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VALUES AND MINIMAL SPECTRUM OF AN ALGEBRAIC LATTICE

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ABSTRACT. Algebraic lattices constitute an appropriate setting for generalizing the results existing in particular structures, as l-groups, MV-algebras, etc.. In this paper we study the very large elements and the very large radical of an algebraic lattice. We also define and characterize compactly generated algebraic lattices. Our results are generalizations of some theorems proved for l-groups in [Bigard, A.—Conrad, P.—Wolfenstein, S.: *Compactly generated lattice-ordered groups*, Math. Z. **107** (1968), 201–211], [Conrad, P.—Martinez, J.: *Very large subgroups of lattice-ordered groups*, Comm. Algebra **18** (1990), 2063–2098], [Conrad, P.—Martinez, J.: *Complemented lattice-ordered groups*, Indag. Math. (N.S.) **1** (1990), 281–298] and for MV-algebras in [Di Nola, A.—Georgescu, G.—Sessa, S.: *Closed ideals of MV-algebras*. In: Advances in Contemporary Logic and Computer Science (W. A. Carnielli, I. M. L. D'Ottaviano eds.). Contemp. Math. 235, Amer. Math. Soc., Providence, RI, 1999, pp. 99–111].

If G is an l-group, then the set $C(G)$ of its convex l-subgroups is an algebraic, distributive lattice ([1]). In fact, $C(G)$ is a relatively normal lattice (IRN, in terms of [8], [9]). A classical problem in the l-group theory is to express the l-group notions and theorems in lattice-theoretical terms. The algebraic, distributive lattices constitute good abstract candidates (see [6], [7]), but some other results in l-groups (for example the finite basis theorem) necessitate to work in relatively normal lattices ([8], [9]). We remark that this kind of abstraction is a general problem in universal algebra: to formulate and to prove some results in an abstract lattice-theoretical context instead of in some particular lattice of congruences (see for example [6]). This generality often brings more light on the content of the theorems and the relations between the structures. The present paper is a contribution to this program. We formulate and prove in the setting of algebraic lattices some results of l-groups ([2], [3], [4]) and of

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MV-algebras ([5]). We introduce the notion of very large element in algebraic lattices; we study the very large radical and very large basis in an algebraic lattice and we characterize compactly generated algebraic lattices.

Let A be an algebraic, distributive lattice with the least element 0 and the greatest element 1 and $\mathrm{Com}(A)$ the join-subsemilattice of compact elements of A. Throughout this paper we assume that $\mathrm{Com}(A)$ is a sublattice of A.

An element $p < 1$ is *meet-irreducible* if $p = x \wedge y$ implies $p = x$ or $p = y$; an element $p < 1$ is *meet-prime* if $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$.

As A is distributive, meet-irreducible and meet-prime elements are the same. These definitions can be extended to arbitrary meets and we obtain the concepts of completely meet-irreducible and completely meet-prime elements, which are no longer equivalent.

The set of all meet-prime elements of A will be denoted by $\mathrm{Spec}\, A$ and the set of all minimal meet-prime elements of A by $\mathrm{Min}\, A$.

Every algebraic lattice contain a lot of (completely) meet-irreducible elements. In fact, every element is the meet of a set of completely meet-irreducible elements. If $c \in \mathrm{Com}(A)$ and $x \in A$, $c \not\leq x$, then there is a maximal element $p \in A$ with $c \not\leq p$, $x \leq p$. Every such maximal element is completely meet-irreducible.

Every element maximal with respect to not exceeding c is called a *value* of c. The set of all values of c will be denoted by $\mathrm{Val}(c)$.

Every completely meet-irreducible element is a value of some compact element. Thus, $\mathrm{Val}(A)$ (the set of all values of A) coincides with the set of all completely meet-irreducible elements.

It is well known that every distributive algebraic lattice is pseudocomplemented. For $a \in A$, let a^* denote the psedocomplement of a.

Lemma 1. ([7; 2.5.1], [9]) *If $p \in \mathrm{Spec}\, A$, then the following are equivalent:*

(1) $p \in \mathrm{Min}\, A$.
(2) *For any $c \in \mathrm{Com}(A)$, $c \leq p$ if and only if $c^* \not\leq p$.*
(3) $p = \bigvee \{c^* :\ c \in \mathrm{Com}(A),\ \ c \not\leq p\}$.

By (2), if $c \leq p$, then $c^* \not\leq p$ and $c^* \wedge c^{**} \leq p$, hence $c^{**} \leq p$. Thus, we have the following assertion.

Remark 2. For any $c \in \mathrm{Com}(A)$ and $p \in \mathrm{Min}\, A$, $c \leq p$ if and only if $c^{**} \leq p$.

Lemma 3. ([7; 2.5])

(i) *If* $p \in \operatorname{Min} A$, *then* $N(p) = \{c \in \operatorname{Com}(A) :\ c \not\leq p\}$ *is an ultrafilter of the lattice* $\operatorname{Com}(A)$.

(ii) *If* M *is an ultrafilter of* $\operatorname{Com}(A)$, *then* $p_M = \bigvee\{c^* :\ c \in M\}$ *is an element of* $\operatorname{Min} A$.

(iii) *The functions* $p \mapsto N(p)$ *and* $M \mapsto p_M$ *establish a bijective correspondence between* $\operatorname{Min} A$ *and the set of ultrafilters of* $\operatorname{Com}(A)$.

Lemma 4. *For any* $a \in A$, $a^* = \bigwedge\{p \in \operatorname{Min} A :\ a \not\leq p\}$. *In particular,* $0 = 1^* = \bigwedge \operatorname{Min} A$.

Proof. If $p \in \operatorname{Min} A$, $a \not\leq p$, then $0 = a \wedge a* \leq p$, hence $a^* \leq p$. Thus, $a^* \leq \bigwedge\{p \in \operatorname{Min} A :\ a \not\leq p\} = d$. For contradiction, suppose that $a^* < d$. Then $a \wedge d > 0$, so $0 < c \leq a \wedge d$ for some $c \in \operatorname{Com}(A)$. There is an ultrafilter M of $\operatorname{Com}(A)$ with $c \in M$. By Lemma 3, $M = N(p_M)$, hence $c \not\leq p_M$, which implies $a \not\leq p_M$ and $d \not\leq p_M$. However, $a \not\leq p_M \in \operatorname{Min} A$ implies $d \leq p_M$, a contradiction. ☐

For $a \in A$ denote

$$D(a) = \{p \in \operatorname{Min} A :\ a \not\leq p\}, \quad V(a) = \{p \in \operatorname{Min} A :\ a \leq p\}.$$

It is easy to see that $D\Big(\bigvee\limits_{i\in I} a_i\Big) = \bigcup\limits_{i\in I} D(a_i)$ and $D(a_1 \wedge a_2) = D(a_1) \cap D(a_2)$ for any $a_i \in A$. Thus, $\operatorname{Min} A$ has a canonical structure of topological space whose open sets are the sets $D(a)$, $a \in A$. The next assertion follows from Lemma 1 and Lemma 4. It implies that $\operatorname{Min} A$ is a zero-dimensional Hausdorff space.

Lemma 5. *Let* $a \in A$ *and* $c \in \operatorname{Com}(A)$. *Then*

(i) $D(c) = D(c^{**}) = V(c^*)$,

(ii) $\operatorname{cl} D(a) = V(a^*)$,

(iii) $\operatorname{int} V(a) = D(a^*)$.

An element $a \in A$ is *dense* if $a^* = 0$; it is *very large* if $D(a) = \operatorname{Min} A$. This notion extends the concept of very large convex l-subgroup of an l-group ([3]).

Obviously, every non-minimal meet-prime element is very large.

Lemma 6. *Any very large element* a *of* A *is dense.*

Proof. Let $a \in A$ be not dense. Then $a^* \neq 0$. Since $\bigwedge \operatorname{Min} A = 0$, there exists $n \in \operatorname{Min} A$ such that $a^* \not\leq n$. Since $a \wedge a^* = 0 \leq n$ and n is meet-prime, we have $a \leq n$, which shows that a is not very large. ☐

Since $D(a \wedge b) = D(a) \cap D(b)$, the set $\mathcal{V}(A)$ of very large elements in A is a lattice filter of A. Let us denote $r(A) = \bigwedge \mathcal{V}(A)$. This extends the notion of very large radical of an l-group ([3]).

For any $a \in A$, $D(a)$ is a clopen set if and only if $a \vee a^*$ is very large. Indeed $D(a)$ is closed if and only if $D(a) = V(a^*)$ if and only if $a \vee a^* \not\leq p$ for each $p \in \operatorname{Min} A$. Thus, Lemma 5 (i) has the following consequence.

Lemma 7. *For every $c \in \operatorname{Com}(A)$, the element $c \vee c^*$ is very large.*

Proposition 8. $r(A) = \bigwedge\{c \vee c^* : \ c \in \operatorname{Com}(A)\}$.

Proof. If a is very large, then $a = \bigwedge P$ for some $P \subseteq \operatorname{Spec} A \setminus \operatorname{Min} A$. By (2) of Lemma 1, for every $p \in P$ there is $c \in \operatorname{Com}(A)$ with $c \vee c^* \leq p$. Hence, $r(A) \geq \bigwedge\{c \vee c^* : \ c \in \operatorname{Com}(A)\}$. The converse follows from Lemma 7. $\square$

Lemma 9. *If $r(A) \geq x \in \operatorname{Com}(A)$, then $\operatorname{Val}(x) \subseteq \operatorname{Min} A$.*

Proof. Let $x \in \operatorname{Com}(A)$, $x \leq r(A)$ and $p \notin \operatorname{Min} A$ for some $p \in \operatorname{Val}(x)$. Thus $p_0 < p$ for some $p_0 \in \operatorname{Spec} A$, so there exists $y \in \operatorname{Com}(A)$ such that $y \leq p$ and $y \not\leq p_0$. By Proposition 8, $x \leq y \vee y^*$, so $x \leq y \vee z$ for some $z \in \operatorname{Com}(A)$ with $z \wedge y = 0$, hence $z \leq p_0 < p$. This yields $x \leq y \vee z \leq p$, contradicting $p \in \operatorname{Val}(x)$. $\square$

Proposition 10. $r(A) = \bigvee\{x \in \operatorname{Com}(A) : \ \operatorname{Val}(x) \subseteq \operatorname{Min} A\}$.

Proof. Let $x \in \operatorname{Com}(A)$ be such that $\operatorname{Val}(x) \subseteq \operatorname{Min} A$. Let a be very large. If $x \not\leq a$, then there exists $p \in \operatorname{Val}(x)$ such that $a \leq p$, which is a contradiction because $p \in \operatorname{Min} A$. Hence, $x \leq a$ and therefore $x \leq r(A)$. Thus, $r(A) \geq \bigvee\{x \in \operatorname{Com}(A) : \ \operatorname{Val}(x) \subseteq \operatorname{Min} A\}$. The inverse inequality follows from Lemma 9. $\square$

Proposition 11. *The following conditions are equivalent:*

(1) *There exists a minimal very large element of A.*
(2) *$r(A)$ is a very large element.*
(3) *For any $p \in \operatorname{Min} A$ there exists $x \in \operatorname{Com}(A)$ such that $p \in \operatorname{Val}(x)$ and $\operatorname{Val}(x) \subseteq \operatorname{Min} A$.*

Proof.

(1) $\implies$ (2): Trivial.

(2) $\implies$ (3):
For any $p \in \operatorname{Min} A$ we have $r(A) \not\leq p$, so there exists $x \in \operatorname{Com}(A)$ such that $x \leq r(A)$ and $x \not\leq p$. One can find an element $p_1 \in \operatorname{Val}(x)$ and $p \leq p_1$. By Lemma 9 we have $\operatorname{Val}(x) \subseteq \operatorname{Min} A$ and $p_1 \in \operatorname{Min} A$, so $p = p_1$ and $p \in \operatorname{Val}(x)$.

(3) $\implies$ (1):
Assume $r(A)$ is not a very large element, so $r(A) \leq p$ for some $p \in \operatorname{Min} A$. By the hypothesis (3), there exists $x \in \operatorname{Com}(A)$ such that $p \in \operatorname{Val}(x)$ and $\operatorname{Val}(x) \subseteq \operatorname{Min} A$. By Proposition 10, $x \leq r(A) \leq p$, contradicting $p \in \operatorname{Val}(x)$. □

An element $x \neq 0$ is called *linear* (or *basic* in the terminology of [7]) if 0 is meet-prime in the lattice $(x] = \{y \in A : \ y \leq x\}$. Denote by $P(A) = \{a^* : a \in A\}$ the set of polars (pseudocomplements) of A.

Lemma 12. ([7; 2.1]) *For any element $a > 0$ the following are equivalent:*

(1) *a is linear.*
(2) *a^{**} is linear.*
(3) *If $0 < x \leq a$, then $x^* = a^*$.*
(4) *$a^* \in \operatorname{Spec} A$.*
(5) *$a^* \in \operatorname{Min} A$.*
(6) *a^* is a maximal polar.*
(7) *a^{**} is a minimal polar.*
(8) *a^{**} is a maximal linear element.*

Recall that a subset B of A is a basis of A if it is a maximal orthogonal (= disjoint) set in A and every element of B is linear. A basis B is very large if $\bigvee\{b^{**} : \ b \in B\}$ is a very large element of A.

Proposition 13. *The following assertions are equivalent:*

(1) *Any dense element of A is a very large element.*
(2) $\operatorname{Min} A \subseteq P(A)$.
(3) *A has a very large basis.*

Proof.
(1) $\implies$ (2):
Let $p \in \operatorname{Min} A$. We have $p^* \wedge p^{**} \leq p$, so $p^{**} = p$ or $p^* \leq p$. If $p^* \leq p$, then $p^* = 0$, which means that p is dense and, by (1), very large. For $p \in \operatorname{Min} A$ this is impossible. Hence, $p = p^{**} \in P(A)$.
(2) $\implies$ (3):
If $p \in \operatorname{Min} A$, then $p = p^{**} \neq 1$ by (2). Lemma 12 implies that $p^* = p^{***}$ is linear. (Notice that $p^* \neq 0$ because $p^{**} \neq 1$.) We claim that $\{p^* : \ p \in \operatorname{Min} A\}$ is a basis. If $p^* \wedge q^* \neq 0$, then by Lemma 12(3), $p = p^{**} = (p^* \wedge q^*)^* = q^{**} = q$. Hence, $p^* \wedge q^* = 0$ for any distinct $p, q \in \operatorname{Min} A$. To show the maximality, let x be a linear element such that $x \wedge p^* = 0$ for every $p \in \operatorname{Min} A$. Then $x \leq p^{**} = p$ for every p, hence $x \leq \bigwedge \operatorname{Min} A = 0$, a contradiction. Thus, $\{p^* : \ p \in \operatorname{Min} A\}$ is a basis. For any $q \in \operatorname{Min} A$ we have $q^{***} = q^* \not\leq q$, hence $\bigvee\{(p^*)^{**} : \ p \in \operatorname{Min} A\} \not\leq q$, so our basis is very large.

$(3) \implies (1)$:
Let $\{c_i : i \in I\}$ be a very large basis. Then, for every $p \in \operatorname{Min} A$, $\bigvee_{i \in I} c_i^{**} \not\leq p$, hence $c_i^{**} \not\leq p$ for some i. Since $0 = c_i^* \wedge c_i^{**} \leq p \in \operatorname{Spec} A$, we have $c_i^* \leq p$. By Lemma 12, the linearity of c_i implies $c_i^* \in \operatorname{Min} A$, hence $c_i^* = p$. Then $p^{**} = c_i^{***} = c_i^* = p \neq 1$, hence $p^* \neq 0$.

Now let $a \in A$ be dense. Then, for every $p \in \operatorname{Min} A$, $a \leq p$ would imply $p^* \leq a^* = 0$, which is impossible. Thus, a is very large. $\square$

A is compactly generated if for any $C \subseteq \operatorname{Com}(A)$, $\bigwedge C = 0$ implies there exists $D \subseteq C$ finite such that $\bigwedge D = 0$.

Proposition 14. *The following are equivalent:*

(1) *A is compactly generated.*
(2) *A is atomic and* $\operatorname{Min} A \subseteq P(A)$.
(3) *For any $m \in \operatorname{Min} A$ there exists an atom $a \not\leq m$.*
(4) *Any ultrafilter of* $\operatorname{Com}(A)$ *is principal.*

Proof.

$(1) \implies (2)$:
Let $x \in A$, $x > 0$. Then $x \geq c > 0$ for some $c \in \operatorname{Com}(A)$. By Zorn axiom, there exists a maximal chain in $\operatorname{Com}(A) \setminus \{0\}$ containing c. Now (1) implies that $a = \bigwedge C \neq 0$. The maximality of C implies that a is an atom of A, $a \leq x$. Thus, A is atomic.

If $m \in \operatorname{Min} A$, then $m = \bigvee\{c^* : c \not\leq m,\ c \in \operatorname{Com}(A)\}$ and $K = N(m) = \{c \in \operatorname{Com}(A) : c \not\leq m\}$ is an ultrafilter of $\operatorname{Com}(A)$. Assume $\bigwedge K = 0$, so there exist $c_1, \ldots, c_n \in K$ such that $c = c_1 \wedge \cdots \wedge c_n = 0$ and $c \in K$. This contradiction yields $\bigwedge K \neq \emptyset$, so there exists an atom $a \leq \bigwedge K$, so $a \leq c$ for any $c \in K$. Then $K = \{x \in \operatorname{Com}(A) : a \leq x\}$ because K is maximal.

Now we shall prove that $m = a^*$. If $c \in \operatorname{Com}(A)$ and $c \wedge a = 0$, then $c \notin K$, so $c \leq m$, hence $a^* \leq m$. Conversely, if $c \in \operatorname{Com}(A)$ and $c \not\leq m$, then $c \in K$, hence $a \leq c$ and $c^* \leq a^*$. thus, $m = \bigvee\{c^* : c \not\leq m,\ c \in \operatorname{Com}(A)\} \leq a^*$.

$(2) \implies (3)$:
Assume $m \in \operatorname{Min} A$, so $m = x^*$ for some $x \in A$. We have $x \neq 0$ (because $x^* = m \neq 1$), so there exists an atom a such that $a \leq x$. If $a \leq m$, then $a \leq x \wedge x^* = 0$. This contradiction shows that $a \not\leq m$.

$(3) \implies (4)$:
If K is an ultrafilter, then $m = \bigvee\{c^* : c \in K\}$ is in $\operatorname{Min} A$. By our hypothesis there exists an atom $a \not\leq m$. Thus $a \in K = \{c \in \operatorname{Com}(A) : c \not\leq m\}$. But K is an ultrafilter and a is an atom, so $K = \{c \in \operatorname{Com}(A) : a \leq c\}$.

(4) $\Longrightarrow$ (1):
Let $C \subseteq \mathrm{Com}(A)$ such that for any finite $D \subseteq C$, $\bigwedge D \neq 0$. Then there exists an ultrafilter K such that $C \subseteq K$. But K is principal, so $K = \{c \in \mathrm{Com}(A) : a \leq c\}$ for some $a \in \mathrm{Com}(A)$, so $0 < a \leq \bigwedge K \leq \bigwedge C$. $\square$

The previous proposition extends results proved for l-groups in [2] and for MV-algebras in [5].

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