Balanced d-lattices are complemented*

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According to Chajda and Eigenthaler ([1]), a d-lattice is a bounded lattice L satisfying for all $a, c \in L$ the implications

(i) $(a, 1) \in \theta(0, c) \ \rightarrow \ a \vee c = 1;$

(ii) $(a, 0) \in \theta(1, c) \ \rightarrow \ a \wedge c = 0;$

where $\theta(x, y)$ denotes the least congruence on L containing the pair (x, y). Every bounded distributive lattice is a d-lattice. The 5-element nonmodular lattice N_5 is a d-lattice.

Theorem 1 *A bounded lattice is a d-lattice if and only if all maximal ideals and maximal filters are prime.*

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P r o o f. Let I be a maximal ideal in a d-lattice L. Let $x, y \in L \setminus I$. We need to show that $x \wedge y \in L \setminus I$. Since I is maximal, there are $c_1, c_2 \in I$ such that $c_1 \vee x = c_2 \vee y = 1$. For $c = c_1 \vee c_2 \in I$ we have $c \vee x = c \vee y = 1$. Then $(x, 1) = (0 \vee x, c \vee x) \in \theta(0, c)$ and similarly $(y, 1) \in \theta(0, c)$, hence $(x \wedge y, 1) \in \theta(0, c)$. By (i) we have $(x \wedge y) \vee c = 1$, hence $x \wedge y \notin I$. The primality of maximal filters can be proved similarly.

Conversely, assume that all maximal ideals and filters in L are prime. To show (i), assume that $a, c \in L$, $a \vee c \neq 1$. By the Zorn lemma, there exists a maximal ideal I containing $a \vee c$. By our assumption, I is prime. Then $\alpha = I^2 \cup (L \setminus I)^2$ is a congruence on L. Since $c \in I$, we have $(0, c) \in \alpha$, which implies that $\theta(0, c) \subseteq \alpha$. Since $a \in I$, we have $(a, 1) \notin \alpha$, hence $(a, 1) \notin \theta(0, c)$. This shows (i). The proof of (ii) is similar. ∎

By [1], a bounded lattice is called "balanced", if the 0-class of any congruence determines the 1-class, and conversely. They showed that complemented lattices are balanced, and they asked:

(∗) Is there a d-lattice which is balanced but not complemented?

We use the above characterization of d-lattices to answer this question.

If A is a subset of an algebra, write θ_A for the smallest congruence that identifies all elements of A; if ϕ is a congruence, x an element, write x/ϕ for the ϕ-congruence class of x.

Further, a congruence ϕ (on an algebra with constants 0 and 1) is called balanced if $0/\phi = 0/\theta_{(1/\phi)}$ and $1/\phi = 1/\theta_{(0/\phi)}$; an algebra is called balanced iff all its congruence relations are balanced, or equivalently if: for any congruence relations ϕ, ϕ' we have:

$$0/\phi = 0/\phi' \text{ iff } 1/\phi = 1/\phi'.$$

Fix a d-lattice $(L, \vee, \wedge, 0, 1)$. For $a \in L$ we denote $F_a := \{x : x \vee a = 1\}$, and $I_a := \{x : x \wedge a = 0\}$.

Fact 2 *F_a is a filter, I_a is an ideal.*

P r o o f. Let $x, y \in F_a$. Similarly as in the proof of Theorem 1, $(x, 1) \in \theta(0, a)$, $(y, 1) \in \theta(0, a)$, hence $(x \wedge y, 1) \in \theta(0, a)$, which by the definition of a d-lattice implies $x \wedge y \in F_a$. The proof for I_a is similar. ∎

Fact 3 *If I is an ideal disjoint to F_a, and $a \notin I$, then also the ideal generated by $I \cup \{a\}$ is disjoint to F_a.*

P r o o f. If $x \leq i \vee a$ for some $i \in I$, and $x \in F_a$, then also $i \vee a \in F_a$, hence $i \vee a = (i \vee a) \vee a = 1$. Thus, $i \in F_a$, so $F_a \cap I \neq \emptyset$. ∎

Fact 4 *If $f : L_1 \to L_2$ is a homomorphism from L_1 onto L_2, and L_1 is balanced, then L_2 is balanced.*

P r o o f. In fact, this holds "level-by-level": If ϕ is an unbalanced congruence on L_2, then the preimage of ϕ is unbalanced on L_1. ∎

Theorem 5 *The following are equivalent (for a d-lattice L):*

1. *There is a maximal (hence prime) filter whose complement is not a maximal ideal.*
2. *There is a maximal (hence prime) ideal whose complement is not a maximal filter.*
3. *There are two prime ideals in L, one properly containing the other.*
4. *There are two prime filters in L, one properly containing the other.*
5. *There is a homomorphism from L onto the 3-element lattice $\{0, d, 1\}$.*
6. *L is not balanced.*
7. *L is not complemented.*

In particular a d-lattice is balanced iff it is complemented.

P r o o f.

$$\begin{array}{ccccccccccc} & & (1) & \to & (3) & & & & & & \\ & \nearrow & & & & \searrow & & & & & \\ (7) & & & & & & (5) & \to & (6) & \to & (7) \\ & \searrow & & & & \nearrow & & & & & \\ & & (2) & \to & (4) & & & & & & \end{array}$$

(1) → (3): By 1, the complement of a maximal filter is a (necessarily prime) ideal. If this ideal is not maximal, it can be properly extended to a maximal (hence prime) ideal. The proof of (2) → (4) is similar (dual).

(3) → (5): Let $I_1 \subset I_2 \subset L$ be prime ideals. Map I_1 to 0, $I_2 \setminus I_1$ to d, and $L \setminus I_2$ to 1. Check that this is a lattice homomorphism. The proof of (4) → (5) is dual.

(5) → (6) follows from fact 4, since the three-element lattice is not balanced.

(6) → (7) is from [1].

Now we show (7) → (1). (Again, (7) → (2) is dual.) Assume that L is not complemented, so there is some a such that $F_a \cap I_a = \emptyset$. Let F_1 be the filter generated by $F_a \cup \{a\}$. We have $F_1 \cap I_a = \emptyset$ by the dual of Fact 3, so F_1 is proper. By the Zorn lemma, F_1 can be extended to a maximal filter F. Let $I_1 = L \setminus F$. It is enough to see that I_1 is not maximal. Let I be the ideal generated by $I_1 \cup \{a\}$. By Fact 3, $I \cap F_a = \emptyset$, so I is a proper ideal properly extending I_1. ∎

References

[1] I. Chajda, G. Eigenthaler, *Balanced congruences.* Discussiones Mathematicae (General Algebra and Applications) *21*(2001), 105–114.