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On order-polynomial completeness of lattices

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Abstract. In this note we prove that the cardinality of an infinite order-polynomially complete lattice (if such a lattice exists) must be greater than each κ_n, $n \geq 0$, where κ_n is defined such that $\kappa_0 = \aleph_0$ and $\kappa_n = 2^{\kappa_{n-1}}$ for $n \geq 1$. This strengthens the result in [4].

A lattice $\boldsymbol{L}$ is said to be *order-polynomially complete* (*OPC*) if every order-preserving function on $\boldsymbol{L}$ is a polynomial function of $\boldsymbol{L}$. It is well-known that a finite lattice is OPC if and only if it is tolerance-free (see [5]; see also [6], [8], [1]); the lattices M_n of height 2 having n atoms provide good examples of finite OPC lattices. There is a well-known conjecture that there are no infinite OPC lattices. In [4] it was shown that an OPC lattice must be bounded and that it cannot be infinite with cardinality $\aleph_0$. For other contributions to this topic see [2], [7].

LEMMA 1. *Let X be a set of cardinality κ. Then the power set $P(X)$ contains an antichain of cardinality 2^κ* (*with respect to set inclusion*).

Proof. Let X_1, X_2 be two copies of the set X and let f: $X_1 \to X_2$ be a bijective map. For any subset $S \subseteq X_1$ let S' denote the set $f(X_1 \backslash S) \subseteq X_2$. Obviously, all the sets $S \cup S' \subseteq X_1 \cup X_2$, $S \in P(X_1)$, are mutually incomparable, thus they give an antichain of cardinality 2^κ in the power set $P(X_1 \cup X_2)$. The proof is complete. □

We shall say that two polynomials of a lattice $\boldsymbol{L}$ are of the same *polynomial type* if one of them can be obtained from the other only by replacing the constant symbols, i.e. their written forms differ at most in constant symbols used; e.g. $p(x_1, x_2) = (((x_1 \wedge a) \vee x_2) \wedge b)$ and $q(x_1, x_2) = (((x_1 \wedge c) \vee x_2) \wedge d)$, $a, b, c, d \in \boldsymbol{L}$, are of the same polynomial type. It is clear that, for any lattice $\boldsymbol{L}$, one can create just $\aleph_0$ different polynomial types of polynomials of $\boldsymbol{L}$.

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We order the set $P_n(\boldsymbol{L})$ of all n-ary polynomial functions of a lattice $\boldsymbol{L}$ in a natural way: $p(x_1, \ldots, x_n) \leq q(x_1, \ldots, x_n)$ iff $p(a_1, \ldots, a_n) \leq q(a_1, \ldots, a_n)$ for all $(a_1, \ldots, a_n) \in \boldsymbol{L}^n$.

LEMMA 2. *Let* $\boldsymbol{L}$ *be an OPC lattice. Let* κ *be an infinite cardinal. Suppose that, for some* $k \geq 1$, $\boldsymbol{L}^k$ *contains an antichain of cardinality* κ. *Then there exists n such that* $\boldsymbol{L}^n$ *contains an antichain of cardinality* 2^κ.

Proof. Let $A \subseteq \boldsymbol{L}^k$ be an antichain of cardinality κ. By Lemma 1 there are 2^κ mutually incomparable subsets of A. For every such subset S (of elements of $\boldsymbol{L}^k$) we define an k-ary function $f_S: \boldsymbol{L}^k \to \boldsymbol{L}$ such that for every $\tilde{x} = (x_1, \ldots, x_k) \in \boldsymbol{L}^k$

$$f_S(\tilde{x}) = \begin{cases} 1 & \text{if } \tilde{x} \geq \tilde{s} \text{ for some } \tilde{s} \in S \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that all such functions are order-preserving and mutually incomparable. Since $\boldsymbol{L}$ is an OPC lattice, we have 2^κ mutually incomparable (k-ary) polynomial functions of $\boldsymbol{L}$. Since there are only $\aleph_0$ different polynomial types, there is a polynomial type, say having n ($n \geq 1$) constant symbols, which comprises 2^κ of the constructed polynomial functions. (Recall that 2^κ has an uncountable cofinality.) This yields that the n-tuples of the constants of these mutually incomparable polynomial functions are, as elements of $\boldsymbol{L}^n$, mutually incomparable, too. Thus, $\boldsymbol{L}^n$ contains an antichain of cardinality 2^κ. □

THEOREM. *The cardinality of an infinite OPC lattice* $\boldsymbol{L}$ (*if such a lattice exists*) *must be greater than each* κ_n, $n \geq 0$, *where* κ_n *is defined such that* $\kappa_0 = \aleph_0$ *and* $\kappa_n = 2^{\kappa_{n-1}}$ *for* $n \geq 1$.

Proof. Suppose that $\boldsymbol{L}$ is an infinite OPC lattice. It is clear that $\boldsymbol{L}$ contains an infinite antichain or an infinite chain. In the first case, Lemma 2 implies that, for every $n \geq 1$, some finite power of $\boldsymbol{L}$ contains an antichain of cardinality κ_n. Since all the finite powers of $\boldsymbol{L}$ have the same cardinality as $\boldsymbol{L}$, this cardinality must be greater than each κ_n.

If $\boldsymbol{L}$ contains an infinite chain of cardinality $\aleph_0$, the proof of Theorem 2 in [4] gives a method how to construct $2^{\aleph_0}$ mutually incomparable (unary) functions on $\boldsymbol{L}$. Then some polynomial type, say having k ($k \geq 1$) constant symbols, comprises $2^{\aleph_0}$ mutually incomparable (unary) polynomial functions of $\boldsymbol{L}$. This yields that there is an antichain of cardinality $2^{\aleph_0}$ in $\boldsymbol{L}^k$ and Lemma 2 applies again. □

Our method yields a natural question if there exists a lattice $\boldsymbol{L}$ and an infinite cardinal κ with the following property: *some power* $\boldsymbol{L}^k$ *of* $\boldsymbol{L}$ ($k \geq 3$) *contains an* *antichain of the cardinality κ, but $\boldsymbol{L}^{k-1}$ does not contain such an antichain*. We do not know the answer even in the case when $\boldsymbol{L}$ is a chain and $k = 3$.

PROBLEM. *Is there a chain $\boldsymbol{L}$ such that $\boldsymbol{L}^3$ has an infinite antichain of cardinality κ while all antichains in $\boldsymbol{L}^2$ have cardinality less then κ?*

REFERENCES

[1] DORNINGER, D., *A note on local polynomial functions over lattices*, Algebra univers. *11* (1980), 135–138.

[2] ERNÉ, M. and SCHWEIGERT, D., *Pre-fixed points of polynomial functions of lattices*, Algebra univers. *31* (1994), 298–300.

[3] GRÄTZER, G., *Lattice theory*, W. H. Freeman and Co., San Francisco, Calif., 1971.

[4] KAISER, H. F. and SAUER, N., *On order polynomially complete lattices*, Algebra univers. *30* (1993), 171–176.

[5] KINDERMANN, M., *Über die Äquivalenz von Ordnungspolynomvollständigkeit und Toleranzeinfachheit endlicher Verbände*, Contributions to General Algebra *2* (1979), 145–149.

[6] SCHWEIGERT, D., *Über endliche, ordnungspolynomvollständige Verbände*, Monatsh. f. Math. *78* (1974), 68–76.

[7] RIVAL, I. and ZAGUIA, N., *Images of simple lattice polynomials*, Algebra univers. *33* (1995), 10–14.

[8] WILLE, R., *Eine Charakterisierung endlicher ordnungspolynomvollständiger Verbände*, Arch. d. Math. *28* (1977), 557–560.

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