AFFINE COMPLETENESS OF KLEENE ALGEBRAS II

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Abstract. A characterization of affine complete algebras in the variety of all Kleene algebras was given in [8]. Also local polynomial functions of Kleene algebras and locally affine complete algebras were characterized there. In this paper alternative proofs to these three main results of [8] are presented. Also examples illustrated the results are given.

1. Introduction

A *polynomial function* of an algebra A is a function that can be obtained by composition of the basic operations of A, the projections and the constant functions. *A local polynomial function* of A is a function which can be represented by a polynomial function on any finite subset of its domain. A well-known fact about polynomial and local polynomial functions of any algebra A is that they are *compatible* functions in the following sense: a function $f : A^n \to A$ is compatible if, for any congruence θ of A, $(a_i, b_i) \in \theta$, $i = 1, \dots, n$, implies that

$$(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \in \theta.$$

An algebra in which (local) polynomial functions are the only compatible functions is called *(locally) affine complete.* (The concept 'locally affine complete' has sometimes also another meaning in the literature - see e.g. [11].) The problem of characterizing algebras which are affine complete was originally formulated in [6]. Since every algebra is a reduct of an affine complete algebra (for example, of that which contains all its compatible functions among the basic operations) and hence affine complete algebras are in general very diverse, in [3] the problem was reducted into the following formulation: *characterize affine complete algebras in your favourite variety.* Many varieties for which the problem has already been solved are mentioned in [3] or [9].

In [8] we characterized (locally) affine complete algebras in the variety of all Kleene algebras. Previously, only a finite case was entirely solved: a finite Kleene algebra is affine complete if and only if it is a Boolean algebra (see [7]). Moreover, in [8] we characterized locally polynomial functions of Kleene algebras as those which preserve the congruences and one important binary relation called 'uncertainty order'.

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The aim of this paper is to give alternative proofs to the main three results of the preceding paper [8] which are presented here in Theorems 3.3, 3.4 and 3.9. Close to our considerations are some ideas of the papers [5], [7]-[10] and [12]-[13]. We use several preliminary results of [8] which are summarized in section 2. Our alternative approach to the main results of [8] starts in section 3 with crucial Lemmas 3.1 and 3.2. In addition to [8] we present examples at the end which illustrate the results.

2. Preliminaries

First we recall a few basic facts about Kleene algebras. For more information we refer the reader, for example, to [1] or [2].

A Kleene algebra is an algebra $(K, \vee, \wedge, ', 0, 1)$ where $(K, \vee, \wedge, 0, 1)$ is a bounded distributive lattice, $'$ is a unary operation of complementation and the identities

$$0' = 1\,,\ x'' = x\,,\ (x \vee y)' = x' \wedge y'\,,\ (x \wedge x') \vee (y \vee y') = y \vee y'$$

and their duals are satisfied. Every Boolean algebra is clearly a Kleene algebra, a smallest Kleene algebra which is not Boolean is $\mathbf{3} = \{0, a, 1\}$ with $0 < a < 1$ and $a' = a$. The algebra $\mathbf{3}$ is subdirectly irreducible and generates the variety of Kleene algebras.

Two subsets of a Kleene algebra K often play an important role: a subset $K^\vee = \{x \vee x' | x \in K\}$, which is a filter of the distributive lattice K, and a dually defined ideal $K^\wedge$. The complementation operation clearly induces an antiisomorphism between $K^\vee$ and $K^\wedge$. Further, the union $K^\vee \cup K^\wedge$ is a subalgebra of the Kleene algebra K. The variety of Kleene algebras has the congruence extension property and we have the following lemma.

1.1 Lemma ([8; 1.1])**.** *For every Kleene algebra K, any congruence of the lattice $K^\vee$ is a restriction of some congruence of the Kleene algebra K.*

In [7] it was proved that a Kleene algebra K with a finite filter $K^\vee$ is (locally) affine complete if and only if it is a Boolean algebra. To characterize affine complete Kleene algebras in general, we will need the following generalization of affine completeness: if A is a subalgebra of an algebra B then A is *affine complete in B* if every compatible function on A can be interpolated by a polynomial of B. (Hence we allow elements of B to be used as constants to represent compatible functions of A.)

We can establish a canonical way of defining any n-ary polynomial function of a Kleene algebra in the following way: to every pair of subsets $\alpha_0, \alpha_1 \subseteq \underline{n} = \{1, \dots, n\}$ we assign the n-ary Kleene term

$$C_\alpha(x_1, \dots, x_n) = (\bigvee_{i \in \alpha_0} x_i) \vee (\bigvee_{i \in \alpha_1} x_i')\,.$$

¿From the axioms of Kleene algebras it follows that that every n-ary Kleene polynomial can be represented as a meet of so-called elementary polynomials $k_\alpha \vee C_\alpha$ where k_α are constants from K.

Let K be an affine complete Kleene algebra. Let $g\colon (K^\vee)^n \to K^\vee$ be a compatible function of the lattice $K^\vee$. We can extend the function g to a compatible function
$f\colon K^n \to K$ by

$$f(x_1, \dots, x_n) = g(x_1 \vee x_1', \dots, x_n \vee x_n') \quad \text{for all } x_1, \dots, x_n \in K.$$

This function must be polynomial, hence representable as a meet of elementary polynomials of K. One can show that its restriction to $K^\vee$, which is the function g, is therefore a lattice polynomial (obtained by omitting all x_i' in the representation of f). Hence we get:

2.1 Lemma. *Let K be a Kleene algebra. If K is affine complete, then $K^\vee$ and $K^\wedge$ are (as lattices) affine complete in K.*

The following lemma, which is a special case of [7; Theorem 1], can be proved similarly.

2.2 Lemma ([8; 2.2]). *If K is a locally affine complete Kleene algebra then the lattices $K^\vee$ and $K^\wedge$ are locally affine complete.* $\square$

To describe situations in which the lattices $K^\vee$ and $K^\wedge$ are affine complete in the lattice K we will use the following two concepts introduced in [12] (see also [9]): a filter F of a distributive lattice L is *almost principal* if for every $x \in L$ the filter $F \cap \uparrow x = \{y \in F \mid y \geq x\}$ is principal, i.e. has a smallest element. An *almost principal ideal* of L is defined dually. Further, a filter or an ideal of L is proper if it is not equal to L while an interval of L is proper if it contains at least two elements.

2.3 Lemma ([8; 2.3]). *Let D be a sublattice of a distributive lattice L. Suppose that D is affine complete in L. Then*

($\mathcal{B}$) *D does not contain a proper Boolean interval;*
($\mathcal{F}$) *for every proper almost principal filter F in D there exists $b \in L$ such that $F = D \cap \uparrow b$;*
($\mathcal{I}$) *for every proper almost principal ideal I in D there exists $c \in L$ such that $I = D \cap \downarrow c$.* $\square$

Let us summarize the known results for distributive lattices.

2.4 Theorem.

(1) *A bounded distributive lattice is affine complete if and only if it does not contain a proper Boolean interval (*[5]*).*
(2) *A distributive lattice is locally affine complete if and only if it does not contain a proper Boolean interval (*[4; p. 102]*).*
(3) *A distributive lattice is affine complete if and only if the following conditions are satisfied:*
(i) it does not contain a proper Boolean interval;
(ii) it does not contain a proper almost principal ideal without a largest element;
(iii) it does not contain a proper almost principal filter without a smallest element ([13; 2.7]*).* $\square$

Throughout the paper we assume that the Kleene algebra K is embedded in $\mathbf{3}^I$, a power of the Kleene algebra $\mathbf{3} = \{0, a, 1\}$. Accordingly, we will write the elements of K in the form $x = (x_i)_{i \in I}$. Clearly, $s \in K^\vee$ iff $s_i \in \{a, 1\}$ for every $i \in I$.

As the proofs of the following two lemmas from [8] are very short, we include them here.

2.5 Lemma. *Let $f: K^n \to K$ be compatible, $x_j, y_j \in K$, $j = 1, \dots, n$ and $i \in I$. Then $x_{1i} = y_{1i}, \dots, x_{ni} = y_{ni}$ implies $f(x_1, \dots, x_n)_i = f(y_1, \dots, y_n)_i$.*

Proof. Consider the compatibility relative to kernel congruence of the i-th projection. $\square$

For any $s \in K^\vee$ we define the subalgebra K^s of K:

$$K^s = \{x \in K | x \vee x' \geq s\}\,.$$

2.6 Lemma. *Let $s \in K^\vee$. If two n-ary compatible functions of K coincide on $\{0, s, 1\}^n$ then they coincide on $(K^s)^n$.*

Proof. Let f and g coincide on $\{0, s, 1\}^n$. We prove that

$$f(x_1, \dots, x_n)_i = g(x_1, \dots, x_n)_i$$

for every $x_1, \dots, x_n \in K^s$ and $i \in I$.

First we define for every x_j the element $y_j \in \{0, s, 1\}$ having the same i-th component as x_j:

$$y_j = \begin{cases} 0 & \text{if} \quad x_{ji} = 0; \\ 1 & \text{if} \quad x_{ji} = 1; \\ s & \text{if} \quad x_{ji} = a\,. \end{cases}$$

Now by Lemma 2.5,

$$f(x_1, \dots, x_n)_i = f(y_1, \dots, y_n)_i = g(y_1, \dots, y_n)_i = g(x_1, \dots, x_n)_i. \quad \square$$

The *uncertainty order* of a Kleene algebra K is the binary relation $\sqsubseteq$ defined by

$$x \sqsubseteq y \quad \Longleftrightarrow \quad x \wedge s \leq y \leq x \vee s' \text{ for some } s \in K^\vee.$$

Hence the uncertainty order on $K = \mathbf{3}$ is the relation

$$\{(0, 0), (a, a), (1, 1), (0, a), (1, a)\}.$$

This relation on $\mathbf{3}$ can really be found under the name 'uncertainty order' in the literature.

2.7 Lemma ([8; 3.4])**.** *The uncertainty order on K is inherited from the uncertainty order on $\mathbf{3}$, i.e. $x \sqsubseteq y$ iff $x_i \sqsubseteq y_i$ for every $i \in I$.* $\square$

It can easily be seen that $\sqsubseteq$ is indeed a partial order relation on K which is a subalgebra of $K \times K$. Hence every local polynomial function preserves $\sqsubseteq$.

2.8 Lemma ([8; 3.7])**.** *If all compatible functions on the lattice $K^\vee$ are order preserving then all compatible functions on the Kleene algebra K preserve $\sqsubseteq$.* $\square$

3. The results - an alternative approach

In this section we give alternative proofs to the three main results presented in [8]. Our approach is based on calculations presented in the following two lemmas.

3.1 Lemma. *If a compatible function $f : K \to K$ preserves $\sqsubseteq$ then, for every $s, t \in K^{\vee}$,*

(1) $f(1) \wedge s \leq f(s) \leq f(1) \vee s'$;
(2) $f(0) \wedge s' \leq f(s) \leq f(0) \vee s$;
(3) $f(s') \wedge s' \leq f(s) \leq f(s') \vee f(1)$;
(4) $f(s) \leq s \vee f(t)$;
(5) *if* $s \geq f(0)$ *or* $s \geq f(1)$ *then* $s \geq f(s)$.

Proof. We prove that $f(1)_i \wedge s_i \leq f(s)_i \leq f(1)_i \vee s_i'$ for every $i \in I$. If $s_i = 1$ then $f(s)_i = f(1)_i$ by 2.5. Let $s_i = a$. Now the case $f(s)_i = a$ is trivial, let $f(s)_i \in \{0, 1\}$. Since $1 \sqsubseteq s$, we have $f(1)_i \sqsubseteq f(s)_i$, which is only possible if $f(s)_i = f(1)_i$. Thus, (1) is proved.

(2) is trivial on those components i where $s_i = 1$ or $f(s)_i = a$. The remaining case is $s_i = a$ and $f(s)_i \in \{0, 1\}$. Then, by 2.5, $f(s)_i = f(s')_i$ and from $0 \sqsubseteq s'$ we deduce that $f(0)_i = f(s)_i$.

To see (3), notice that $s_i = a$ implies $f(s')_i = f(s)_i$, while $s_i = 1$ implies $f(s)_i = f(1)_i$.

It is clear that $f(s)_i \leq s_i \vee f(t)_i$ if $s_i = 1$ or $s_i = t_i$ or $f(s)_i \leq a$. The remaining case is $s_i = a$, $f(s_i) = 1$ and $t_i = 1$. Then $1 \sqsubseteq s$ implies that $f(1)_i \sqsubseteq f(s)_i = 1$, hence $f(t)_i = f(1)_i = 1 = f(s)_i$. This proves (4).

(5) follows from (2) and (4). $\square$

3.2. Lemma. *If a compatible function $f : K \to K$ preserves $\sqsubseteq$ then, for every $s \in K^{\vee}$,*

$$\begin{aligned} f(s) =& (f(s) \wedge f(0) \wedge f(1)) \vee (f(1) \wedge s) \vee ((f(s') \vee f(0) \vee f(1)) \wedge s') = \\ & (f(1) \vee s') \wedge (f(s') \vee f(0) \vee f(1)) \wedge ((f(s) \wedge f(0) \wedge f(1)) \vee s). \end{aligned}$$

Proof. The equality of the last two expressions follows from the distributivity, since $s' \leq s$ and $f(s) \wedge f(0) \wedge f(1) \leq f(1) \leq f(s') \vee f(0) \vee f(1)$.

Obviously, $f(s) \geq f(s) \wedge f(0) \wedge f(1)$. By 3.1 we have $f(s) \geq f(1) \wedge s \geq f(1) \wedge s'$, $f(s) \geq f(0) \wedge s'$ and $f(s) \geq f(s') \wedge s'$, hence $f(s) \geq (f(s) \wedge f(0) \wedge f(1)) \vee (f(1) \wedge s) \vee ((f(s') \vee f(0) \vee f(1)) \wedge s')$.

It remains to prove the inverse inequality. By 3.1, $f(s) \leq f(1) \vee s'$, $f(s) \leq f(s') \vee f(1)$, $f(s) \leq f(0) \vee s$ and $f(s) \leq f(1) \vee s$ and obviously $f(s) \leq f(s) \vee s$, which completes the proof. $\square$

The previous lemma will be used to characterize local polynomial functions of Kleene algebras and consequently also locaally affine complete Kleene algebras.

3.3 Theorem ([8; 4.1]**).** *Let f be an n-ary compatible function on a Kleene algebra K. Then the following conditions are equivalent:*

(1) *f is a local polynomial function of K;*
(2) *f preserves the uncertainity order of K ;*
(3) *f can be interpolated by a polynomial on K^s for every $s \in K^{\vee}$.*

Proof. Since $\sqsubseteq$ is a subalgebra of $K \times K$, we have (1) $\Longrightarrow$ (2). Clearly, every finite subset of K is contained in some K^s , $s \in K^{\vee}$, which yields (3) $\Longrightarrow$ (1). Hence the key implication is (2) $\Longrightarrow$ (3).

By 2.6 it suffices to interpolate f on the set $\{0, s, 1\}$. We proceed by induction on arity n of f. The claim is obviously true for $n = 0$. Suppose now that $n > 0$ and that the implication (2) $\Longrightarrow$ (3) is true for all functions of arity less than n. Hence, the $(n-1)$-ary functions $f(0, x_2, \dots, x_n)$, $f(s, x_2, \dots, x_n)$, $f(s', x_2, \dots, x_n)$, $f(1, x_2, \dots, x_n)$ (of variables $x_2, \dots, x_n$) are representable by polynomials $p_0, p_s, p_{s'}, p_1$, respectively. Let us set

$$p(x_1, \dots, x_n) = (p_s \wedge p_0 \wedge p_1) \vee (p_1 \wedge x_1) \vee (p_0 \wedge x_1') \vee ((p_{s'} \vee p_0 \vee p_1) \wedge x_1 \wedge x_1').$$

We claim that p represents f on $\{0, s, 1\}^n$. Let $x_1, \dots, x_n \in \{0, s, 1\}$. It is easy to see that
$p(x_1, \dots, x_n) = f(x_1, \dots, x_n)$ whenever $x_1 \in \{0, 1\}$. Finally, for $x_1 = s$ we have $s' \leq s$ and therefore $p(s, x_2, \dots, x_n) = (p_s \wedge p_0 \wedge p_1) \vee (p_1 \wedge s) \vee ((p_{s'} \vee p_0 \vee p_1) \wedge s')$, which is equal to $f(s, x_2, \dots, x_n)$ by 3.2. (Apply 3.2 to the unary function g defined by $g(y) = f(y, x_2, \dots, x_n)$.) □

3.4 Theorem ([8; 4.2])**.** *Let K be a Kleene algebra. The following are equivalent:*

(1) *K is locally affine complete;*
(2) *$K^\vee$ is a locally affine complete lattice;*
(3) *$K^\vee$ does not contain a proper Boolean interval.*

Proof. The equivalence of (2) and (3) was given by 2.4(2). We stated (1) $\Longrightarrow$ (2) in 2.2. By (2), every compatible function of the lattice $K^\vee$ is order preserving and by 2.8 and the previous theorem, every compatible function on the Kleene algebra K is a local polynomial function. □

Before characterizing affine complete Kleene algebras in general we can already state the following special result.

3.5 Proposition ([8; 4.3])**.** *Let K be a Kleene algebra such that $K^\vee$ has a smallest element. The following are equivalent:*

(1) *K is affine complete;*
(2) *$K^\vee$ is an affine complete lattice;*
(3) *$K^\vee$ does not contain a proper Boolean interval.*

Proof. The equivalence of (2) and (3) was given by 2.4(1). The implication (1) $\Longrightarrow$ (3) follows the fact that every affine complete algebra is locally affine complete and from Theorem 3.4. If (3) holds, then the algebra K is locally affine complete by 3.4 and hence by 2.6 every compatible function of K can be interpolated by a polynomial function on any K^s. But clearly $K = K^s$ where s is the smallest element of $K^\vee$, which completes the proof. □

For a subset Y of an ordered set X we denote $\uparrow !Y = \{x \in X \mid x \geq y \text{ for some } y \in Y\}$ and $\downarrow !Y = \{x \in X \mid x \leq y \text{ for some } y \in Y\}$.

3.6 Lemma ([8; 5.1])**.** *Let $f : K \longrightarrow K$ be a local polynomial function of a Kleene algebra K. Then $K^\vee \cap \uparrow f(K^\vee)$ is an almost principal filter in $K^\vee$ and $K^\wedge \cap \downarrow f(K^\wedge)$ is an almost principal ideal in $K^\wedge$.*

Proof. Denote $F = K^\vee \cap \uparrow f(K^\vee) = \{x \in K^\vee \mid f(z) \leq x \text{ for some } z \in K^\vee\}$. We show that, for $x \in K^\vee$,

$$x \vee f(x) = \min\{y \in F \mid x \leq y\}.$$

Clearly, $x \leq x \vee f(x) \in F$. Conversely, let $x \leq y \in F$. Then $y \geq f(z)$ for some $z \in K^\vee$. By Lemma 3.1, $f(x) \leq x \vee f(z)$, hence $x \vee f(x) \leq x \vee f(z) \leq y$.

It remains to show that F is closed under meets. Let $x, y \in F$, $z = x \wedge y$, $t = \min\{u \in F \mid z \leq u\}$. Then $z \leq t \leq x$, $t \leq y$, thus $z = t \in F$. We showed that F is an almost principal filter in $K^\vee$.

The other statement can be proved dually. $\square$

Let P denote the set of all pairs $\alpha = (\alpha_0, \alpha_1)$ with $\alpha_0, \alpha_1 \subseteq \underline{n}$, $\alpha_0 \cap \alpha_1 = \emptyset$. We introduce an order relation on P by $\alpha \leq \beta$ iff $\alpha_0 \subseteq \beta_0$ and $\alpha_1 \subseteq \beta_1$.

Suppose now that a Kleene algebra K satisfies the following conditions:

($\mathcal{B}$) $K^\vee$ does not contain a proper Boolean interval;

($\mathcal{F}$) for every proper almost principal filter F in $K^\vee$ there exists $b \in K$ such that $F = K^\vee \cap \uparrow b$.

Since $'$ is a dual automorphism of the lattice K, ($\mathcal{F}$) is equivalent to the dual condition

($\mathcal{I}$) for every proper almost principal ideal I in $K^\wedge$ there exists $c \in K$ such that $I = K^\wedge \cap \downarrow c$.

Let $f : K^n \to K$ be a compatible function. By ($\mathcal{B}$) and 3.4, f is a local polynomial function. For every $\alpha \in P$ we define a unary function $f_\alpha : K \to K$ by the rule

$$f_\alpha(y) = f(x_1, \ldots, x_n), \text{ where } x_i = \begin{cases} 0 & \text{if } \quad i \in \alpha_0; \\ 1 & \text{if } \quad i \in \alpha_1; \\ y & \text{otherwise.} \end{cases}$$

It is clear that the functions f_α are compatible. Therefore by ($\mathcal{F}$) and ($\mathcal{I}$) we have constants b_α, c_α such that

$$(*) \qquad \begin{aligned} K^\vee \cap \uparrow f_\alpha(K^\vee) &= K^\vee \cap \uparrow b_\alpha; \\ K^\wedge \cap \downarrow f_\alpha(K^\wedge) &= K^\wedge \cap \downarrow c_\alpha. \end{aligned}$$

¿From the proof of Lemma 3.6 we see that

$$\begin{aligned} x \vee f_\alpha(x) &= \min\{y \in K^\vee \cap \uparrow f_\alpha(K^\vee) \mid x \leq y\} = b_\alpha \vee x; \\ z \wedge f_\alpha(z) &= \max\{y \in K^\wedge \cap \downarrow f_\alpha(K^\wedge) \mid z \geq y\} = c_\alpha \wedge z \end{aligned}$$

for every $x \in K^\vee$, $z \in K^\wedge$.

3.7 Lemma. *If $\alpha \leq \beta$ then $K^\vee \cap \uparrow f_\alpha(K^\vee) \supseteq K^\vee \cap \uparrow f_\beta(K^\vee)$.*

Proof. It suffices to deal with the case when $(\beta_0 \cup \beta_1) \setminus (\alpha_0 \cup \alpha_1)$ is a one-element set, say, $\{j\}$. Let $x \in K^\vee \cap \uparrow f_\beta(K^\vee)$. Then $x \geq f_\beta(y)$ for some $y \in K^\vee$ and by 3.1 we have $x \geq f_\beta(y) \vee x \geq f_\beta(x)$. Let us define a unary compatible function g as follows:

$$g(y) = f(x_1, \ldots, x_n), \text{ where } x_i = \begin{cases} 0 & \text{if } \quad i \in \alpha_0; \\ 1 & \text{if } \quad i \in \alpha_1; \\ y & \text{if } \quad i = j; \\ x & \text{otherwise.} \end{cases}$$

If $j \in \beta_0$ then $g(0) = f_\beta(x)$. If $j \in \beta_1$ then $g(1) = f_\beta(x)$. Hence, either $x \geq g(0)$ or $x \geq g(1)$. By 3.1(5) then $x \geq g(x) = f_\alpha(x)$, hence $x \in K^\vee \cap \uparrow f_\alpha(K^\vee)$. $\square$

3.8 Lemma. *The constants b_α, c_α in (*) can be chosen in such a way that*

(i) *if $\alpha_0 \cup \alpha_1 = \underline{n}$ then both b_α and c_α are equal to the value of the constant function f_α;*
(ii) *if $\alpha \le \beta$ then $b_\alpha \le b_\beta \le c_\beta \le c_\alpha$.*

Proof. If $\alpha_0 \cup \alpha_1 = \underline{n}$ then f_α is a constant function equal to some $k \in K$. We set $b_\alpha = c_\alpha = k$. Clearly, (*) is satisfied.

Let b_α, c_α be arbitrary elements satisfying (*). We set $b'_\alpha = \bigwedge_{\beta \ge \alpha} b_\beta$, $c'_\alpha = \bigvee_{\beta \ge \alpha} c_\beta$. Now the constants b'_α, c'_α fulfil (ii) (notice that for $\beta_0 \cup \beta_1 = \underline{n}$ we have $b_\beta = c_\beta$) and it remains to show that (*) is valid when we replace b_α, c_α by b'_α, c'_α.

For any $x, y \in K$ we have $K^\vee \cap \uparrow(x \wedge y) = (K^\vee \cap \uparrow x) \vee (K^\vee \cap \uparrow y)$, i.e. $K^\vee \uparrow (x \wedge y)$ is the least filter containing both $K^\vee \cap \uparrow x$ and $K^\vee \cap \uparrow y$. By induction we obtain that, for any $\alpha \in P$,

$$K^\vee \cap \uparrow b'_\alpha = \bigvee_{\alpha \le \beta} K^\vee \cap \uparrow b_\beta = \bigvee_{\alpha \le \beta} K^\vee \cap \uparrow f_\beta(K^\vee) = K^\vee \cap \uparrow f_\alpha(K^\vee)$$

using Lemma 3.7. Hence, the elements b'_α fulfil (*). The proof for c'_α is analogous. $\square$

3.9 Theorem. *Let K be a Kleene algebra. The following conditions are equivalent:*

(1) *K is affine complete;*
(2) *$K^\vee$ is affine complete in K;*
(3) *$K^\wedge$ is affine complete in K;*
(4) *$K^\vee$ does not contain proper Boolean intervals and for every proper almost principal filter F in $K^\vee$ there exists $b \in K$ such that $F = K^\vee \cap \uparrow b$.*
(5) *$K^\wedge$ does not contain proper Boolean intervals and for every proper almost principal ideal I in $K^\wedge$ there exists $c \in K$ such that $F = K^\wedge \cap \downarrow c$.*

Proof. The existence of the dual automorphism $'$ for the lattice K yields that the conditions (2) and (3) and similarly the conditions (4) and (5) are equivalent. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (4) follow from Lemmas 2.1 and 2.3. So we have to prove only the implication (4) $\Longrightarrow$ (1). Let K be a Kleene algebra satisfying (4) and $f : K^n \longrightarrow K$ a compatible function. Hence, we have the constants b_α, c_α that satisfy (*) and 3.8(i),(ii). For $\alpha \in P$ we define polynomials C_α, D_α by the rule

$$C_\alpha = \bigvee_{j \in \alpha_0} x_j \vee \bigvee_{j \in \alpha_1} x'_j; \qquad D_\alpha = \bigvee_{j \in \underline{n} \setminus \alpha_1} x_j \vee \bigvee_{j \in \underline{n} \setminus \alpha_0} x'_j.$$

Let us set

$$p(x_1, \dots, x_n) = \bigwedge_{\alpha \in P} (c_\alpha \vee C_\alpha) \wedge \bigwedge_{\alpha \in P} (b_\alpha \vee D_\alpha).$$

To prove that p represents f, it suffices to show that $p(x_1, \dots, x_n) = f(x_1, \dots, x_n)$ for $x_1, \dots, x_n \in \{0, s, 1\}$, where s is an arbitrary element of $K^\vee$. Without loss of generality, $x_1 = \dots = x_k = 0$, $x_{k+1} = \dots = x_l = s$ and $x_{l+1} = \dots = x_n = 1$. Let us denote $\beta = (\underline{k}, \underline{n} \setminus \underline{l})$, $\gamma = (\underline{k}, \underline{n} \setminus \underline{k})$. If $k = l$ then $p(x_1, \dots, x_n) = c_\beta = f(x_1, \dots, x_n)$ by 3.8. Suppose that $k < l$. We claim that

$$p(x_1, \dots, x_n) = c_\beta \wedge (b_\beta \vee s) \wedge (b_\gamma \vee s').$$

Clearly, $p(x_1,\dots,x_n) \le c_\beta \wedge (b_\beta \vee s) \wedge (b_\gamma \vee s')$, because $C_\beta = 0$, $D_\beta = s$ and $D_\gamma = s'$. The other inequality follows from the facts that

$$c_\alpha \vee C_\alpha \ge \begin{cases} C_\alpha = 1 & \text{if } \alpha_0 \nsubseteq \underline{l} \text{ or } \alpha_1 \nsubseteq \underline{n} \setminus \underline{k}; \\ c_\beta & \text{if } \alpha \le \beta; \\ c_\alpha \vee s' \ge c_\gamma \vee s' = b_\gamma \vee s' & \text{if } \gamma \ge \alpha \nleq \beta; \\ c_\alpha \vee s \ge c_{\alpha\cup\beta} \vee s \ge b_{\alpha\cup\beta} \vee s \ge b_\beta \vee s & \text{otherwise} \end{cases}$$

and

$$b_\alpha \vee D_\alpha \ge \begin{cases} D_\alpha = 1 & \text{if } \beta \nleq \alpha; \\ b_\alpha \vee s' = b_\gamma \vee s' & \text{if } \beta \le \alpha = \gamma; \\ b_\alpha \vee s \ge b_\beta \vee s & \text{if } \beta \le \alpha \ne \gamma. \end{cases}$$

We wish to show that $f(x_1,\dots,x_n) = f_\beta(s) = p(x_1,\dots,x_n)$. We have $c_\beta \ge c_\gamma = f_\beta(1)$ and also $c_\beta \ge f_\beta(s') \wedge s'$, thus, by 3.1, $c_\beta \ge (f_\beta(1) \vee f_\beta(s')) \wedge (f_\beta(1) \vee s') \ge f_\beta(s)$. Further, $b_\beta \vee s = f_\beta(s) \vee s \ge f_\beta(s)$ and $b_\gamma \vee s' = f_\beta(1) \vee s' \ge f_\beta(s)$ by 3.1. Hence, $f_\beta(s) \le p(x_1,\dots,x_n)$.

By the distributivity (using inequalities $b_\beta \le b_\gamma = c_\gamma \le c_\beta$) we can write

$$p(x_1,\dots,x_n) = b_\beta \vee (b_\gamma \wedge s) \vee (c_\beta \wedge s').$$

Using the equalities $b_\beta \vee s = f_\beta(s) \vee s$, $c_\beta \wedge s' = f_\beta(s') \wedge s'$ and the inequalities from 3.1 we have $b_\beta \le (f_\beta(s) \vee s) \wedge f_\beta(1) = (f_\beta(s) \wedge f_\beta(1)) \vee (s \wedge f_\beta(1)) \le f_\beta(s)$, $b_\gamma \wedge s = f_\beta(1) \wedge s \le f_\beta(s)$ and $c_\beta \wedge s' \le f_\beta(s)$. Hence, $p(x_1,\dots,x_n) \le f_\beta(s)$. □

3.10 Examples.

(1) Let $K_1 = \{(-\infty,-\infty)\} \cup ((R \times R) \setminus \{(0,0)\}) \cup \{(\infty,\infty)\}$ be the Kleene algebra with the complementation defined by $(x,y)' = (-x,-y)$. Then $K_1^\vee = (\{(x,y) \in R \times R \mid x \ge 0, y \ge 0\} \setminus \{(0,0)\}) \cup \{(\infty,\infty)\}$ is obviously an affine complete distributive lattice by 2.4(3). Hence $K_1^\vee$ is affine complete in the lattice K_1 and by 3.9, the Kleene algebra K_1 is affine complete.

Note that for the same reason, the Kleene algebra $K_1^\wedge \oplus K_1^\vee$, where $\oplus$ means the linear sum, is affine complete. In general, for every affine complete distributive lattice D, the Kleene algebra $D \oplus D^d$ is affine complete where D^d is a dual of D and the complementation operation is the antiisomorphism between D and D^d.

(2) Let $K_2 = K_1 \setminus \{(0,y) \mid y \in R\}$ be a subalgebra of K_1. Then $K_2^\vee = \{(x,y) \in R \times R \mid x > 0, y \ge 0\} \cup \{(\infty,\infty)\}$ is not, according to 2.4(3), an affine complete lattice because $F = \{(x,y) \in R \times R \mid x > 0, y \ge 1\} \cup \{(\infty,\infty)\}$ is a proper almost principal filter in $K_2^\vee$ without a smallest element. One can verify that the unary function $g\colon K_2^\vee \to K_2^\vee$ given by $g(x) = \min\{y \in F \mid x \le y\}$ is a compatible function of the lattice $K_2^\vee$ but cannot be represented by a polynomial function of $K_2^\vee$ (see a similar verification in [12; 2.2]). However, note that there is an element b in K_2, for example, $b = (-1,1)$ such that $F = K_2^\vee \cap \uparrow b$. It can easily be seen that the condition (4) of 3.9 is satisfied, hence again, $K_2^\vee$ is affine complete in the lattice K_2 and the Kleene algebra K_2 is affine complete.

(3) Let $K_3 = K_2 \setminus \{(x,y) \in R \times R \mid x \cdot y < 0\}$ be a Kleene subalgebra of K_2. Then $K_3^\vee = \{(x,y) \in R \times R \mid x > 0, y \ge 0\} \cup \{(\infty,\infty)\} = K_2^\vee$ is again not an affine complete lattice. But note that for the almost principal filter without a smallest element F defined in (2) there is now no element $b \in K_3$ such that $F = K_3^\vee \cap \uparrow b$.

Hence $K_3^{\vee}$ is not affine complete in the lattice K_3 and the Kleene algebra K_3 is not affine complete. It can be verified that the unary function $f\colon K_3 \to K_3$ given by $f(x) = \min\{y \in F \mid x \vee x' \leq y\}$ is a compatible function of the Kleene algebra K_3 but cannot be represented by a polynomial of K_3.

However, by 3.4 it is clear that K_3 is a locally affine complete Kleene algebra.

(4) Every finite Kleene algebra which is not a Boolean algebra is not affine complete. $\square$

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