set $\mathbf{A}^{**}$ is a subuniverse of $\mathbf{A}$; in fact $\mathbf{A}^{**}$ is a Kleene algebra.

Lemma 1.4. *For any $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$, the set $\mathbf{A}^{**} \cap \mathbf{A}^{\vee}$ is a filter.*

Proof. It is easy to see that $\mathbf{A}^{**} \cap \mathbf{A}^{\vee}$ is closed under intersections. Now, let $x \geq a \in \mathbf{A}^{**} \cap \mathbf{A}^{\vee}$. Obviously, $x \in \mathbf{A}^{\vee}$. We have $a^{**} = a \geq a^*$. Since $\mathbf{A}$ satisfies (6), we obtain that $x \geq x \vee a^* \vee a^{**} \geq x^{**}$. By (3), $x = x^{**} \in \mathbf{A}^{**}$. □

The *uncertainty order* of the $\mathcal{K} \vee \mathcal{S}$-algebra $\mathbf{A}$ is a binary relation $\sqsubseteq$, defined by

$$x \sqsubseteq y \quad \Leftrightarrow \quad x \wedge y^* \leq y \leq x \vee y^*.$$

This relation generalizes one which for Kleene algebras was introduced by M. Haviar, K. Kaarli and M. Ploščica in [4]. It is always a partial order relation. In the algebra $\mathbf{K_3}$ we have $0 \sqsubseteq a$, $1 \sqsubseteq a$, while in $\mathbf{S_3}$ the only nontrivial related pair is $1 \sqsubseteq b$. In general, $\mathbf{A}$ can be embedded into a direct product of algebras isomorphic to $\mathbf{S_3}$ or $\mathbf{K_3}$ and inherits the uncertainty order from this direct product.

A function f on an algebra $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ is called *uncertainty preserving* if it preserves the uncertainty relation of $\mathbf{A}$. It is easy to check that $\sqsubseteq$ is a diagonal subalgebra of $\mathbf{A}^2$ for every $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ and hence it is preserved by all local polynomial functions. In the next section we will show that, conversely, every compatible, uncertainty preserving function is a local polynomial function.

2. Local polynomial functions

An ideal I of a lattice $\mathbf{L}$ is said to be *principal* if it is of the form $\downarrow u = \{x \in L \mid x \leq u\}$, for some $u \in L$. We say that I is *almost principal* if its intersection with every principal ideal of $\mathbf{L}$ is a principal ideal of $\mathbf{L}$. If $\mathbf{L}$ has a largest element, then every almost principal ideal is principal. In general, there are almost principal ideals which are not principal (See [7] or [9].) The notions of principal and almost principal filter are defined dually. The whole lattice $\mathbf{L}$ is also regarded as an (almost principal) ideal and filter.

Let $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$. Let $\mathcal{I}(\mathbf{A})$ and $\mathcal{F}(\mathbf{A})$ denote the set of all almost principal ideals of the lattice $\mathbf{A}^{\wedge}$ and the set of all almost principal filters of the lattice $\mathbf{A}^{\vee}$, respectively.

Every $F \in \mathcal{F}(\mathbf{A})$ determines a function $\hat{F}\colon A \to A$ given by $\hat{F}(x) = \min(F \cap {\uparrow}(x \vee x^*))$. If F is principal, then the function $\hat{F}$ is polynomial, because $\hat{F}(x) = x \vee x^* \vee \min F$. In general, we have the following assertion.

Lemma 2.1. *For every $F \in \mathcal{F}(\mathbf{A})$ the function $\hat{F}$ is a local polynomial.*

Proof. Let $x_1, \dots, x_n \in A$. Since F is a filter, it is possible to choose $u \in F$ with $u \leq \hat{F}(x_i)$ for every $i = 1, \dots, n$. Then clearly $\hat{F}(x_i) = u \vee x_i \vee x_i^*$, so the polynomial $p(x) = u \vee x \vee x^*$ interpolates $\hat{F}$ on $\{x_1, \dots, x_n\}$. □

In the sequel we often assume that $\mathbf{A}$ is a subalgebra of a direct product

$$\Pi_{i \in I} \mathbf{A}_i, \tag{11}$$

where each $\mathbf{A}_i$ is isomorphic to $\mathbf{S_3}$ or $\mathbf{K_3}$. We consider the elements of $\mathbf{A}$ in the form $a = (a_i)_{i \in I}$. Then $a \sqsubseteq b$ iff $a_i \sqsubseteq b_i$ for every $i \in I$. Further, let $f\colon A^n \to A$ be compatible and $i \in I$. Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ be elements of $\mathbf{A}^n$ such that $(x_k)_i = (y_k)_i$, for every $k = 1, \dots, n$. Then $f(\mathbf{x})_i = f(\mathbf{y})_i$. (It follows from the fact that the projection on the i-th coordinate is a homomorphism, so f must preserve its kernel.)

We are going to prove that every compatible, uncertainty preserving function on $\mathbf{A}$ is a composition of polynomials and functions of the type $\hat{F}$. First we prove it for a special case. Let $f\colon A^n \to A$ be a compatible, uncertainty preserving function whose range is contained in $\mathbf{A}^{\vee}$.

Consider the set $\mathcal{Q}$ of all ordered pairs $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 \subseteq \{1, \dots, n\}$, $\alpha_1 \cap \alpha_2 = \emptyset$. For $y \in A$, $\alpha \in \mathcal{Q}$ let $\mathbf{y}^{\alpha}$ denote the n-tuple $(y_1^{\alpha}, \dots, y_n^{\alpha})$, where

$$y_k^{\alpha} = \begin{cases} 0 \text{ if } k \in \alpha_1; \\ 1 \text{ if } k \in \alpha_2; \\ y \text{ otherwise.} \end{cases}$$

Further, let

$$F_{\alpha} = \{u \in A \mid u \geq f(\mathbf{y}^{\alpha}) \text{ for some } y \in \mathbf{A}^{\vee}\}.$$

Lemma 2.2. *Let f be an n-ary compatible, uncertainty preserving function on an algebra $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ with the range contained in $\mathbf{A}^{\vee}$. Then $F_{\alpha} \in \mathcal{F}(\mathbf{A})$, for every $\alpha \in \mathcal{Q}$. Moreover, $\hat{F}_{\alpha}(x) = x \vee f(\mathbf{x}^{\alpha})$, for every $x \in \mathbf{A}^{\vee}$.*

Proof. Let $\alpha \in \mathcal{Q}$, $x \in \mathbf{A}^{\vee}$. We claim that

$$\min(F_{\alpha} \cap {\uparrow}x) = x \vee f(\mathbf{x}^{\alpha}). \tag{12}$$

Certainly, $x \vee f(\mathbf{x}^{\alpha}) \in F_{\alpha} \cap {\uparrow}x$. Now let s be an arbitrary element of $F_{\alpha} \cap {\uparrow}x$. Let $g\colon A \to A$ be defined by $g(y) = f(\mathbf{y}^{\alpha})$. This function is also compatible and uncertainty preserving. We need to show that $s \geq x \vee g(x)$. Obviously, $s \geq x$. We use the decomposition (11) and show the inequality $s_i \geq g(x)_i$, for every $i \in I$. The latter is trivial if we have that $s_i = 1$ or $g(x)_i \in \{a, b\}$. Assume that $s_i \in \{a, b\}$,

$g(x)_i = 1$. Since $x \leq s$, we have $x_i = s_i$. Further, $s \in F_\alpha$ implies that $s \geq g(y)$ for some $y \in \mathbf{A}^\vee$. If $y_i \in \{a, b\}$, then $y_i = x_i$, which implies that $s_i \geq g(y)_i = g(x)_i = 1$, a contradiction. If $y_i = 1$, then $g(y)_i = g(1)_i$. Since $1 \sqsubseteq x$, we have $g(1)_i \sqsubseteq g(x)_i = 1$, hence $s_i \geq g(y)_i = g(1)_i = 1$, a contradiction again.

Thus, (12) is proved. Now we show that F_α is closed with respect to meets. Let $x, y \in F_\alpha$. By the first part of this proof, there exists $u = \min(F_\alpha \cap \uparrow(x \wedge y))$. Clearly, $u \leq x$, $u \leq y$ and $u \in F_\alpha$, which implies that $u \leq x \wedge y \in F_\alpha$.

Finally, the formula $\hat{F}_\alpha(x) = x \vee f(\mathbf{x}^\alpha)$ follows from (12) and the definition of $\hat{F}_\alpha$. □

Let $\underline{n} = \{1, \ldots, n\}$. For $\mathbf{x} = (x_1, \ldots, x_n) \in A^n$ we define

$$g^\alpha(\mathbf{x}) = \begin{cases} \bigvee_{k\in\alpha_1} x_k^{**} \vee \bigvee_{k\in\alpha_2} x_k^* \vee \bigvee_{k\notin\alpha_1\cup\alpha_2} \hat{F}_\alpha(x_k) & \text{if } \alpha_1 \cup \alpha_2 \neq \underline{n} \\ \bigvee_{k\in\alpha_1} x_k^{**} \vee \bigvee_{k\in\alpha_2} x_k^* \vee \min F_\alpha & \text{if } \alpha_1 \cup \alpha_2 = \underline{n} \end{cases} \tag{13}$$

Notice that if $\alpha_1 \cup \alpha_2 = \underline{n}$, then $f(\mathbf{y}^\alpha)$ is a constant and hence F_α is a principal ideal.

Lemma 2.3. *Let f be an n-ary compatible, uncertainty preserving function on an algebra $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ with the range contained in $\mathbf{A}^\vee$. Then*

$$f(\mathbf{x}) = \bigwedge_{\alpha\in\mathcal{Q}} g^\alpha(\mathbf{x}),$$

for every $\mathbf{x} \in A^n$.

Proof. We use the decomposition (11). Let $i \in I$. We show that

$$f(\mathbf{x})_i = \bigwedge_{\alpha\in\mathcal{Q}} g^\alpha(\mathbf{x})_i. \tag{14}$$

Let $\beta_1 = \{k \mid (x_k)_i = 0\}$, $\beta_2 = \{k \mid (x_k)_i = 1\}$. We have $c \in \{a, b\}$ such that $(x_k)_i = c$ for every $k \in \underline{n} \setminus (\beta_1 \cup \beta_2)$. If $\beta_1 \cup \beta_2 \neq \underline{n}$, then choose $j \in \underline{n} \setminus (\beta_1 \cup \beta_2)$ arbitrarily and put $y = x_j \vee x_j^*$. If $\beta_1 \cup \beta_2 = \underline{n}$, then put $y = \min F_\beta$. In both cases, $(y_k^\beta)_i = (x_k)_i$ for every k. Since f and all g^α are compatible, we have $f(\mathbf{x})_i = f(\mathbf{y}^\beta)_i$, $g^\alpha(\mathbf{x})_i = g^\alpha(\mathbf{y}^\beta)_i$.

It is easy to check that $g^\beta(\mathbf{y}^\beta) = \hat{F}_\beta(y)$. By Lemma 2.2, we have $\hat{F}_\beta(y) = y \vee f(\mathbf{y}^\beta)$. Further, $f(\mathbf{y}^\beta) \in \mathbf{A}^\vee$ implies that $y_i \leq f(\mathbf{y}^\beta)_i$. (If $\beta_1 \cup \beta_2 \neq \underline{n}$, then $y_i = c = \min \mathbf{A}_i^\vee$.) Hence, we have

$$f(\mathbf{x})_i = f(\mathbf{y}^\beta)_i = y_i \vee f(\mathbf{y}^\beta)_i = \hat{F}_\beta(y)_i = g^\beta(\mathbf{y}^\beta)_i = g^\beta(\mathbf{x})_i,$$

which shows that the left hand side of (14) is greater than or equal to the right hand side. For the opposite inequality we need to show that

$$f(\mathbf{y}^\beta)_i \leq g^\alpha(\mathbf{y}^\beta)_i, \tag{15}$$

for every $\alpha \in \mathcal{Q}$. This is clear if $f(\mathbf{y}^\beta)_i \neq 1$ (since $g^\alpha(\mathbf{y}^\beta) \in \mathbf{A}^\vee$) or $\beta_1 \nsubseteq \alpha_1$ or $\beta_2 \nsubseteq \alpha_2$ (because then $g^\alpha(\mathbf{y}^\beta) = 1$). Suppose now that $f(\mathbf{y}^\beta)_i = 1$, $\beta_1 \subseteq \alpha_1$, $\beta_2 \subseteq \alpha_2$. We distinguish two cases.

a) Let $c = b$. If $\beta_1 \neq \alpha_1$, then $g^\alpha(\mathbf{y}^\beta)_i = 1$ and (15) holds. Let $\beta_1 = \alpha_1$. Then $y_k^\alpha \sqsubseteq y_k^\beta$ for every $k = 1, \dots, n$, hence $f(\mathbf{y}^\alpha)_i \sqsubseteq f(\mathbf{y}^\beta)_i = 1$, which implies that $f(\mathbf{y}^\alpha)_i = 1$.

Now we compute $g^\alpha(\mathbf{y}^\beta)$. If $\alpha_1 \cup \alpha_2 \neq \underline{n}$ then $g^\alpha(\mathbf{y}^\beta) \geq \hat{F}_\alpha(y) = y \vee f(\mathbf{y}^\alpha) \geq f(\mathbf{y}^\alpha)$, hence $g^\alpha(\mathbf{y}^\beta)_i \geq f(\mathbf{y}^\alpha)_i = 1$ and (15) holds. If $\alpha_1 \cup \alpha_2 = \underline{n}$, then $g^\alpha(\mathbf{y}^\beta) \geq \min F_\alpha = f(\mathbf{y}^\alpha)$, so $g^\alpha(\mathbf{y}^\beta)_i = 1$, and (15) holds.

b) Let $c = a$. Then $y_k^\alpha \sqsubseteq y_k^\beta$ for every $k = 1, \dots, n$, so $f(\mathbf{y}^\alpha)_i \sqsubseteq f(\mathbf{y}^\beta)_i = 1$, and we can use the same arguments as above. □

Theorem 2.4. *For a function $f\colon A^n \to A$ on $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$, the following conditions are equivalent.*

(i) *f is compatible, uncertainty preserving;*
(ii) *f is a composition of polynomial functions and functions $\hat{F}$ for $F \in \mathcal{F}(\mathbf{A})$;*
(iii) *f is a local polynomial.*

Proof. (i)$\Longrightarrow$ (ii) Let f be compatible, uncertainty preserving. By (9) we have the identity

$$f(\mathbf{x}) = f(\mathbf{x})^{**} \wedge (f(\mathbf{x}) \vee f(\mathbf{x})^*)\,.$$

By Lemma 2.3, we only need to prove our statement for the function $f(\mathbf{x})^{**}$. Our claim is that every compatible, uncertainty preserving function $g\colon A^n \to A$ whose range is in $\mathbf{A}^{**}$ is a composition of polynomial functions and functions of the type $\hat{F}$. Every such function preserves the Glivenko congruence

$$\Phi = \{(x, y) \in A^2 \mid x^* = y^*\}\,,$$

so $g(\mathbf{x}) = g(\mathbf{x}^{**})$ for every $\mathbf{x} \in A^n$. We prove our claim by induction on n.

The claim is obviously true for nullary functions. Suppose now that $n > 0$. Let g_0, g_1 be the $(n-1)$-ary functions defined by

$$g_0(x_2, \dots, x_n) = g(0, x_2, \dots, x_n), \quad g_1(x_2, \dots, x_n) = g(1, x_2, \dots, x_n).$$

We show the following equality for every $\mathbf{x} \in A^n$:

$$\begin{aligned} g(\mathbf{x}) = {} & ((g(\mathbf{x}) \wedge x_1^* \wedge x_1^{**}) \vee g_0(\mathbf{y}) \vee g_1(\mathbf{y})) \wedge (g_0(\mathbf{y}) \vee x_1^{**}) \\ & \wedge (g_1(\mathbf{y}) \vee x_1^*) \wedge (g(\mathbf{x}) \vee x_1^* \vee x_1^{**}), \end{aligned} \tag{16}$$

where $\mathbf{y} = (x_2, \dots, x_n)$. Denote by $h(\mathbf{x})$ the right hand side of (16). On both sides of (16) we have compatible functions whose range is contained in $\mathbf{A}^{**}$, so it suffices to prove (16) for all $x_1, \dots, x_n \in \mathbf{A}^{**}$. We consider the decomposition (11) and show that $g(\mathbf{x})_i = h(\mathbf{x})_i$ for all $i \in I$.

If $(x_1)_i = 0$ then the compatibility of g yields $g(\mathbf{x})_i = g_0(\mathbf{y})_i$ and it is easy to compute that $h(\mathbf{x})_i = g_0(\mathbf{y})_i$. Similarly, if $(x_1)_i = 1$, then $g(\mathbf{x})_i = g_1(\mathbf{y})_i = h(\mathbf{x})_i$.

The remaining case is $(x_1)_i = a$. Obviously, $0 \sqsubseteq x_1 \wedge x_1^*$ and $1 \sqsubseteq x_1 \vee x_1^*$. Since $(x_1 \wedge x_1^*)_i = (x_1 \vee x_1^*)_i = a = (x_1)_i$, we obtain that $g_0(\mathbf{y})_i \sqsubseteq g(x_1 \wedge x_1^*, x_2, \dots, x_n)_i = g(\mathbf{x})_i$ and similarly $g_1(\mathbf{y})_i \sqsubseteq g(\mathbf{x})_i$. If $g(\mathbf{x})_i \in \{0, 1\}$, then necessarily $g_0(\mathbf{y})_i = g_1(\mathbf{y})_i = g(\mathbf{x})_i$ and it is easy to check that also $h(\mathbf{x})_i = g(\mathbf{x})_i$. If $g(\mathbf{x})_i = a$, then it is easy to compute that $h(\mathbf{x})_i = a$.

Thus, (16) is proved. The functions $k(\mathbf{x}) = (g(\mathbf{x}) \wedge x_1^* \wedge x_1^{**})^*$ and $g(\mathbf{x}) \vee x_1^* \vee x_1^{**}$ have the range contained in $\mathbf{A}^\vee$, so, by Lemma 2.3, they are compositions of polynomial functions and functions of the type $\hat{F}$. The same is then true for the function $g(\mathbf{x}) \wedge x_1^* \wedge x_1^{**} = k(\mathbf{x})^*$ and, by the induction hypothesis, for the functions $g_0(\mathbf{y})$ and $g_1(\mathbf{y})$. Consequently, the statement holds for the function g.

(ii)$\Longrightarrow$(iii) This follows from Lemma 2.1.

(iii)$\Longrightarrow$(i) This follows from the fact that the uncertainty relation as well as all congruences are diagonal subalgebras of $\mathbf{A}^2$. □

The equivalence of (i) and (iii) is known for Kleene algebras ([4]) as well as for Stone algebras ([7]). Thus, we have generalized both of these results. Moreover, the equivalence of (ii) and (iii) gives solution to Problem 1.1 (in the case of local polynomial functions) for Stone and Kleene algebras (in fact, for the variety $\mathcal{K} \vee \mathcal{S}$).

The affine completeness of Kleene and Stone algebras has been characterized as follows. (See [4], [5] or [7].)

Theorem 2.5. *An algebra $\mathbf{A} \in \mathcal{K} \cup \mathcal{S}$ is affine complete iff the following conditions hold.*

(i) $\mathbf{A}^\vee$ *does not contain a nontrivial Boolean interval.*
(ii) *For every $F \in \mathcal{F}(\mathbf{A})$ there is $u \in A$ such that $F = {\uparrow} u \cap \mathbf{A}^\vee$.*

The most difficult part of the proof is the statement that, assuming (ii), every compatible, uncertainty preserving function is a polynomial. Now this statement follows easily from 2.4, because (ii) implies that $\hat{F}(x) = u \vee x \vee x^*$ for every $F \in \mathcal{F}(\mathbf{A})$, so all functions $\hat{F}$ are polynomial.

3. Ideal-filter extensions of Ockham algebras

Let $\mathbf{A}$ be an Ockham algebra. Let $\mathrm{I}(\mathbf{A})$ and $\mathrm{F}(\mathbf{A})$ denote its ideal lattice and its filter lattice respectively. The set $\mathrm{I}(\mathbf{A})$ is ordered by the usual set inclusion, while $\mathrm{F}(\mathbf{A})$ is ordered by the inverse set inclusion ($F \leq G$ iff $G \subseteq F$). Thus, for every $I, J \in \mathrm{I}(\mathbf{A})$, $F, G \in \mathrm{F}(\mathbf{A})$, we have

$$\begin{aligned} I \wedge J &= I \cap J; \\ F \vee G &= F \cap G; \\ I \vee J &= \{x \in A \mid x \leq y \vee z \text{ for some } y \in I,\ z \in J\}; \\ F \wedge G &= \{x \in A \mid x \geq y \wedge z \text{ for some } y \in F,\ z \in G\}. \end{aligned}$$

We use the same lattice operations symbols in $\mathbf{A}$, $\mathrm{I}(\mathbf{A})$ and $\mathrm{F}(\mathbf{A})$, hoping that no confusion arises. Since $\mathbf{A}$ is a distributive lattice, $\mathrm{I}(\mathbf{A})$ and $\mathrm{F}(\mathbf{A})$ are distributive too. Further, for every $I \in \mathrm{I}(\mathbf{A})$, $F \in \mathrm{F}(\mathbf{A})$ we define

$$\begin{aligned} I^* &= \{x \in A \mid x \geq a^* \text{ for some } a \in I\}; \\ F^* &= \{x \in A \mid x \leq a^* \text{ for some } a \in F\}. \end{aligned}$$

For any $x, y \in I^*$ we have $a, b \in I$ with $x \geq a^*$, $y \geq b^*$, and then $x \wedge y \geq a^* \wedge b^* = (a \vee b)^*$, hence $x \wedge y \in I^*$. This shows that $I^* \in \mathrm{F}(\mathbf{A})$. Similarly, $F^* \in \mathrm{I}(\mathbf{A})$.

Lemma 3.1. *Let $\mathbf{A}$ be an Ockham algebra and let $I, J \in \mathrm{I}(\mathbf{A})$, $F, G \in \mathrm{F}(\mathbf{A})$. Then*

(i) $(I \vee J)^* = I^* \wedge J^*$;

(ii) $(F \vee G)^* = F^* \wedge G^*$;
(iii) $(I \wedge J)^* = I^* \vee J^*$;
(iv) $(F \wedge G)^* = F^* \vee G^*$.

Proof. (1) Clearly, $(I \vee J)^* \supseteq I^* \cup J^*$. By definition, $I^* \wedge J^*$ is the smallest filter containing both I^* and J^*. Thus, $(I \vee J)^* \supseteq I^* \wedge J^*$. Conversely, let $x \in (I \vee J)^*$. Then $x \geq a^*$ for some $a \in I \vee J$. There are $b \in I$, $c \in J$ with $a \leq b \vee c$. Then $a^* \geq b^* \wedge c^*$, so $x \geq b^* \wedge c^*$. Since $b^* \in I^*$, $c^* \in J^*$, we obtain that $x \in I^* \vee J^*$.

(2) We have $F \vee G \subseteq F$, $F \vee G \subseteq G$, so $(F \vee G)^* \subseteq F^* \cap G^* = F^* \wedge G^*$. Conversely, let $x \in F^* \wedge G^*$. There are $b \in F$, $c \in G$ with $x \leq b^*$, $x \leq c^*$, hence $x \leq b^* \wedge c^* = (b \vee c)^*$. Since $b \vee c \in F \cap G = F \vee G$, we obtain that $x \in (F \vee G)^*$.

(3) and (4) are dual to (1) and (2), respectively. □

Consider now the direct product $\mathrm{I}(\mathbf{A}) \times \mathrm{F}(\mathbf{A})$. This is a distributive lattice. Let us define the operation $*$ on $\mathrm{I}(\mathbf{A}) \times \mathrm{F}(\mathbf{A})$ by

$$(I, F)^* = (F^*, I^*).$$

Theorem 3.2. *Let* $\mathbf{A}$ *be an Ockham algebra. Then* $\mathrm{I}(\mathbf{A}) \times \mathrm{F}(\mathbf{A})$ *is an Ockham algebra.*

Proof. We only need to check de Morgan laws (1) and (2). Let us compute: $((I, F) \wedge (J, G))^* = (I \wedge J, F \wedge G)^* = ((F \wedge G)^*, (I \wedge J)^*) = (F^* \vee G^*, I^* \vee J^*) = (F^*, I^*) \vee (G^*, J^*) = (I, F)^* \vee (J, G)^*$. The proof for (2) is dual. □

Lemma 3.3. *Let* $\mathbf{A}$ *be an Ockham algebra. For any* $x, y \in A$,

(i) $\downarrow(x \vee y) = \downarrow x \vee \downarrow y$;
(ii) $\downarrow(x \wedge y) = \downarrow x \wedge \downarrow y$;
(iii) $\uparrow(x \vee y) = \uparrow x \cap \uparrow y = \uparrow x \vee \uparrow y$;
(iv) $\uparrow(x \wedge y) = \uparrow x \wedge \uparrow y$;
(v) $(\uparrow x)^* = \downarrow x^*$;
(vi) $(\downarrow x)^* = \uparrow x^*$.

Proof. (i)-(iv) are obvious. Further, $y \in (\uparrow x)^*$ iff $y \leq z^*$ for some $z \geq x$ iff $y \leq x^*$ iff $y \in \downarrow x^*$, so (v) holds and (vi) is its dual. □

Theorem 3.4. *The map* $f\colon A \to \mathrm{I}(\mathbf{A}) \times \mathrm{F}(\mathbf{A})$ *defined by* $f(x) = (\downarrow x, \uparrow x)$ *is an embedding of Ockham algebras.*

Proof. Let $x, y \in A$. We compute: $f(x \vee y) = (\downarrow(x \vee y), \uparrow(x \vee y)) = (\downarrow x, \uparrow x) \vee (\downarrow y, \uparrow y) = f(x) \vee f(y)$, and similarly for meets. Further, we have $f(x)^* = (\downarrow x, \uparrow x)^* = ((\uparrow x)^*, (\downarrow x)^*) = (\downarrow x^*, \uparrow x^*) = f(x^*)$. The injectivity of f is obvious. □

Thus, $\mathrm{I}(\mathbf{A}) \times \mathrm{F}(\mathbf{A})$ can be regarded as an extension of $\mathbf{A}$. We call it the *ideal-filter extension* of $\mathbf{A}$.

Lemma 3.5. *Suppose that the Ockham algebra* $\mathbf{A}$ *satisfies the following two assumptions:*

(i) $\mathbf{A}^\wedge$ *is an ideal and* $\mathbf{A}^\vee$ *is a filter;*
(ii) $x^* \vee x^{**} \vee y \geq y^{**}$ *and* $x^* \wedge x^{**} \wedge y \leq y^{**}$, *for every* $x, y \in A$.

Then, for every $x \in A$, $a \in \mathbf{A}^\wedge$, $b \in \mathbf{A}^\vee$, $I, J \in \mathcal{I}(\mathbf{A})$, $F, G \in \mathcal{F}(\mathbf{A})$, the following holds.

(iii) *The sets $\downarrow a$, $I \wedge J$, $I \vee J$ and $\downarrow x \wedge I$ belong to $\mathcal{I}(\mathbf{A})$.*
(iv) *The sets $\uparrow b$, $F \vee G$, $F \wedge G$ and $\uparrow x \vee F$ belong to $\mathcal{F}(\mathbf{A})$.*
(v) *$I^* \in \mathcal{F}(\mathbf{A})$, $F^* \in \mathcal{I}(\mathbf{A})$.*

Proof. Recall that $\mathcal{I}(\mathbf{A})$ contains ideals that are almost principal in $\mathbf{A}^\wedge$ (not necessarily in $\mathbf{A}$). For every $y \in \mathbf{A}^\wedge$ we have $\downarrow y \cap \downarrow a = \downarrow(y \wedge a)$, which shows that $\downarrow a \in \mathcal{I}(\mathbf{A})$.

Clearly, $I \wedge J \subseteq \mathbf{A}^\wedge$. By (i) we also have $I \vee J \subseteq \mathbf{A}^\wedge$. Since I and J are almost principal, we have $u, v \in \mathbf{A}^\wedge$ with $\downarrow y \cap I = \downarrow u$ and $\downarrow y \cap J = \downarrow v$. It is not difficult to check that $\downarrow y \cap (I \wedge J) = \downarrow(u \wedge v)$ and $\downarrow y \cap (I \vee J) = \downarrow(u \vee v)$. Thus, $I \wedge J$, $I \vee J \in \mathcal{I}(\mathbf{A})$.

Finally, we note that for every $y \in \mathbf{A}^\wedge$ we have $\downarrow y \cap (\downarrow x \cap \mathbf{A}^\wedge) = \downarrow(y \wedge x)$ and hence $\downarrow x \cap \mathbf{A}^\wedge \in \mathcal{I}(\mathbf{A})$. Now, obviously, $\downarrow x \wedge I = (\downarrow x \cap \mathbf{A}^\wedge) \wedge I \in \mathcal{I}(\mathbf{A})$.

The proof of (iv) is analogous.

Clearly, $I^* \in \mathrm{F}(\mathbf{A})$ and $I^* \subseteq \mathbf{A}^\vee$. Let $y \in \mathbf{A}^\vee$. Then $y^* \in \mathbf{A}^\wedge$, so $\downarrow y^* \cap I = \downarrow c$, for some $c \in I$. We claim that $\uparrow y \cap I^* = \uparrow(c^* \vee y)$. If $z \in \uparrow(c^* \vee y)$ then clearly $z \in \uparrow y \cap I^*$. Conversely, let $z \in \uparrow y \cap I^*$. Then $z \geq y$ and $z \geq d^*$ for some $d \in I$. Then $d \wedge y^* \in \downarrow y^* \cap I$, hence $d \wedge y^* \leq c$, which implies that $d^* \vee y^{**} \geq c^*$. Since $d^* \in \mathbf{A}^\vee$, we have $d^* \geq d^{**}$. By (ii) we obtain that $d^* \vee y = d^* \vee d^{**} \vee y \geq y^{**}$ and hence $z \geq (d^* \vee y) \geq (d^* \vee y^{**}) \vee y \geq c^* \vee y$, so $z \in \uparrow(c^* \vee y)$.

The proof of $F^* \in \mathcal{I}(\mathbf{A})$ is analogous. □

Definition 3.6. *Let $\mathbf{T}(\mathbf{A})$ be the subset of $\mathrm{I}(\mathbf{A}) \times \mathrm{F}(\mathbf{A})$ consisting of all pairs of the form $(\downarrow x \vee I, \uparrow x \wedge F)$, where $x \in A$, $I \in \mathcal{I}(\mathbf{A})$, $F \in \mathcal{F}(\mathbf{A})$.*

Theorem 3.7. *If $\mathbf{A}$ is an Ockham algebra satisfying conditions (i),(ii) in Lemma 3.5, then $\mathbf{T}(\mathbf{A})$ is a subalgebra of $\mathrm{I}(\mathbf{A}) \times \mathrm{F}(\mathbf{A})$.*

Proof. We have

$$(\downarrow x \vee I, \uparrow x \wedge F) \wedge (\downarrow y \vee J, \uparrow y \wedge G) = ((\downarrow x \vee I) \wedge (\downarrow y \wedge J), \uparrow x \wedge F \wedge \uparrow y \wedge G)\,.$$

Since the lattice $\mathrm{I}(\mathbf{A})$ is distributive, we obtain

$$(\downarrow x \vee I) \wedge (\downarrow y \vee J) = \downarrow(x \wedge y) \vee (\downarrow x \wedge J) \vee (\downarrow y \wedge I) \vee (I \wedge J),$$

$$\uparrow x \wedge F \wedge \uparrow y \wedge G = \uparrow(x \wedge y) \wedge (F \wedge G).$$

We set $z = x \wedge y$, $K = (\downarrow x \wedge J) \vee (\downarrow y \wedge I) \vee (I \wedge J)$, $H = F \wedge G$. By Lemma 3.5, $K \in \mathcal{I}(A)$, $H \in \mathcal{F}(\mathbf{A})$, so $(\downarrow x \vee I, \uparrow x \wedge F) \wedge (\downarrow y \vee J, \uparrow y \wedge G) = (\downarrow z \vee K, \uparrow z \wedge H) \in \mathbf{T}(\mathbf{A})$.

The proof for joins is analogous.

Finally, $(\downarrow x \vee I, \uparrow x \wedge F)^* = ((\uparrow x)^* \vee F^*, (\downarrow x)^* \wedge I^*) = (\downarrow x^* \vee F^*, \uparrow x^* \wedge I^*)$. By Lemma 3.5, $F^* \in \mathcal{I}(\mathbf{A})$, $I^* \in \mathcal{F}(\mathbf{A})$, hence $(\downarrow x \vee I, \uparrow x \wedge F)^* \in \mathbf{T}(\mathbf{A})$. □

For every $a \in A$ we have $(\downarrow a, \uparrow a) = (\downarrow a \vee \downarrow 0, \uparrow a \wedge \uparrow 1) \in \mathbf{T}(\mathbf{A})$. Thus, the assignment $a \mapsto (\downarrow a, \uparrow a)$ is a natural embedding $\mathbf{A} \to \mathbf{T}(\mathbf{A})$.

4. Affine completions in $\mathcal{K} \vee \mathcal{S}$

In this section we assume that $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$. Then the conditions (i) and (ii) in Lemma 3.5 are satisfied, so we have the Ockham algebra $\mathbf{T}(\mathbf{A})$.

For any set $X \subseteq A$ let $X^\wedge = X \cap \mathbf{A}^\wedge$, $X^\vee = X \cap \mathbf{A}^\vee$.

Let us define the binary relation $\sim$ on $\mathbf{T}(\mathbf{A})$ by

$$(I,F) \sim (J,G) \\ \Updownarrow \\ I \wedge \downarrow a = J \wedge \downarrow a \text{ and } F \vee \uparrow b = G \vee \uparrow b \text{ for some } b \in I^\wedge, a \in F^\vee . \tag{17}$$

Since $\mathbf{A}$ satisfies the identity (5), we have $\mathbf{A}^\wedge \subseteq \downarrow a$ for every $a \in \mathbf{A}^\vee$. Consequently, (17) implies that $I^\wedge = (I \wedge \downarrow a) \wedge \mathbf{A}^\wedge = (J \wedge \downarrow a) \wedge \mathbf{A}^\wedge = J^\wedge$, and similarly, $F^\vee = G^\vee$. This shows that the relation $\sim$ is symmetric.

Lemma 4.1. *The relation $\sim$ is a congruence relation on $\mathbf{T}(\mathbf{A})$.*

Proof. The relation $\sim$ is clearly reflexive and symmetric. If $(I,F) \sim (J,G) \sim (K,H)$, then

$$I \wedge \downarrow a_1 = J \wedge \downarrow a_1 \text{ and } J \wedge \downarrow a_2 = K \wedge \downarrow a_2,$$

for some $a_1, a_2 \in F^\vee = G^\vee = H^\vee$. Since $\mathbf{A}^\vee$ is a filter, we have $a = a_1 \wedge a_2 \in F^\vee$ and

$$I \wedge \downarrow a = J \wedge \downarrow a = K \wedge \downarrow a.$$

This and the dual argument (for filters) show the transitivity of $\sim$. Thus, $\sim$ is an equivalence. Now we show its compatibility with the operations of $\mathbf{T}(\mathbf{A})$.

Let $(I,F),(J,G),(K,H) \in \mathbf{T}(\mathbf{A})$, $(I,F) \sim (J,G)$, $(K,H) = (K_0 \vee \downarrow z, H_0 \wedge \uparrow z)$, for some $z \in A$, $K_0 \in \mathcal{I}(\mathbf{A})$, $H_0 \in \mathcal{F}(\mathbf{A})$. Let $a \in F^\vee$, $b \in I^\wedge$ such that (17) holds. Clearly, $(I \vee K)^\wedge = I^\wedge \vee K^\wedge = (J \vee K)^\wedge$, $(F \vee H)^\vee = F^\vee \vee H^\vee = (G \vee H)^\vee$. Let us set $c = z \vee a$. Then $c \in F^\vee \vee H = F^\vee \vee H^\vee = (F \vee H)^\vee$. We claim that

$$(I \vee K) \wedge \downarrow c = (J \vee K) \wedge \downarrow c . \tag{18}$$

Since $I \wedge \downarrow z \subseteq \downarrow z \subseteq K$ and $I \wedge \downarrow a \subseteq J$, we have $I \wedge \downarrow c = (I \wedge \downarrow z) \vee (I \wedge \downarrow a) \subseteq J \vee K$ and also $I \wedge \downarrow c \subseteq (J \vee K) \wedge \downarrow c$. Hence, $(I \vee K) \wedge \downarrow c = (I \wedge \downarrow c) \vee (K \wedge \downarrow c) \subseteq (J \vee K) \wedge \downarrow c$. The proof of the inverse inclusion is analogous, so (18) holds.

Further, $b \in I^\wedge \subseteq (I \vee K)^\wedge$ and obviously $(F \vee H) \vee \uparrow b = (F \vee \uparrow b) \vee H = (G \vee \uparrow b) \vee H = (G \vee H) \vee \uparrow b$. This, together with (18) shows that $(I,F) \vee (K,H) \sim (J,G) \vee (K,H)$. The proof for meets is similar.

It remains to show that $(F^*, I^*) = (I,F)^* \sim (J,G)^* = (G^*, J^*)$. We have $b^* \in I^*$ and $b \in \mathbf{A}^\wedge$, hence $b^* \in (I^*)^\vee$. From $F \vee \uparrow b = G \vee \uparrow b$ we obtain that $F^* \wedge \downarrow b^* = G^* \wedge \downarrow b^*$. Dually, we have $I^* \vee \uparrow a^* = J^* \vee \uparrow a^*$ and $a^* \in (F^*)^\wedge$. $\square$

Therefore, we can consider the quotient algebra $\overline{\mathbf{T}(\mathbf{A})} = \mathbf{T}(\mathbf{A})/\sim$.

Lemma 4.2. *Let $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$. Then for $x \in A$, $I \in \mathcal{I}(\mathbf{A})$ and $F \in \mathcal{F}(\mathbf{A})$ the following holds:*

(i) $I^{**} = I$, $F^{**} \geq F$;
(ii) $(\uparrow x^* \wedge I^*) \vee (\uparrow x^{**} \wedge F^{**}) \subseteq \mathbf{A}^{**} \cap \mathbf{A}^\vee$;

Proof. By the definitions of I^* and F^* we have

$$\begin{aligned} I^{**} &= \{x \in A \mid x \leq a^{**} \text{ for some } a \in I\}, \\ F^{**} &= \{x \in A \mid x \geq a^{**} \text{ for some } a \in F\}. \end{aligned}$$

Since (4) holds in $\mathbf{A}$, for any $a \in \mathbf{A}^\wedge$ we have $a = a^{**}$ and hence $I^{**} = I$. To prove $F^{**} \subseteq F$, take $x \in F^{**}$. Since $x \geq a^{**}$ for some $a \in F$ and (3) holds in $\mathbf{A}$, we have $x \geq a$ implying $x \in F$. Thus $F^{**} \subseteq F$ and $F^{**} \geq F$.

Let $z \in (\uparrow x^* \wedge I^*) \vee (\uparrow x^{**} \wedge F^{**})$. Then

$$z \geq t = (x^* \wedge a^*) \vee (x^{**} \wedge b^{**}) = (x^* \vee x^{**}) \wedge (x^* \vee b^{**}) \wedge (a^* \vee x^{**}) \wedge (a^* \vee b^{**}),$$

for some $a \in I$, $b \in F$. Clearly, $t \in \mathbf{A}^{**}$. Since a^*, b^{**} and $x^* \vee x^{**}$ belong to the filter $\mathbf{A}^\vee$, we also have $t \in \mathbf{A}^\vee$. By Lemma 1.4, $\mathbf{A}^{**} \cap \mathbf{A}^\vee$ is a filter, so $z \in \mathbf{A}^{**} \cap \mathbf{A}^\vee$. □

Theorem 4.3. *If $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$, then $\mathbf{T}(\mathbf{A}) \in \mathcal{K} \vee \mathcal{S}$ and $\overline{\mathbf{T}(\mathbf{A})} \in \mathcal{K} \vee \mathcal{S}$. Moreover, if $\mathbf{A}$ is a Kleene (Stone) algebra then $\mathbf{T}(\mathbf{A})$ and $\overline{\mathbf{T}(\mathbf{A})}$ are Kleene (Stone) algebras.*

Proof. It suffices to prove the statements for $\mathbf{T}(\mathbf{A})$. Let $u, v \in A$, $I, J \in \mathcal{I}(\mathbf{A})$ and $F, G \in \mathcal{F}(\mathbf{A})$.

To prove that $\mathbf{T}(\mathbf{A}) \in \mathcal{K} \vee \mathcal{S}$, we need to check the identities (3) – (6) for $x = (\downarrow u \vee I, \uparrow u \wedge F)$, $y = (\downarrow v \vee J, \uparrow v \wedge G)$.

We have $x^{**} = (\downarrow u^{**} \vee I^{**}, \uparrow u^{**} \wedge F^{**})$. Since (3) holds in $\mathbf{A}$ we have $\downarrow u^{**} \geq \downarrow u$ and $\uparrow u^{**} \geq \uparrow u$. Now Lemma 4.2(i) implies $x^{**} \geq (\downarrow u \vee I, \uparrow u \wedge F) = x$. Thus, (3) holds in $\mathbf{T}(\mathbf{A})$.

Further, using the distributivity, the element $x^* \wedge x^{**}$ is equal to

$$(\downarrow(u^{**} \wedge u^*) \vee (\downarrow u^{**} \wedge F^*) \vee (\downarrow u^* \wedge I^{**}) \vee (I^{**} \wedge F^*), \uparrow u^* \wedge \uparrow u^{**} \wedge I^* \wedge F^{**}).$$

Clearly, $(\downarrow u^* \wedge I^{**}) \vee (F^* \wedge I^{**}) \leq I \leq \downarrow u \vee I$. Since (4) holds in $\mathbf{A}$ we also have $\downarrow(u^{**} \wedge u^*) \leq \downarrow u$. By Lemma 3.5(iii) and (v), the ideals F^* and $\downarrow u \wedge F^*$ belong to $\mathcal{I}(\mathbf{A})$. Using Lemma 4.2(i) we obtain that $F^{***} = F^*$ and $\downarrow u^{**} \wedge F^* = (\downarrow u \wedge F^*)^{**} = \downarrow u \wedge F^* \leq \downarrow u$. Thus,

$$\downarrow(u^{**} \wedge u^*) \vee (\downarrow u^{**} \wedge F^*) \vee (\downarrow u^* \wedge I^{**}) \vee (I^{**} \wedge F^*) \leq \downarrow u \vee I.$$

By (5) we have $u^* \wedge u^{**} \leq w$ for every $w \in F \subseteq \mathbf{A}^\vee$. By (4), also $u^* \wedge u^{**} \leq u$. Hence,

$$\uparrow u^{**} \wedge \uparrow u^* \wedge I^* \wedge F^{**} \leq \uparrow(u^{**} \wedge u^*) \leq \uparrow u \wedge F.$$

Thus (4) holds in $\mathbf{T}(\mathbf{A})$.

Since $\downarrow u \wedge \downarrow u^* \leq \mathbf{A}^\wedge$ and $\uparrow u \wedge \uparrow u^* \leq \mathbf{A}^\vee$ (and, of course, $I, F^* \leq \mathbf{A}^\wedge$), we obtain

$$x \wedge x^* = ((\downarrow u \vee I) \wedge (\downarrow u^* \vee F^*), \uparrow u \wedge F \wedge \uparrow u^* \wedge I^*) \leq (\mathbf{A}^\wedge, \mathbf{A}^\vee).$$

Dually, $y \vee y^* \geq (\mathbf{A}^\wedge, \mathbf{A}^\vee)$, hence (5) holds in $\mathbf{T}(\mathbf{A})$.

Finally, we prove (6). The element $x \vee y^* \vee y^{**}$ is equal to

$$(\downarrow u \vee I \vee \downarrow v^* \vee G^* \vee \downarrow v^{**} \vee J^{**}, (\uparrow u \wedge F) \vee (\uparrow v^* \wedge J^*) \vee (\uparrow v^{**} \wedge G^{**})).$$

We need to show that this element is greater than or equal to $x^{**} = (\downarrow u^{**} \vee I^{**}, \uparrow u^{**} \wedge F^{**})$. By Lemma 4.2(i), $I^{**} = I$. Since (6) holds in $\mathbf{A}$ we have $\downarrow u \vee \downarrow v^* \vee \downarrow v^{**} \geq \downarrow u^{**}$. Thus,

$$\downarrow u \vee I \vee \downarrow v^* \vee G^* \vee \downarrow v^{**} \vee J^{**} \geq \downarrow u^{**} \vee I^{**}\,.$$

By Lemma 4.2(ii) we have

$$\begin{aligned}(\uparrow u \wedge F) \vee (\uparrow v^* \wedge J^*) \vee (\uparrow v^{**} \wedge G^{**}) &\subseteq (\uparrow u \wedge F) \cap \mathbf{A}^{**} \cap \mathbf{A}^{\vee} \\ &\subseteq (\uparrow u \wedge F) \cap \mathbf{A}^{**} \subseteq (\uparrow u \wedge F)^{**} = \uparrow u^{**} \wedge F^{**},\end{aligned}$$

which completes the proof of $\mathbf{T}(\mathbf{A}) \in \mathcal{K} \vee \mathcal{S}$.

Further, assume that $\mathbf{A} \in \mathcal{K}$. We need to check the identity (7) for $x = (\downarrow u \vee I, \uparrow u \wedge F)$. By Lemma 4.2(i) we have $I = I^{**}$. Since (7) holds in $\mathbf{A}$, we have $u = u^{**}$ and $F = F^{**}$. Thus

$$x = (\downarrow u \vee I, \uparrow u \wedge F) = (\downarrow u^{**} \vee I^{**}, \uparrow u^{**} \wedge F^{**}) = x^{**}$$

implying $\mathbf{T}(\mathbf{A}) \in \mathcal{K}$.

Finally, suppose that $\mathbf{A} \in \mathcal{S}$. We need to show that the identity (8) holds for $x = (\downarrow u \vee I, \uparrow u \wedge F)$. Since (8) holds in $\mathbf{A}$ we have $u \wedge u^* = 0$ and $\mathcal{I}(\mathbf{A}) = \{\downarrow 0\}$. Now we compute:

$$\begin{aligned} x \wedge x^* &= ((\downarrow u \vee I) \wedge (\downarrow u^* \vee F^*), \uparrow u \wedge F \wedge \uparrow u^* \wedge I^*) \\ &= (\downarrow (u \wedge u^*) \vee (\downarrow u \wedge F^*) \vee (\downarrow u^* \wedge I) \vee (I \wedge F^*), \uparrow (u \wedge u^*) \wedge F \wedge I^*) \\ &= (\downarrow 0, \uparrow 0)\,. \end{aligned}$$

Hence $\mathbf{T}(\mathbf{A}) \in \mathcal{S}$. □

For $(I, F) \in \mathbf{T}(\mathbf{A})$, let $\overline{(I,F)}$ denote the $\sim$-equivalence class containing (I, F).

Lemma 4.4.

(i) *The assignment* $x \mapsto \overline{(\downarrow x, \uparrow x)}$ *is an embedding* $\mathbf{A} \to \overline{\mathbf{T}(\mathbf{A})}$*;*

(ii) *If* $x \in \mathbf{A}^{\vee}$ *then* $(\downarrow 1, \uparrow x) \sim (\downarrow x, \uparrow x)$*;*

(iii) $\overline{\mathbf{T}(\mathbf{A})}^{\vee}$ *is isomorphic to the lattice* $\mathcal{F}(\mathbf{A})$.

Proof. To prove (i), suppose that $x, y \in A$ with $(\downarrow x, \uparrow x) \sim (\downarrow y, \uparrow y)$. Then (17) implies $x \wedge a = y \wedge a$ and $x \vee b = y \vee b$ for some $a \in \uparrow x \cap \mathbf{A}^{\vee}$, $b \in \downarrow x \cap \mathbf{A}^{\wedge}$. Hence $y \wedge a = x$ and $y \vee b = x$ implying $x = y$.

(ii) follows directly from (17), since we can take $a = x$.

The fact that $\mathcal{F}(\mathbf{A})$ is a sublattice of $\mathrm{F}(\mathbf{A})$ follows from Lemma 3.5. For every $G \in \mathcal{F}(\mathbf{A})$ we have $(\downarrow 1, G) = (\downarrow 1 \vee A^{\wedge}, \uparrow 1 \wedge G) \in \mathbf{T}(\mathbf{A})$ and $(\downarrow 1, G)^* = (G^*, \uparrow 0) \leq (\downarrow 1, G)$, hence $\overline{(\downarrow 1, G)} \in \overline{\mathbf{T}(\mathbf{A})}^{\vee}$. The assignment $G \mapsto \overline{(\downarrow 1, G)}$ is clearly a lattice homomorphism, it remains to show its bijectivity.

Every element of $\overline{\mathbf{T}(\mathbf{A})}^{\vee}$ is of the form $\overline{(\downarrow u \vee I, \uparrow u \wedge F)} \vee \overline{(\downarrow u \vee I, \uparrow u \wedge F)}^*$ for some $u \in A$, $I \in \mathcal{I}(\mathbf{A})$ and $F \in \mathcal{F}(\mathbf{A})$. Using the distributivity and the inequality (5) it is easy to calculate that

$$(\downarrow u \vee I, \uparrow u \wedge F) \vee (\downarrow u^* \vee F^*, \uparrow u^* \wedge I^*) = (\downarrow v, H)\,,$$

where $v = u \vee u^* \in A^\vee$ and $H = \uparrow v \wedge (\uparrow u^* \vee F) \wedge (\uparrow u \vee I^*) \wedge (F \vee I^*) \in \mathcal{F}(\mathbf{A})$. Since $v \in H = H^\vee$, by (17), we have $(\downarrow v, H) \sim (\downarrow 1, H)$. Hence, our assignment is surjective.

Assume, that $\overline{(\downarrow 1, G_1)} = \overline{(\downarrow 1, G_2)}$, for some $G_1, G_2 \in \mathcal{F}(\mathbf{A})$. Then, by (17), there exists $t \in \downarrow 1 \cap A^\wedge = A^\wedge$ such that $G_1 \vee \uparrow t = G_2 \vee \uparrow t$. By (5) we have $G_i \subseteq \uparrow t$, $i = 1, 2$. Thus $G_i \vee \uparrow t = G_i$, $i = 1, 2$ implying $G_1 = G_2$, which completes the proof. □

Now we are ready to prove our second main theorem, which solves Problem 1.2 for local polynomial functions on $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$. The equivalence of (ii) and (iii) also shows that our construction is, in some sense, the best possible.

Theorem 4.5. *Let $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$. For $f\colon A^n \to A$ the following statements are equivalent:*

(i) *f is a local polynomial function;*
(ii) *f can be interpolated by a polynomial of $\overline{\mathbf{T}(\mathbf{A})}$;*
(iii) *f can be interpolated by a polynomial of some extension $\mathbf{B} \in \mathcal{K} \vee \mathcal{S}$ of $\mathbf{A}$.*

Proof. The implication (ii)$\Longrightarrow$(iii) follows from Theorem 4.3 and Lemma 4.4. Suppose that (iii) holds. Every polynomial of $\mathbf{B}$ preserves the uncertainty relation and all congruences on $\mathbf{B}$. Since Ockham algebras have the congruence extension property, every congruence on $\mathbf{A}$ extends to a congruence on $\mathbf{B}$, so f must be compatible. Since $\mathbf{B} \in \mathcal{K} \vee \mathcal{S}$, the uncertainty order on $\mathbf{A}$ is a restriction of the uncertainty order on $\mathbf{B}$, so f must be uncertainty preserving. By Theorem 2.4, f is local polynomial function.

For the implication (i)$\Longrightarrow$(ii) due to Theorem 2.4 we just need to show that every function $\hat{F}$ can be interpolated by a polynomial of $\overline{\mathbf{T}(\mathbf{A})}$. So, let $F \in \mathcal{F}(\mathbf{A})$. We claim that $\hat{F}$ is interpolated by the polynomial $g(t) = \overline{(\downarrow 1, F)} \vee t \vee t^*$. Clearly, $(\downarrow 1, F) = (\downarrow 1, \uparrow 1 \wedge F) \in \mathbf{T}(\mathbf{A})$. Let $x \in A$. We compute in $\mathbf{T}(\mathbf{A})$:

$$(\downarrow 1, F) \vee (\downarrow x, \uparrow x) \vee (\downarrow x^*, \uparrow x^*) = (\downarrow 1, F \cap \uparrow x \cap \uparrow x^*) = (\downarrow 1, \uparrow \hat{F}(x)).$$

Since $\hat{F}(x) \in \mathbf{A}^\vee$, we have $(\downarrow 1, \uparrow \hat{F}(x)) \sim (\downarrow \hat{F}(x), \uparrow \hat{F}(x))$, by Lemma 4.4(ii). By identifying every $y \in A$ with $\overline{(\downarrow y, \uparrow y)}$ we obtain that $g(x) = \hat{F}(x)$. □

For compatible functions (which are not local polynomial functions) on Stone and Kleene algebras the Problems 1.1 and 1.2 are still open.

If $\mathbf{A}^\vee$ does not contain nontrivial Boolean intervals, then every compatible function of $\mathbf{A}$ preserves the uncertainty order. (This was proved in [5] for Stone algebras, in [4] for Kleene algebras and in [8] for $\mathcal{K} \vee \mathcal{S}$-algebras.) Thus, we have the following consequence.

Theorem 4.6. *If $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ is such that $\mathbf{A}^\vee$ does not contain nontrivial Boolean intervals, then for any function f on $\mathbf{A}$, the following conditions are equivalent.*

(i) *f is compatible;*
(ii) *f is a composition of polynomials and the functions $\hat{F}$, where $F \in \mathcal{F}(\mathbf{A})$;*
(iii) *f can be interpolated by a polynomial of $\overline{\mathbf{T}(\mathbf{A})}$.*

To justify the title of this Section, we prove the following result.

Theorem 4.7. *If $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ is such that $\mathbf{A}^\vee$ does not contain nontrivial Boolean intervals, then $\overline{\mathbf{T}(\mathbf{A})}$ is affine complete.*

Proof. If $\mathbf{A}^\vee$ does not contain a nontrivial Boolean interval, then the same is true for the lattice $\mathcal{F}(\mathbf{A})$. (See [9], Lemma 3.8.) By Lemma 4.4(iii), $\overline{\mathbf{T}(\mathbf{A})}^\vee$ does not contain a nontrivial Boolean interval. Then, by [8], every compatible function on $\overline{\mathbf{T}(\mathbf{A})}$ is uncertainty-preserving. By Theorem 2.4, every such function is a composition of polynomials and functions of the type $\hat{F}$. However, $\overline{\mathbf{T}(\mathbf{A})}^\vee$ has a smallest element (namely $\overline{(\downarrow 1, \mathbf{A}^\vee)}$), so every almost principal filter of $\overline{\mathbf{T}(\mathbf{A})}^\vee$ is principal and all functions $\hat{F}$ are polynomials. □

If $\mathbf{A}^\vee$ does contain a nontrivial Boolean interval $[u, v]$, then the function $f(x) = ((x \vee u) \wedge v)'$ is compatible but not uncertainty-preserving. (Here $'$ denotes the complement in $[u, v]$.) Our conjecture is that every compatible function is a composition of local polynomial functions and the functions of the above type.

It is still possible that every $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ has a CEP-extension $\mathbf{B}$ with the property that every compatible function on $\mathbf{A}$ can be interpolated by a polynomial of $\mathbf{B}$. However, if $\mathbf{A}^\vee$ contains a nontrivial Boolean interval, such extension cannot be found inside $\mathcal{K} \vee \mathcal{S}$ (by Theorem 4.5).

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