Dual spaces of some congruence lattices

Miroslav Ploščica *

Abstract

For any set X and any $n \geq 3$ we define a topological space $\overline{L_n(X)}$ and characterize its closed subspaces if $|X| \leq \aleph_1$. As an application we obtain a characterization of congruence lattices of algebras in some varieties of lattices. The spaces $\overline{L_n(X)}$ are close to Boolean spaces, but they are not Hausdorff.

keywords: open compact set, Boolean space, congruence lattice
1991 AMS subject classification: 54F65, 54H10, 08A30

1 Introduction

The motivation for this paper comes from universal algebra. The general problem is as follows: *Characterize congruence lattices of algebras belonging to a variety (equational class) V.* There are rather few relevant classes V for which we have a satisfactory answer.

It seems that in many cases topological tools are convenient for such a description. If $\mathrm{Con}(A)$ (the congruence lattice of an algebra A) is distributive then $\mathrm{Con}(A)$ is isomorphic to the lattice $\mathcal{O}(T)$ of all open subsets of some topological space T. (See [2],[1] or [6].) Thus, if the class V is a congruence distributive variety, the following strategy was proposed in [6]:

1. For every set X describe a topological space $T(X)$ such that $\mathcal{O}(T(X)) \cong \mathrm{Con}(F(X))$, where $F(X)$ denotes the free algebra in V with X as the set of free generators.

2. Describe closed subspaces of the spaces $T(X)$.

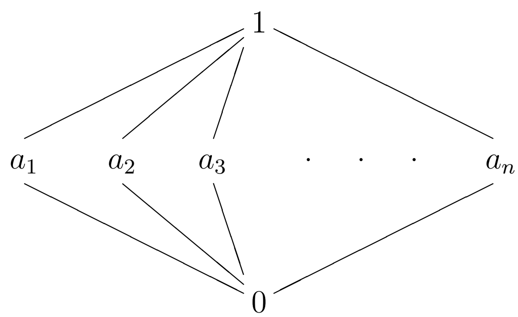
The general result is that lattices $\mathrm{Con}(A)$ (for $A \in V$) are precisely the lattices $\mathcal{O}(Y)$ (for closed subspaces Y of some $T(X)$). Let us remark that

*Supported by VEGA Grant 5125/98.

if we have a good knowledge of subdirectly irreducible algebras in V then the explicit definition of the spaces $T(X)$ is not a problem. Hence, the original algebraic problem is transferred to the topological problem of abstract characterization of closed subspaces of $T(X)$.

As an example, consider the class B of Boolean algebras. Then the proposed topological representation is in fact the well-known Stone duality and $T(X)$ is $\{0,1\}^X$ (the power space of the 2-element discrete space). The closed subspaces of $\{0,1\}^X$ are precisely all Boolean (compact Hausdorff zerodimensional) spaces. Hence, a lattice L is isomorphic to $\mathrm{Con}(A)$ for some Boolean algebra A if and only if $L \cong \mathcal{O}(Y)$ for some Boolean space Y.

In this paper we consider the case not far from Boolean algebras, which is however much more complicated. For $n \geq 3$ let $\mathcal{M}_n^{01}$ denote the equational class generated by the $(n+2)$-element bounded lattice M_n depicted below. (That is, we consider the constants 0, 1 as nullary operations.)



The spaces $T(X)$ corresponding to varieties $\mathcal{M}_n^{01}$ were constructed in [6]. In this paper we denote them $\overline{L_n(X)}$. (For the definition see the next section and the remark in section 4.) Our main result is a characterization of closed subspaces of $\overline{L_n(X)}$ for $|X| \leq \aleph_1$. It turns out that this characterization does not depend on n, which in particular means that any $\overline{L_n(X)}$ is homeomorphic to a closed subspace of $\overline{L_3(X)}$. In the last section we discuss the algebraic consequences of these results.

We use the topological terminology from [4] and [3]. In some places we find it convenient to work with nets, cluster points and limit points. The closure of a set A is denoted by $\mathrm{cl}(A)$. Our compact spaces need not be Hausdorff. Consequently, compact sets need not be closed. The following assertion is a simple exercise.

Lemma 1.1 *Let C be a compact subset of a topological space T and $W \subseteq C$. Let $w \in \mathrm{cl}(W) \setminus C$. Then there is $u \in C$ such that for every open sets $A, B \subseteq C$, $u \in A$ and $w \in B$ implies $A \cap B \cap W \neq \emptyset$. Moreover, there is a net N in W such that N converges to both u and w.*

Our reference books for the lattice theory and universal algebra are [2] and [5]. If f is a function then $\mathrm{dom}(f)$ and $\mathrm{rng}(f)$ stand for its domain and range, respectively. If $Y \subseteq \mathrm{dom}(f)$ then $f \restriction Y$ means the restriction of f to Y.

2 The spaces $L_n(X)$ and $\overline{L_n(X)}$

Let us consider the set $M_n = \{0, 1, a_1, \dots, a_n\}$ as a discrete topological space. For any set X the usual power M_n^X is a compact Hausdorff zerodimensional space.

Let Π_n be the set of all permutations π on M_n such that $\pi(0) = 0$, $\pi(1) = 1$. Let R_0 be the set of all functions $r : X_0 \to \{0, 1\}$ such that X_0 is a finite subset of X. Let R_n be the set of all functions $r : X_0 \to M_n$ such that $\mathrm{rng}(r)$ contains at least three of a_i $(i = 1, \dots, n)$. For $r \in R_0$ we define

$$K_r = \{f \in M_n^X \mid \begin{array}{l} r^{-1}(0) \subseteq (\pi f)^{-1}(0, a_1), \\ r^{-1}(1) \subseteq (\pi f)^{-1}(1, a_2) \text{ for some } \pi \in \Pi_\mathrm{n}\}. \end{array}$$

For $r \in R_n$ we set

$$K_r = \{f \in M_n^X \mid r = \pi f \restriction X_0 \text{ for some } \pi \in \Pi_n\}.$$

It is clear that every K_r is a clopen (and hence compact) subset of M_n^X. The following assertion is obvious.

Lemma 2.1 *If $r, s \in R_0 \cup R_n$, $r \subseteq s$, then $K_s \subseteq K_r$. If $f \in K_r$ and $g \restriction \mathrm{dom}(r) = \pi f \restriction \mathrm{dom}(r)$ for some $\pi \in \Pi_n$ then $g \in K_r$.*

Now we define the space $L_n(X)$. The points of $L_n(X)$ are all maps $f : X \to M_n$ such that $f(X) \subseteq \{0, 1\}$ or $|\mathrm{rng}(f) \cap \{a_1, \dots, a_n\}| \geq 3$. For every $r \in R_0 \cup R_n$ we denote $G_r = K_r \cap L_n(X)$. Let us endow $L_n(X)$ with the topology whose base of open set is $\mathcal{G} = \{G_r \mid r \in R_0 \cup R_n\}$. The correctness of this definition is assured by the following assertion.

Lemma 2.2 *For any $G_r, G_s \in \mathcal{G}$, the set $G_r \cap G_s$ is a union of some sets from $\mathcal{G}$. Hence, $\mathcal{G}$ is indeed a base of some topology on $L_n(X)$.*

P r o o f. Let $f \in G_r \cap G_s$. Suppose first that $\mathrm{rng}(f) \subseteq \{0,1\}$. This is only possible if $r, s \in R_0$, $r = f \restriction \mathrm{dom}(r)$, $s = f \restriction \mathrm{dom}(s)$. Choose a finite set $X_0 \subseteq X$ with $X_0 \supseteq \mathrm{dom}(r) \cup \mathrm{dom}(s)$ and let $t = f \restriction X_0$. Clearly, $f \in G_t \subseteq G_r \cap G_s$.

Now let $\mathrm{rng}(f) \not\subseteq \{0,1\}$. Choose a finite set $X_0 \subseteq X$ such that $X_0 \supseteq \mathrm{dom}(r) \cup \mathrm{dom}(s)$ and $f(X_0) = \mathrm{rng}(f)$. Let $t = f \restriction X_0$. Then $t \in R_n$ and $f \in G_t$. For every $g \in G_t$ there is $\pi \in \Pi_n$ with $g \restriction X_0 = \pi f \restriction X_0$, which by 2.1 implies that $g \in G_r \cap G_s$. ∎

Lemma 2.3 *The space $L_n(X)$ is compact. Every $G_r \in \mathcal{G}$ is compact. If $r \in R_n$, then G_r is clopen.*

P r o o f. Suppose that the set $G_r \in \mathcal{G}$ is covered by sets $G_{r_i} \in \mathcal{G}$, $i \in I$. We claim that the set K_r is covered by the sets K_{r_i}. Let $f \in K_r$. If $f \in L_n(X)$ then clearly $f \in G_{r_i} \subseteq K_{r_i}$ for some $i \in I$. Let $f \in K_r \setminus L_n(X)$. Then necessarily $r \in R_0$ and $\mathrm{rng}(f) \subseteq \{0, 1, a_i, a_j\}$ for some $a_i, a_j \in M_n$. Let us define $g: X \to \{0,1\}$ by $g(x) = 0$ if $f(x) \in \{0, a_i\}$ and $g(x) = 1$ if $f(x) \in \{1, a_j\}$. Then $f \in K_r$ implies $g \in K_r$ and hence $g \in G_r$. By our assumption, there exists $i \in I$ with $g \in G_{r_i}$, which implies $f \in K_{r_i}$. Hence, the set K_r is covered by K_{r_i}. Since K_r is compact, there is a finite set $I_0 \subseteq I$ with $K_r \subseteq \bigcup\{K_{r_i} \mid i \in I_0\}$. Then $G_r = K_r \cap L_n(X) \subseteq \bigcup\{G_{r_i} \mid i \in I_0\}$, which shows that G_r is compact.

The compactness of the whole $L_n(X)$ can be shown by exactly the same argument. For the last part of our assertion, let $r \in R_n$, $X_0 = \mathrm{dom}(r)$ and let $f \in L_n(X) \setminus G_r$. We wish to find $G_s \in \mathcal{G}$ with $f \in G_s$ and $G_s \cap G_r = \emptyset$. Choose a finite set $X_1 \subseteq X$ with $X_1 \supseteq X_0$ and $f(X_1) = f(X)$ and set $s = f \restriction X_1$. Then clearly $s \in R_0 \cup R_n$ and $f \in G_s$. Let $h \in G_s$. If $s \in R_0$, then $h(X_0) \subseteq h(X_1) \subseteq \{0, 1, a_i, a_j\}$ for some $a_i, a_j \in M_n$. Since $r \in R_n$, such h cannot belong to G_r. If $s \in R_n$ then $\pi h \restriction X_1 = s = f \restriction X_1$ for some $\pi \in \Pi_n$. Then also $\pi h \restriction X_0 = f \restriction X_0$. Since $f \notin G_r$, we have $\pi h \notin G_r$ and hence $h \notin G_r$. ∎

Lemma 2.4 $\{G_r \in \mathcal{G} \mid r \in R_0,\ |\mathrm{dom}(r)| \leq 2\}$ *is a subbase for the space $L_n(X)$.*

P r o o f. Let $G_r \in \mathcal{G}$, $\mathrm{dom}(r) = X_0$. Let S be the collection of all $s \in R_0$ with $\mathrm{dom}(s) \subseteq X_0$, $|\mathrm{dom}(s)| \leq 2$ and $G_r \subseteq G_s$. Then clearly $G_r \subseteq \bigcap\{G_s \mid s \in S\}$. We prove the other inclusion. Let $f \in L_n(X) \setminus G_r$. We distinguish several cases. In every case we find $s \in S$ with $f \notin G_s$.

1. Let $r(x), f(x) \in \{0,1\}$ and $r(x) \neq f(x)$ for some $x \in X_0$. We set $s = r \restriction \{x\}$.

2. Let $r(x) = r(y) \in \{0,1\}$, $f(x) = a_i$, $f(y) = a_j$, $i \neq j$ for some $x, y \in X_0$. We set $s = r \restriction \{x, y\}$.

3. Let $r(x) = 0$, $r(y) = 1$, $f(x) = f(y) = a_i$ for some $x, y \in X_0$. We set $s = r \restriction \{x, y\}$.

The above three cases cover the case $r \in R_0$. Now we handle the case $r \in R_n$

4. Let $r \in R_n$ and $r(x) = 0$, $f(x) = a_i$ for some $x \in X_0$. (The case $r(x) = 1$ is similar.) Choose $y \in X_0$ such that $r(y) \in \{a_1, \ldots, a_n\}$. We define $s :\ \{x, y\} \to \{0,1\}$ as follows. If $f(y) = f(x)$ or $f(y) = 0$, then $s(x) = 0$ and $s(y) = 1$. If $f(y) \notin \{f(x), 0\}$ then $s(x) = s(y) = 0$.

5. Let $r \in R_n$, $r(x) = a_i$, $f(x) \in \{0,1\}$ for some $x \in X_0$. We define $s :\ \{x\} \to \{0,1\}$ such that $s(x) \neq f(x)$.

6. Let $r(x) = r(y) = a_i$, $f(x) = a_m \neq a_l = f(y)$ for some $x, y \in X_0$. We define $s :\ \{x, y\} \to \{0,1\}$ by $s(x) = s(y) = 0$.

7. Let $f(x) = f(y) = a_i$, $r(x) = a_m \neq a_l = r(y)$ for some $x, y \in X_0$. We define $s :\ \{x, y\} \to \{0,1\}$ by $s(x) = 0$, $s(y) = 1$. ∎

The space $L_n(X)$ is not T_0. For $f, g \in L_n(X)$ we write $f \sim g$ if $f = \pi g$ for some permutation $\pi \in \Pi_n$. If $f \sim g$ then the points $f, g \in L_n(X)$ are topologically indistinguishable. (If an open set contains one of them, it contains both.) It is therefore reasonable to consider the space $\overline{L_n(X)}$ whose points are $\sim$-equivalence classes (with $\overline{f}$ denoting the class containing f) and the base of open sets is $\overline{\mathcal{G}} = \{\overline{G} \mid G \in \mathcal{G}\}$, where $\overline{G} = \{\overline{f} \mid f \in G\}$.

Lemma 2.5 *Let $f, g \in L_n(X)$, $f \not\sim g$, $\mathrm{rng}(f) \not\subseteq \{0,1\}$. Then there is $r \in R_n$ with $f \in G_r$, $g \notin G_r$.*

P r o o f. Since Π_n is finite, there is a finite set $Y \subseteq X$ such that $f \restriction Y \neq \pi g \restriction Y$ for every $\pi \in \Pi_n$. Choose a finite set $X_0 \subseteq X$ such that $Y \subseteq X_0$ and $\mathrm{rng}(f) \subseteq f(X_0)$ and set $r := f \restriction X_0$. ∎

Because of the application to congruence lattices, we would like to have an abstract characterization of closed subspaces of $\overline{L_n(X)}$. The space $\overline{L_n(X)}$

is a disjoint union of two sets $H_0 = \{\overline{f} \mid f \in \{0,1\}^X\}$, $H_n = \overline{L_n(X)} \setminus H_0$. Similarly, every $Z \subseteq \overline{L_n(X)}$ is composed of two subspaces $Z_0 = Z \cap H_0$ and $Z_n = Z \cap H_n$.

It is easy to see that H_0 is homeomorphic to $\{0,1\}^X$ with the usual product topology. (We consider $\{0,1\}$ as a discrete space.) Hence, it is compact, Hausdorff, zerodimensional.

Lemma 2.6 *H_n is zerodimensional.*

P r o o f. It suffices to prove that $H_n \cap G_r$ is closed in H_n for every $r \in R_0 \cup R_n$. Let $\overline{f} \in H_n \setminus G_r$. By 2.5, for every $\overline{g} \in G_r$ there is a clopen set C_g with $\overline{f} \in C_g$, $\overline{g} \notin C_g$. The sets $L_n(X) \setminus C_g$ cover the compact set G_r. Hence, there are $g_1, \ldots, g_k \in G_r$ with $G_r \subseteq L_n(X) \setminus (C_{g_1} \cap \ldots \cap C_{g_k})$. For the open set $C = C_{g_1} \cap \ldots C_{g_k}$ we have $\overline{f} \in C$, $C \cap G_r = \emptyset$. ∎

Lemma 2.7 *Let $Z = Z_0 \cup Z_n$ be a closed subspace of $\overline{L_n(X)}$.*

(i) *Z is compact and has a base of compact open sets.*

(ii) *Z_0 is a closed subspace of Z.*

(iii) *Z_0 and Z_n are Hausdorff, zerodimensional.*

(iv) *If $\overline{f} \in Z_n$, $\overline{g} \in Z \setminus \{\overline{f}\}$ then there exists a clopen set G with $\overline{f} \in G \subseteq Z_n$, $\overline{g} \notin G$.*

(v) *If $\overline{f}, \overline{g}, \overline{h} \in Z$ are mutually different, then there exist open sets A, B, C such that $\overline{f} \in A$, $\overline{g} \in B$, $\overline{h} \in C$ and $A \cap B \cap C = \emptyset$.*

P r o o f. It is easy to see that the properties (i)–(v) are preserved by closed subspaces. Thus, it suffices to prove them for $Z = \overline{L_n(X)}$. In this case, (i) follows from 2.3 and (ii) and (iv) from 2.5. (Notice that $G_r \cap Z_0 = \emptyset$ whenever $r \in R_n$.) Further, Z_n is zerodimensional by 2.6 and Hausdorff by 2.5. For Z_0, (iii) was discussed before 2.6. Finally, the proof of (v) is identical with the proof of 5.5 in [6]. ∎

The compactness of Z and (iv) have the following consequence.

Lemma 2.8 *Suppose that $\overline{f} \in Z_n$, $W \subseteq Z$ is closed and $\overline{f} \notin W$. Then there exists a clopen set G with $\overline{f} \in G \subseteq Z_n$, $G \cap W = \emptyset$.*

If X is infinite, then $\overline{L_n(X)}$ is not Hausdorff. In fact, no two points of H_0 can be separated. However, (v) says that every three points can be separated. It implies that if a net N converges to two distinct limit points, then N has no other cluster points.

As a simple example of a space Z satisfying (i)–(v) of 2.7 one may consider a discrete sequence converging to two different limit points.

3 The embedding theorem

Throughout this section we assume that a space $Z = Z_0 \cup Z_n$ satisfies (i)–(v) of 2.7. We wish to construct an embedding $Z \to \overline{L_n(X)}$ for some X.

Lemma 3.1 *For every $S \subseteq Z_0$ which is clopen in Z_0 and every $W \subseteq Z$ with $\mathrm{cl}(W) \cap S = \emptyset$ there is a compact open set $C \subseteq Z$ with $C \cap Z_0 = S$ and $C \cap W = \emptyset$.*

P r o o f. Since S is open in Z_0, there is an open set C' with $C' \cap Z_0 = S$ and $C' \cap \mathrm{cl}(W) = \emptyset$. By (i), $C' = \bigcup\{C_\alpha \mid \alpha \in I\}$, where all C_α are compact open. Since S is closed, it is compact and hence $S \subseteq C_{\alpha_1} \cup \ldots \cup C_{\alpha_m}$ for some $\alpha_1, \ldots, \alpha_m \in I$. The set $C = C_{\alpha_1} \cup \ldots \cup C_{\alpha_m}$ has the required properties. ∎

Lemma 3.2 *For every compact open set $A \subseteq Z$, the set $A \cap Z_n$ is closed in Z_n.*

P r o o f. The same as the proof of 2.6, using (iv) instead of 2.5. ∎

Lemma 3.3 *If C is compact open and $Z_0 \cap C = \emptyset$ or $Z_0 \subseteq C$ then C is clopen.*

P r o o f. The case $Z_0 \cap C = \emptyset$ is an easy consequence of (iv). Let $Z_0 \subseteq C$. By 3.2, $Z \setminus C$ is open in Z_n. Since Z_n is open in Z, the set $Z \setminus C$ is open in Z and hence C is closed. ∎

Lemma 3.4 *Let $A \subseteq Z$ be clopen and $B \subseteq Z$ open compact. Then $A \cap B$ is open compact. Moreover, if $A \subseteq Z_n$ then $A \cap B$ is clopen.*

P r o o f. Since A is closed and B compact, $A \cap B$ is obviously compact. By 3.2 we have $\mathrm{cl}(B) \subseteq B \cup Z_0$. If $A \subseteq Z_n$ then $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(B) \subseteq A \cap (B \cup Z_0) = A \cap B$. ∎

Lemma 3.5 *Let B_1, B_2, B_3, B_4 be disjoint subsets of Z_0 such that $B_1 \cup B_2 \cup B_3 \cup B_4 = Z_0$. Let $C_{12}, C_{13}, C_{14}, C_{23}, C_{24}, C_{34}$ be compact open subsets of Z such that $C_{ij} \cap Z_0 = B_i \cup B_j$. Then the set $W = C_{12} \cap \ldots \cap C_{34}$ is closed.*

P r o o f. It is easy to check that $W \cap Z_0 = \emptyset$. Suppose that W is not closed and $w \in \mathrm{cl}(W) \setminus W$. By 3.2, W is closed in Z_n, hence $w \in Z_0$. Without loss of generality, $w \in B_1$. Then $w \in \mathrm{cl}(W) \setminus C_{23}$. By 1.1 there is $v \in C_{23}$ and a net N in W converging to both v and w. From (v) it follows that N has no other cluster point in Z than v and w. Now, either $v \in B_2$ or $v \in B_3$. Consequently, either C_{34} or C_{24} contains no cluster point of N, which contradicts the compactness. ∎

We wish to construct a set X and an embedding $Z \to \overline{L_n(X)}$. It turns out that we are able to do it if the Boolean space Z_0 contains at most $\aleph_1$ clopen sets. (Equivalently, Z_0 is homeomorphic to a closed subspace of $\{0,1\}^{\aleph_1}$.) Thus, in the sequel we assume that $\mathcal{B} = \{B^\alpha \mid \alpha \in I\}$ with $|I| \leq \aleph_1$ is the family of all clopen subsets of Z_0.

By 3.1, there are compact open sets C_0^α, C_1^α such that $C_0^\alpha \cap Z_0 = B^\alpha$, $C_1^\alpha \cap Z_0 = Z_0 \setminus B^\alpha$. By 3.3 the set $Z \setminus (C_0^\alpha \cup C_1^\alpha)$ is compact clopen, so we can assume that $C_0^\alpha \cup C_1^\alpha = Z$. We denote $D^\alpha = C_0^\alpha \cap C_1^\alpha$. Hence $D^\alpha \subseteq Z_n$. In general, $D^\alpha \neq \emptyset$.

The next lemma contains the core of our construction. It claims the existence of some sets $E_{\{\alpha,\beta\}}$, $F_{\{\alpha,\beta\}}$ indexed by two-element subsets of I. (For simplification, we write $E_{\alpha\beta}$, $F_{\alpha\beta}$.)

Lemma 3.6 *There are families of compact open sets $\{E_{\alpha\beta} \mid \alpha, \beta \in I,\ \alpha \neq \beta\}$, $\{F_{\alpha\beta} \mid \alpha, \beta \in I,\ \alpha \neq \beta\}$ such that the following conditions are satisfied for every distinct $\alpha, \beta, \gamma \in I$:*

(1) $E_{\alpha\beta} \cap Z_0 = ((C_0^\alpha \cap C_0^\beta) \cup (C_1^\alpha \cap C_1^\beta)) \cap Z_0$;

(2) $F_{\alpha\beta} \cap Z_0 = ((C_0^\alpha \cap C_1^\beta) \cup (C_1^\alpha \cap C_0^\beta)) \cap Z_0$;

(3) $E_{\alpha\beta} \cap F_{\alpha\beta} \cap D^\alpha \cap D^\beta = \emptyset$;

(4) $E_{\alpha\beta} \cup F_{\alpha\beta} = Z$;

(5) $E_{\alpha\beta} \cap E_{\beta\gamma} \cap D^{\alpha} \cap D^{\beta} \cap D^{\gamma} \subseteq E_{\alpha\gamma}$;

(6) $F_{\alpha\beta} \cap F_{\beta\gamma} \cap D^{\alpha} \cap D^{\beta} \cap D^{\gamma} \subseteq E_{\alpha\gamma}$.

P r o o f. Notice that (3),(4) and (5) imply

(7) $E_{\alpha\beta} \cap F_{\beta\gamma} \cap D^{\alpha} \cap D^{\beta} \cap D^{\gamma} \subseteq F_{\alpha\gamma}$.

We proceed by induction. Suppose that I is well ordered of the type at most $\aleph_1$. Let $\delta \in I$ and suppose that we have constructed $E_{\alpha\beta}$, $F_{\alpha\beta}$ for every $\alpha, \beta < \delta$ and (1)-(7) are satisfied whenever $\alpha, \beta, \gamma < \delta$. The set $\{\alpha \in I \mid \alpha < \delta\}$ is countable and can be arranged as a sequence $\{\alpha_0, \alpha_1, \ldots\}$. We will define $E_{\alpha_i\delta}$, $F_{\alpha_i\delta}$ by induction on i. Suppose that $k \in \omega$ and we have defined $E_{\alpha_i\delta}$, $F_{\alpha_i\delta}$ for all $i < k$. Thus, our induction hypothesis is that (1)-(7) are true whenever $\alpha, \beta, \gamma < \delta$ or $\max\{\alpha, \beta, \gamma\} = \delta$ and $\{\alpha, \beta, \gamma\} \setminus \{\delta\} \subseteq \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$.

For every $i < k$ denote
$W_i = ((E_{\alpha_i\delta} \cap E_{\alpha_i\alpha_k}) \cup (F_{\alpha_i\delta} \cap F_{\alpha_i\alpha_k})) \cap D^{\alpha_i} \cap D^{\alpha_k} \cap D^{\delta}$,
$V_i = ((E_{\alpha_i\delta} \cap F_{\alpha_i\alpha_k}) \cup (F_{\alpha_i\delta} \cap E_{\alpha_i\alpha_k})) \cap D^{\alpha_i} \cap D^{\alpha_k} \cap D^{\delta}$.
Further, let $B_1 = ((C_0^{\delta} \cap C_1^{\alpha_k}) \cup (C_0^{\alpha_k} \cap C_1^{\delta})) \cap Z_0$, $B_2 = ((C_0^{\delta} \cap C_0^{\alpha_k}) \cup (C_1^{\alpha_k} \cap C_1^{\delta})) \cap Z_0$.

Claim $V_i \cap W_j = \mathrm{cl}(V_i) \cap B_2 = \mathrm{cl}(W_j) \cap B_1 = \emptyset$ *for every* $i, j < k$.

Proof of the claim. Denote $D = D^{\alpha_i} \cap D^{\alpha_j} \cap D^{\alpha_k} \cap D^{\delta}$. The equality $V_i \cap W_j = \emptyset$ requires to show the following:

(i) $(E_{\alpha_i\alpha_k} \cap F_{\alpha_i\delta}) \cap (E_{\alpha_j\alpha_k} \cap E_{\alpha_j\delta}) \cap D = \emptyset$;

(ii) $(F_{\alpha_i\alpha_k} \cap E_{\alpha_i\delta}) \cap (E_{\alpha_j\alpha_k} \cap E_{\alpha_j\delta}) \cap D = \emptyset$;

(iii) $(E_{\alpha_i\alpha_k} \cap F_{\alpha_i\delta}) \cap (F_{\alpha_j\alpha_k} \cap F_{\alpha_j\delta}) \cap D = \emptyset$;

(iv) $(F_{\alpha_i\alpha_k} \cap E_{\alpha_i\delta}) \cap (F_{\alpha_j\alpha_k} \cap F_{\alpha_j\delta}) \cap D = \emptyset$.

This is clear if $i = j$ because (3) in our induction hypothesis implies $E_{\alpha_i\delta} \cap F_{\alpha_i\delta} \cap D = E_{\alpha_i\alpha_k} \cap F_{\alpha_i\alpha_k} \cap D = \emptyset$. Suppose that $i \neq j$. By (3), (5) and (7) in our induction hypothesis we have $(E_{\alpha_j\delta} \cap F_{\alpha_i\delta}) \cap (E_{\alpha_i\alpha_k} \cap E_{\alpha_j\alpha_k}) \cap D \subseteq F_{\alpha_i\alpha_j} \cap E_{\alpha_i\alpha_j} \cap D = \emptyset$, hence (i) holds. The argument for (ii), (iii) and (iv) is analogous.

Next we prove that $\mathrm{cl}(W_i) \cap B_1 = \emptyset$. (The proof of $\mathrm{cl}(V_i) \cap B_2 = \emptyset$ is similar.) Let $z \in B_1$. We have either $z \in C_0^{\alpha_i}$ or $z \in C_1^{\alpha_i}$, so we distinguish four cases.

a) Let $z \in C_0^{\alpha_i} \cap C_0^{\delta} \cap C_1^{\alpha_k}$. By (1) and (2), z belongs to the open set $E_{\alpha_i\delta} \cap F_{\alpha_i\alpha_k}$. Since $E_{\alpha_i\delta}$ is disjoint from $F_{\alpha_i\delta} \cap D^{\alpha_i} \cap D^{\delta}$ and $F_{\alpha_i\alpha_k}$ is disjoint from $E_{\alpha_i\alpha_k} \cap D^{\alpha_i} \cap D^{\alpha_k}$, the set $E_{\alpha_i\delta} \cap F_{\alpha_i\alpha_k}$ is disjoint from $(E_{\alpha_i\alpha_k} \cap D^{\alpha_i} \cap D^{\alpha_k}) \cup (F_{\alpha_i\delta} \cap D^{\alpha_i} \cap D^{\delta}) \supseteq W_i$, hence $z \notin \mathrm{cl}(W_i)$.

b) Let $z \in C_0^{\alpha_i} \cap C_0^{\alpha_k} \cap C_1^{\delta}$. Similarly as above z belongs to $E_{\alpha_i\alpha_k} \cap F_{\alpha_i\delta}$ and this open set is disjoint from $(F_{\alpha_i\alpha_k} \cup E_{\alpha_i\delta}) \cap D^{\alpha_i} \cap D^{\alpha_k} \cap D^{\delta} \supseteq W_i$, hence $z \notin \mathrm{cl}(W_i)$.

c) If $z \in C_1^{\alpha_i} \cap C_0^{\alpha_k} \cap C_1^{\delta}$ then $z \in E_{\alpha_i\delta} \cap F_{\alpha_i\alpha_k}$ and $E_{\alpha_i\delta} \cap F_{\alpha_i\alpha_k} \cap W_i = \emptyset$, hence $z \notin \mathrm{cl}(W_i)$.

d) If $z \in C_1^{\alpha_i} \cap C_1^{\alpha_k} \cap C_0^{\delta}$ then $z \in F_{\alpha_i\delta} \cap E_{\alpha_i\alpha_k}$ and $F_{\alpha_i\delta} \cap E_{\alpha_i\alpha_k} \cap W_i = \emptyset$, hence $z \notin \mathrm{cl}(W_i)$. This completes the proof of the claim.

Now we continue the proof of 3.6. Let $W = W_0 \cup \ldots \cup W_{k-1}$, $V = V_0 \cup \ldots \cup V_{k-1}$. As a consequence of the claim we obtain that $V \cap W = \mathrm{cl}(V) \cap B_2 = \mathrm{cl}(W) \cap B_1 = \emptyset$. By 3.1 there are compact open sets $E'_{\alpha_k\delta}$, $F'_{\alpha_k\delta}$ such that $E'_{\alpha_k\delta} \cap Z_0 = B_2$, $F'_{\alpha_k\delta} \cap Z_0 = B_1$, $E'_{\alpha_k\delta} \cap V = \emptyset$, $F'_{\alpha_k\delta} \cap W = \emptyset$. By 3.2, for every $i < k$ the sets $W_i \cap Z_n$ and $V_i \cap Z_n$ are closed in Z_n. Hence, $\mathrm{cl}(W) \subseteq W \cup Z_0$, $\mathrm{cl}(V) \subseteq V \cup Z_0$. Since $Z_0 = B_1 \cup B_2$ and $\mathrm{cl}(W) \cap B_1 = \mathrm{cl}(V) \cap B_2 = \emptyset$, we have $\mathrm{cl}(W) \subseteq W \cup B_2 \subseteq W \cup E'_{\alpha_k\delta}$, $\mathrm{cl}(V) \subseteq V \cup B_1 \subseteq V \cup F'_{\alpha_k\delta}$. The sets $\mathrm{cl}(V)$, $\mathrm{cl}(W)$, $E'_{\alpha_k\delta}$ and $F'_{\alpha_k\delta}$ are compact, thus the sets $E''_{\alpha_k\delta} = \mathrm{cl}(W) \cup E'_{\alpha_k\delta} = W \cup E'_{\alpha_k\delta}$ and $F''_{\alpha_k\delta} = \mathrm{cl}(V) \cup F'_{\alpha_k\delta} = V \cup F'_{\alpha_k\delta}$ are compact. Since V and W are open, $E''_{\alpha_k\delta}$ and $F''_{\alpha_k\delta}$ are open sets. By 3.3 and 3.5, the sets $P = Z \setminus (E''_{\alpha_k\delta} \cup F''_{\alpha_k\delta})$ and $Q = E''_{\alpha_k\delta} \cap F''_{\alpha_k\delta} \cap D^{\alpha_k} \cap D^{\delta}$ are clopen. (In using 3.5, let the sets $Z_0 \cap C_l^{\alpha_k} \cap C_m^{\delta}$, $l, m \in \{0, 1\}$, play the role of $B_1, \ldots, B_4$.) We set $E_{\alpha_k\delta} = E''_{\alpha_k\delta}$, $F_{\alpha_k\delta} = (F''_{\alpha_k\delta} \setminus Q) \cup P$. Clearly, $E_{\alpha_k\delta}$ and $F_{\alpha_k\delta}$ are compact open sets. We need to show that (1)-(6) remain valid.

We have $E'_{\alpha_k\delta} \cap Z_0 = B_2$, $F'_{\alpha_k\delta} \cap Z_0 = B_1$. Since $W_i \cap Z_0 \subseteq D^{\delta} \cap Z_0 = \emptyset$ for every $i < k$, we have $W \cap Z_0 = \emptyset$ and similarly $V \cap Z_0 = \emptyset$. Hence, $E''_{\alpha_k\delta} \cap Z_0 = B_2$, $F''_{\alpha_k\delta} \cap Z_0 = B_1$. Since $P \cap Z_0 = Q \cap Z_0 = \emptyset$, we also have $E_{\alpha_k\delta} \cap Z_0 = B_2$ and $F_{\alpha_k\delta} \cap Z_0 = B_1$, hence (1) and (2) hold.

Further, $E_{\alpha_k\delta} \cap P = \emptyset$, hence $E_{\alpha_k\delta} \cap F_{\alpha_k\delta} \cap D^{\alpha_k} \cap D^{\delta} = E''_{\alpha_k\delta} \cap (F''_{\alpha_k\delta} \setminus Q) \cap D^{\alpha_k} \cap D^{\delta} = (E''_{\alpha_k\delta} \cap F''_{\alpha_k\delta} \cap D^{\alpha_k} \cap D^{\delta}) \setminus Q = Q \setminus Q = \emptyset$, which shows (3).

Next, for every $z \in Z$ we have either $z \in P \subseteq F_{\alpha_k\delta}$ or $z \in E''_{\alpha_k\delta} = E_{\alpha_k\delta}$ or $z \in F''_{\alpha_k\delta} \setminus E''_{\alpha_k\delta} \subseteq F''_{\alpha_k\delta} \setminus Q \subseteq F_{\alpha_k\delta}$. Hence, (4) is valid.

To prove (5) and (6) we have to show the following inclusions for every $i < k$:

(5a) $E_{\alpha_i\alpha_k} \cap E_{\alpha_i\delta} \cap D \subseteq E_{\alpha_k\delta}$;

(5b) $E_{\alpha_i\alpha_k} \cap E_{\alpha_k\delta} \cap D \subseteq E_{\alpha_i\delta}$;

(5c) $E_{\alpha_i\delta} \cap E_{\alpha_k\delta} \cap D \subseteq E_{\alpha_i\alpha_k}$;

(6a) $F_{\alpha_i\alpha_k} \cap F_{\alpha_i\delta} \cap D \subseteq E_{\alpha_k\delta}$;

(6b) $F_{\alpha_i\alpha_k} \cap F_{\alpha_k\delta} \cap D \subseteq E_{\alpha_i\delta}$;

(6c) $F_{\alpha_i\delta} \cap F_{\alpha_k\delta} \cap D \subseteq E_{\alpha_i\alpha_k}$,

where $D = D^{\alpha_i} \cap D^{\alpha_k} \cap D^{\delta}$. Obviously, $W_i \subseteq E''_{\alpha_k\delta} = E_{\alpha_k\delta}$, hence (5a) and (6a) hold. Further, $V_i \subseteq F''_{\alpha_k\delta}$. Since $V_i \cap W = \emptyset$ and $V_i \cap E'_{\alpha_k\delta} = \emptyset$, we have $V_i \cap E''_{\alpha_k\delta} = \emptyset$, which implies $V_i \cap Q = \emptyset$, hence $V_i \subseteq F''_{\alpha_k\delta} \setminus Q \subseteq F_{\alpha_k\delta}$. Thus,

(7a) $E_{\alpha_i\alpha_k} \cap F_{\alpha_i\delta} \cap D \subseteq F_{\alpha_k\delta}$;

(7b) $F_{\alpha_i\alpha_k} \cap E_{\alpha_i\delta} \cap D \subseteq F_{\alpha_k\delta}$.

Because of (3) we have $E_{\alpha_k\delta} \cap F_{\alpha_k\delta} \cap D = \emptyset$, so (7a) implies $E_{\alpha_i\alpha_k} \cap F_{\alpha_i\delta} \cap E_{\alpha_k\delta} \cap D = \emptyset$. By (4), $F_{\alpha_i\delta} \cup E_{\alpha_i\delta} = Z$, hence $E_{\alpha_i\alpha_k} \cap E_{\alpha_k\delta} \cap D \subseteq E_{\alpha_i\delta}$, which is (5b). Similarly, (7b) implies $F_{\alpha_i\alpha_k} \cap E_{\alpha_i\delta} \cap D \cap E_{\alpha_k\delta} = \emptyset$, hence $E_{\alpha_i\delta} \cap E_{\alpha_k\delta} \cap D \subseteq E_{\alpha_i\alpha_k}$, which is (5c). By the same argument, (6a) implies that $F_{\alpha_i\alpha_k} \cap F_{\alpha_i\delta} \cap F_{\alpha_k\delta} \cap D = \emptyset$, which yields (6b) and (6c). The proof is complete. ∎

Thus, we have families $\{E_{\alpha\beta} \mid \alpha, \beta \in I,\ \alpha \neq \beta\}$, $\{F_{\alpha\beta} \mid \alpha, \beta \in I,\ \alpha \neq \beta\}$ with (1)-(6). For every $z \in Z$ let $D(z) = \{\alpha \in I \mid z \in D^{\alpha}\}$. For $\alpha, \beta \in D(z)$ we set $\alpha \sim_z \beta$ if $z \in E_{\alpha\beta}$ or $\alpha = \beta$. Now (5) ensures that $\sim_z$ is an equivalence relation and (6) means that this equivalence has at most two equivalence classes. Now we shall extend the index set I so that (1)-(6) remain valid and for every $z \in Z$ the equivalence $\sim_z$ has either 0 or 2 equivalence classes. (Of course, 0 equivalence classes means that $D(z) = \emptyset$.)

Suppose that $z \in Z$ is such that $\sim_z$ has exactly 1 equivalence class. Then $z \in D^{\delta}$ for some $\delta \in I$. Necessarily, $z \in Z_n$ and by 2.8 there is a clopen set $C \subseteq D^{\delta}$, $z \in C$. Let $I' = I \cup \{z\}$. We set $C_0^z = Z$, $C_1^z = C$, so that $D^z = C$. Further we set $E_{\delta z} = C_0^{\delta} \setminus C$, $F_{\delta z} = C_1^{\delta}$ and for every $\alpha \in I \setminus \{\delta\}$
$E_{\alpha z} = (C \cap F_{\alpha\delta}) \cup (C_0^{\alpha} \setminus (C \cap D^{\alpha}))$,
$F_{\alpha z} = (C \cap E_{\alpha\delta}) \cup (C_1^{\alpha} \setminus (C \cap D^{\alpha}))$.

Since C is clopen, all the defined sets are compact open. (See 3.4.) We check that (1)-(6) remain valid.

Let $\alpha \in I$, $\alpha \neq \delta$. Since $C_0^z \supseteq Z_0$ and $C_1^z \cap Z_0 = C \cap Z_0 = \emptyset$, we can compute:
$((C_0^\alpha \cap C_0^z) \cup (C_1^\alpha \cap C_1^z)) \cap Z_0 = C_0^\alpha \cap Z_0 = E_{\alpha z} \cap Z_0$,
$((C_0^\alpha \cap C_1^z) \cup (C_1^\alpha \cap C_0^z)) \cap Z_0 = C_1^\alpha \cap Z_0 = F_{\alpha z} \cap Z_0$.
The same is true if we write δ instead of α. Hence, (1) and (2) hold. Further, $E_{\delta z} \cap F_{\delta z} \cap D^\delta \cap D^z \subseteq (C_0^\delta \setminus C) \cap C = \emptyset$. Since $E_{\alpha z} \cap D^\alpha \cap D^z = E_{\alpha z} \cap D^\alpha \cap C = F_{\alpha\delta} \cap D^\alpha \cap D^z$ and $F_{\alpha z} \cap D^\alpha \cap D^z = E_{\alpha\delta} \cap D^\alpha \cap D^z$, we obtain that $E_{\alpha z} \cap F_{\alpha z} \cap D^\alpha \cap D^z \subseteq F_{\alpha\delta} \cap E_{\alpha\delta} \cap D^\alpha \cap D^\delta = \emptyset$. (Obviously, $D^z = C \subseteq D^\delta$.) This shows (3).

Further, $E_{\delta z} \cup F_{\delta z} = (C_0^\delta \setminus C) \cup C_1^\delta = C_0^\delta \cup C_1^\delta = Z$ and $E_{\alpha z} \cup F_{\alpha z} = (C \cap (E_{\alpha\delta} \cup F_{\alpha\delta})) \cup (C_0^\alpha \setminus (C \cap D^\alpha)) \cup (C_1^\alpha \setminus (C \cap D^\alpha)) \supseteq C \cup (C_0^\alpha \setminus C) \cup (C_1^\alpha \setminus C) = C_0^\alpha \cup C_1^\alpha = Z$. Thus, (4) holds.

To show (5) and (6) let $\alpha, \beta \in I$, $\alpha \neq \beta \neq \delta \neq \alpha$. Let us denote $D = D^\alpha \cap D^\beta \cap D^z$, $D' = D^\alpha \cap D^\delta \cap D^z$. Notice that $D \subseteq D^\delta$, $E_{\delta z} \cap D' = \emptyset$, $F_{\delta z} \cap D' = D'$. We have:
$E_{\alpha\beta} \cap E_{\beta z} \cap D = E_{\alpha\beta} \cap F_{\beta\delta} \cap D \cap D^\delta \subseteq F_{\alpha\delta} \cap D^\alpha \cap D^z \subseteq E_{\alpha z}$,
$E_{\alpha z} \cap E_{\beta z} \cap D = F_{\alpha\delta} \cap F_{\beta\delta} \cap D \subseteq E_{\alpha\beta}$,
$E_{\alpha\delta} \cap E_{\delta z} \cap D' = E_{\alpha z} \cap E_{\delta z} \cap D' = \emptyset$,
$E_{\alpha\delta} \cap E_{\alpha z} \cap D' = E_{\alpha\delta} \cap F_{\alpha\delta} \cap D' = \emptyset$,
which shows (5). Similarly,
$F_{\alpha\beta} \cap F_{\beta z} \cap D = F_{\alpha\beta} \cap E_{\beta\delta} \cap D \subseteq F_{\alpha\delta} \cap D^\alpha \cap D^z \subseteq E_{\alpha z}$,
$F_{\alpha z} \cap F_{\beta z} \cap D = E_{\alpha\delta} \cap E_{\beta\delta} \cap D \subseteq E_{\alpha\beta}$,
$F_{\alpha\delta} \cap F_{\delta z} \cap D' = F_{\alpha\delta} \cap D' \subseteq E_{\alpha z}$,
$F_{\alpha\delta} \cap F_{\alpha z} \cap D' \subseteq F_{\alpha\delta} \cap E_{\alpha\delta} \cap D' = \emptyset$,
$F_{\alpha z} \cap F_{\delta z} \cap D' \subseteq F_{\alpha z} \cap D^\alpha \cap D^z \subseteq E_{\alpha\delta}$.
This completes the proof of (6).

Since (6) remained valid, $\sim_y$ has at most 2 equivalence classes for every $y \in Z$. Moreover, $\sim_z$ has now 2 equivalence classes, since $z \in C = F_{\delta z}$. Repeating this procedure we obtain families $\{E_{\alpha\beta} \mid \alpha, \beta \in I'',\ \alpha \neq \beta\}$, $\{F_{\alpha\beta} \mid \alpha, \beta \in I'',\ \alpha \neq \beta\}$ satisfying (1)-(6) such that $\sim_z$ has either 0 or 2 equivalence classes for every $z \in Z$. In the sequel we write I instead of I''. (The assumption $|I| \leq \aleph_1$ is no longer essential.)

Lemma 3.7 *There exist families of compact open sets $\{E_{\alpha\beta} \mid \alpha, \beta \in J,\ \alpha \neq \beta\}$, $\{F_{\alpha\beta} \mid \alpha, \beta \in J,\ \alpha \neq \beta\}$ such that (1)-(5) are valid and, moreover,*

(*) *for every $z \in Z$, $\sim_z$ has either 0 or 3 equivalence classes;*

(**) *for every* $z_1, z_2 \in Z$, $z_1 \neq z_2$, *there is* $\alpha \in J$ *such that* $z_1 \in C_0^\alpha \setminus C_1^\alpha$, $z_2 \in C_1^\alpha \setminus C_0^\alpha$.

P r o o f. Let $\mathcal{G}_1$ be the set of all clopen sets in Z that are subsets of $D = \bigcup_{\alpha \in I} D^\alpha \subseteq Z_n$. Let $\mathcal{G}_2$ be the family of all clopen subsets of Z. Let J be the disjoint union of the sets I, $\mathcal{G}_1$ and $\mathcal{G}_2$. We need to define the sets C_0^α, C_1^α, $E_{\alpha\beta}$, $F_{\alpha\beta}$ if $\alpha \in \mathcal{G}_1 \cup \mathcal{G}_2$ or $\beta \in \mathcal{G}_1 \cup \mathcal{G}_2$.

For $\alpha \in \mathcal{G}_1$ we set $C_0^\alpha = Z$, $C_1^\alpha = \alpha$. For $\alpha \in \mathcal{G}_1$, $\beta \in I$ we define $E_{\alpha\beta} = C_0^\beta \setminus (\alpha \cap D^\beta)$, $F_{\alpha\beta} = \alpha \cup C_1^\beta$. If $\alpha, \beta \in \mathcal{G}_1$, we set $E_{\alpha\beta} = Z$, $F_{\alpha\beta} = \emptyset$.

For $\alpha \in \mathcal{G}_2$ we set $C_0^\alpha = Z \setminus \alpha$, $C_1^\alpha = \alpha$. If $\alpha \in \mathcal{G}_2$, $\beta \in J \setminus \{\alpha\}$, we set $E_{\alpha\beta} = (C_0^\alpha \cap C_0^\beta) \cup (C_1^\alpha \cap C_1^\beta)$, $F_{\alpha\beta} = (C_0^\alpha \cap C_1^\beta) \cup (C_1^\alpha \cap C_0^\beta)$.

Since every $\alpha \in \mathcal{G}_1 \cup \mathcal{G}_2$ is a clopen set, all the defined sets are compact open. (Use 3.4.) In all cases, (1), (2) and (4) are easy to see.

(3) is trivial if $\alpha \in \mathcal{G}_2$ or $\beta \in \mathcal{G}_2$, because for $\alpha \in \mathcal{G}_2$ we have $D^\alpha = \emptyset$. If both α and β belong to $\mathcal{G}_1$ then (3) follows from $F_{\alpha\beta} = \emptyset$. Finally, let $\alpha \in \mathcal{G}_1$, $\beta \in I$. Then $E_{\alpha\beta} \cap D^\alpha \cap D^\beta = E_{\alpha\beta} \cap \alpha \cap D^\beta = \emptyset$. Hence, (3) holds.

If some of α, β, γ belong to $\mathcal{G}_2$, then $D^\alpha \cap D^\beta \cap D^\gamma = \emptyset$ and (5) is trivial. If $\alpha \in \mathcal{G}_1$, $\beta \in I$, then $E_{\alpha\beta} \cap D^\alpha \cap D^\beta = \emptyset$, which implies (5). The other cases when $\{\alpha, \beta, \gamma\} \cap \mathcal{G}_1 \neq \emptyset$ and $\{\alpha, \beta, \gamma\} \cap I \neq \emptyset$ are similar. The remaining case is $\alpha, \beta, \gamma \in \mathcal{G}_1$. Then $E_{\alpha\gamma} = Z$ and (5) is trivial.

Now we show (*). If $z \notin D$ then $z \notin D^\alpha$ for every $\alpha \in \mathcal{G}_1 \cup \mathcal{G}_2$ and $\sim_z$ has 0 equivalence classes. Suppose that $z \in D^\alpha$ for some $\alpha \in I$. Then $\sim_z$ restricted to I has two equivalence classes, hence $z \in D^\alpha \cap D^\beta$ and $z \in F_{\alpha\beta}$ for some $\beta \in I$. By 2.8, there is $\gamma \in \mathcal{G}_1$ with $z \in \gamma \subseteq D$. Then $z \in D^\gamma = \gamma \subseteq F_{\alpha\gamma} \cap F_{\beta\gamma}$. Thus, α, β and γ belong to different $\sim_z$-equivalence classes.

Now, let $\delta \in J \setminus \{\alpha, \beta, \gamma\}$, $z \in D^\delta$. We claim that either $z \in E_{\alpha\delta}$ or $z \in E_{\beta\delta}$ or $z \in E_{\gamma\delta}$. This is clear from (6) if $\delta \in I$, since $\alpha, \beta \in I$. If $\delta \in \mathcal{G}_1$, then obviously $z \in E_{\gamma\delta}$. Finally, the case $\delta \in \mathcal{G}_2$ is excluded because $D^\delta = \emptyset$ whenever $\delta \in \mathcal{G}_2$.Thus $\sim_z$ has exactly 3 equivalence classes.

It remains to show (**). Let $z_1 \neq z_2$. If $z_1, z_2 \in Z_0$ then $z_1 \in B^\alpha$, $z_2 \notin B^\alpha$ for some $B^\alpha \in \mathcal{B}$ and therefore $z_1 \in C_0^\alpha$, $z_2 \in C_1^\alpha$ for some $\alpha \in I$. Since $C_0^\alpha \cap C_1^\alpha \cap Z_0 = \emptyset$, we have $z_1 \in C_0^\alpha \setminus C_1^\alpha$, $z_2 \in C_1^\alpha \setminus C_0^\alpha$. If $z_1 \notin Z_0$ then there is a clopen set $C \subseteq Z_n$ with $z_1 \in C$, $z_2 \notin C$. Clearly, $\delta = Z \setminus C \in \mathcal{G}_2$, $z_1 \in C_0^\delta \setminus C_1^\delta$, $z_2 \in C_1^\delta \setminus C_0^\delta$. ∎

Lemma 3.8 *For every* $\alpha, \beta \in J$, *the sets* $C_0^\alpha \cap C_0^\beta \setminus (F_{\alpha\beta} \cap D^\alpha \cap D^\beta)$, $C_1^\alpha \cap C_1^\beta \setminus (F_{\alpha\beta} \cap D^\alpha \cap D^\beta)$, $C_0^\alpha \cap C_1^\beta \setminus (E_{\alpha\beta} \cap D^\alpha \cap D^\beta)$ *are open.*

P r o o f. We prove the statement for the set $Y = C_0^\alpha \cap C_0^\beta \setminus (F_{\alpha\beta} \cap D^\alpha \cap D^\beta)$. By 3.2, the set $Y \cap Z_n$ is open. By (1) and (3), $Y \cap Z_0 \subseteq E_{\alpha\beta}$, hence $Y \cap Z_0 \subseteq Y \cap E_{\alpha\beta} \subseteq Y$ and hence $Y = (Y \cap Z_n) \cup (Y \cap E_{\alpha\beta})$, which is an open set because $Y \cap E_{\alpha\beta} = C_0^\alpha \cap C_0^\beta \cap E_{\alpha\beta}$ is open. ∎

Now we are ready to define an embedding $\varphi: \; Z \to \overline{L_3(J)}$. For every $z \in \bigcup_{i\in J} D^\alpha$ let $A_z,\ B_z,\ C_z$ denote the three equivalence classes of $\sim_z$. For every $z \in Z$ we set $\varphi(z) = \overline{f}$, where $f: \; J \to \{0,1,a_1,a_2,a_3\}$ is defined as follows:

$$f(\alpha) = \begin{cases} 0 & \text{if} \quad z \in C_0^\alpha \setminus C_1^\alpha, \\ 1 & \text{if} \quad z \in C_1^\alpha \setminus C_0^\alpha, \\ a_1 & \text{if} \quad z \in A_z, \\ a_2 & \text{if} \quad z \in B_z, \\ a_3 & \text{if} \quad z \in C_z. \end{cases}$$

Since $A_z, B_z, C_z \neq \emptyset$ or $A_z = B_z = C_z = \emptyset$, $\overline{f}$ is indeed an element of $\overline{L_3(J)}$. In the definition of $\overline{f}$ it does not matter which $\sim_z$-equivalence class is denoted A_z, B_z or C_z respectively.

Lemma 3.9 *φ is injective.*

P r o o f. Let $z_1 \neq z_2$, $\varphi(z_1) = \overline{f}$, $\varphi(z_2) = \overline{g}$. By (**) there is $\delta \in J$ such that $f(\delta) = 0$, $g(\delta) = 1$, hence $f \not\sim g$. ∎

Lemma 3.10 *φ is continuous.*

P r o o f. By 2.4 we have to show that $\varphi^{-1}(\overline{G_r})$ is open for every $r: \{\alpha,\beta\} \to \{0,1\}$, $\alpha,\beta \in J$. If $\alpha = \beta$ then $\varphi^{-1}(\overline{G_r})$ is equal to C_0^α or C_1^α. Let $\alpha \neq \beta$. First let $r(\alpha) = r(\beta) = 0$. By the definition of $\overline{G_r}$, (3) and (4) we have $\varphi^{-1}(\overline{G_r}) = ((C_0^\alpha \setminus C_1^\alpha) \cap (C_0^\beta \setminus C_1^\beta)) \cup (C_0^\alpha \setminus C_1^\alpha) \cap D^\beta) \cup (D^\alpha \cap (C_0^\beta \setminus C_1^\beta)) \cup (D^\alpha \cap D^\beta \cap E_{\alpha\beta}) = ((C_0^\alpha \cap C_0^\beta) \setminus (D^\alpha \cap D^\beta)) \cup (D^\alpha \cap D^\beta \cap E_{\alpha\beta}) = (C_0^\alpha \cap C_0^\beta) \setminus (D^\alpha \cap D^\beta \cap F_{\alpha\beta})$, which is open by 3.8.

If $r(\alpha) = 0$, $r(\beta) = 1$, the calculation is the same, just interchange C_0^β with C_1^β and $E_{\alpha\beta}$ with $F_{\alpha\beta}$. The remaining possibilities for r are similar. ∎

Lemma 3.11 *Let P, Q be open compact subsets of Z such that $P \cap Z_0 = Q \cap Z_0$. Then $P = (Q \setminus B_1) \cup B_2$ for some clopen sets $B_1, B_2 \subseteq Z$.*

P r o o f. By 3.2, the sets $B_1 = Q \setminus P$ and $B_2 = P \setminus Q$ are clopen in Z_n and hence open in Z. Obviously, they are compact. By 3.3, they are closed. ∎

Lemma 3.12 *φ is an open mapping.*

P r o o f. For every $\alpha \in J$ we have $\varphi(C_0^\alpha) = \{\overline{f} \in \overline{L_3(J)} \mid f(\alpha) \neq 1\} \cap \varphi(Z)$, $\varphi(C_1^\alpha) = \{\overline{f} \in \overline{L_3(J)} \mid f(\alpha) \neq 0\} \cap \varphi(Z)$, which are open subsets of $\varphi(Z)$. Further, if C is clopen then $C = \alpha$ for some $\alpha \in \mathcal{G}_2$ and $\varphi(C) = \varphi(C_1^\alpha \setminus C_0^\alpha) = \{\overline{f} \in \varphi(Z) \mid f(\alpha) = 1\}$ and $\varphi(Z \setminus C) = \varphi(C_0^\alpha \setminus C_1^\alpha) = \{\overline{f} \in \varphi(Z) \mid f(\alpha) = 0\}$, which are closed subsets of $\varphi(Z)$. Hence, $\varphi(C)$ is clopen in $\varphi(Z)$.

Now, let B be an arbitrary open compact subset of Z. Then $B \cap Z_0 \in \mathcal{B}$ and hence $B \cap Z_0 = C_0^\alpha \cap Z_0$ for some $\alpha \in J$. By 3.11, $B = (C_0^\alpha \setminus B_1) \cup B_2$ for some clopen sets B_1, B_2. Since φ is injective, we have $\varphi(B) = (\varphi(C_0^\alpha) \setminus \varphi(B_1)) \cup \varphi(B_2)$, which is an open set.

Thus, $\varphi(B)$ is open for every open compact set $B \subseteq Z$. Since these sets form a base of the topology of Z, $\varphi(M)$ is open for every open set $M \subseteq Z$. ∎

Lemma 3.13 *$\varphi(Z)$ is a closed subset of $\overline{L_3(J)}$.*

P r o o f. Since Z is compact, $\varphi(Z)$ is compact too. For contradiction, suppose that $\overline{f} \in \mathrm{cl}(\varphi(Z)) \setminus \varphi(Z)$. By 1.1 there is $\overline{g} = \varphi(z) \in \varphi(Z)$ and a net $N = \{\varphi(x_\alpha) \mid \alpha \in \Lambda\} \subseteq \varphi(Z)$ such that $\overline{f}$ and $\overline{g}$ are the limit points of N. This net has no other cluster points. Necessarily, $f, g : J \to \{0, 1\}$, $f \neq g$. There exists $\gamma \in J$ such that $f(\gamma) \neq g(\gamma)$. Let $\varphi(x_\alpha) = \overline{h_\alpha}$. The sets $A_0 = \{\overline{h} \in \overline{L_3(J)} \mid h(\gamma) \neq 0\}$, $A_1 = \{\overline{h} \in \overline{L_3(J)} \mid h(\gamma) \neq 1\}$ are open and contain the points $\overline{f}$ and $\overline{g}$ respectively. Hence, there is $\beta \in \Lambda$ such that $\overline{h_\alpha} \in A_0 \cap A_1$ for every $\alpha \geq \beta$. Consider the net $M = \{x_\alpha \mid \alpha \in \Lambda,\ \alpha \geq \beta\}$. Since Z is compact, M must have some cluster points. If x is a cluster point of M, then the continuity of φ implies that $\varphi(x)$ is a cluster point of N. It follows that z is the only cluster point of M. Further, $\overline{h_\alpha} \in A_0 \cap A_1$ implies $x_\alpha \in D^\gamma = C_0^\gamma \cap C_1^\gamma$. Since the sets C_0^γ, C_1^γ are compact, we obtain that $z \in D^\gamma$, which means that $g(\gamma) \in \{a_1, a_2, a_3\}$, a contradiction. ∎

Thus, we have completed the proof of our main result.

Theorem 3.14 *Let Z be a topological space satisfying (i)–(v) of 2.7. If Z_0 contains at most $\aleph_1$ clopen sets, then Z is homeomorphic to a closed subspace of $\overline{L_3(X)}$ for some set X.*

Together with 2.7 we have the following consequence.

Theorem 3.15 *If $|X| \leq \aleph_1$ and $n \geq 3$ then $\overline{L_n(X)}$ is homeomorphic to a closed subspace of $\overline{L_3(X)}$.*

On the other hand, in [6] we proved that for $|X| > \aleph_1$ the space $\overline{L_{n+1}(X)}$ is not embeddable in $\overline{L_n(X)}$. To get an abstract characterization of closed subspaces of $\overline{L_3(X)}$ in general, we should state the existence of the sets $E_{\alpha\beta}$, $F_{\alpha\beta}$ with appropriate properties as an additional condition.

4 Congruence lattices of lattices

Let L be a distributive algebraic lattice. An element $x \in L$ is called strictly meet irreducible if $x = \inf X$ implies $x \in X$ for every subset X of L. Let $M(L)$ denote the set of all strictly meet irreducible elements. Endow the set $M(L)$ with the topology whose closed sets are $C_x = \{y \in M(L) \mid y \geq x\}$ for all $x \in L$. We have the following theorem. (See [6], theorems 2.4, 3.2 and 5.1.)

Theorem 4.1 *For a distributive algebraic lattice L, the following conditions are equivalent:*

1. *L is isomorphic to $\mathrm{Con}(A)$ for some $A \in \mathcal{M}_n^{01}$;*
2. *$M(L)$ is homeomorphic to a closed subspace of $\overline{L_n(X)}$ for some set X;*
3. *L is isomorphic to $\mathcal{O}(Z)$ for some set X and a closed subspace Z of $\overline{L_n(X)}$.*

Remark. Instead of $\mathcal{M}_n^{01}$, the paper [6] deals with the varieties $\mathcal{M}_n$ of (possibly unbounded) lattices. Consequently, it investigates the spaces $\overline{H(F_n(X))}$ which are obtained from $\overline{L_n(X)}$ by deleting the points $\overline{o}$, $\overline{i}$, which correspond to the two constant mapping $X \to \{0, 1\}$. That is why we used the results of [6] only in the places where the difference between $\overline{H(F_n(X))}$ and $\overline{L_n(X)}$ is inessential.

The spaces $\overline{H(F_n(X))}$ are not compact. We conjecture that theorems analogous to 3.14 and 3.15 are valid for these spaces if the compactness is replaced by local compactness at appropriate places. However, an attempt to do so leads to a quite nontrivial complications.

The above theorem together with 3.14 and 3.15 has the following consequences.

Theorem 4.2 *Let L be a distributive algebraic lattice containing at most $\aleph_1$ compact elements. Let $n \geq 3$. The following condition are equivalent:*

1. *L is isomorphic to* $\mathrm{Con}(A)$ *for some $A \in \mathcal{M}_n^{01}$;*
2. *$M(L)$ satisfies (i)–(v) of 2.7;*
3. *L is homeomorphic to $\mathcal{O}(Z)$ for some topological space Z satisfying (i)–(v) of 2.7.*

Theorem 4.3 *If $A \in \mathcal{M}_n^{01}$, $n \geq 3$ and $|A| \leq \aleph_1$ then there exists $B \in \mathcal{M}_3^{01}$ such that* $\mathrm{Con}(A)$ *is isomorphic to* $\mathrm{Con}(B)$*.*

References

[1] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott, *A Compendium of Continuous Lattices,* Springer Verlag 1980.

[2] G. Grätzer, *General Lattice Theory (2nd edition),* Birkhäuser Verlag 1998.

[3] P. T. Johnstone, *Stone spaces,* Cambridge Univ. Press 1982.

[4] J. L. Kelley, *General Topology,* Van Nostrand 1955.

[5] R. McKenzie, R. McNulty, W. Taylor, *Algebras, lattices, varieties I,* Wadsworth &Brooks/ Cole 1987.

[6] M. Ploščica, *Separation properties in congruence lattices of lattices.* Colloquium Mathematicum *83*(2000), 71–84.

Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 04001 Košice, Slovakia