AFFINE COMPLETENESS OF KLEENE ALGEBRAS

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Abstract. An algebra is called affine complete if all its compatible (i.e. congruence-preserving) functions are polynomial functions. In this paper we characterize affine complete members in the variety of Kleene algebras. We also characterize local polynomial functions of Kleene algebras and use this result to describe locally affine complete Kleene algebras.

1. Introduction

Let A be a universal algebra. A function $f:\ A^n \to A$ is called *compatible* if, for any congruence θ of A, $(a_i, b_i) \in \theta$, $i = 1, \dots, n$, implies that

$$(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \in \theta.$$

A polynomial function (or simply *a polynomial*) of A is any function that can be obtained by composition of the basic operations of A, the projections and the constant functions. *A local polynomial* of A is any function which can be interpolated by polynomials on all finite subsets of its domain.

Obviously, (local) polynomials are compatible functions. An algebra is called *(locally) affine complete* if the converse holds: every compatible function is a (local) polynomial. (We note that the concept 'locally affine complete' has sometimes another meaning in the literature - see e.g. [10].)

Originally, the problem of characterizing algebras which are affine complete was formulated in [6]. Since every algebra is a reduct of an affine complete algebra (e.g. of that which contains all its compatible functions among the basic operations), in [3] the problem was reformulated as follows: characterize affine complete algebras in your favourite variety.

For various varieties of algebras affine completeness has already been investigated (see introductions in [3] or [8]). The papers [5], [7]-[9] and [11]-[12] contain some ideas that are close to our considerations. In [7] affine completeness of the class of algebras containing Kleene algebras was studied. In particular, it was shown there that a finite Kleene algebra is affine complete if and only if it is a Boolean algebra. The aim of this paper is to characterize affine completeness and local affine completeness for Kleene algebras in general.

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For our purpose, the following generalization of affine completeness will prove useful. Let A be a subalgebra of an algebra B. We say that A is *affine complete in* B if every compatible function on A can be interpolated by a polynomial of B. (That is, we can use all elements of B as constants.) Clearly A is affine complete iff it is affine complete in itself.

In fact the study of affine completeness is motivated by far more general problem of characterizing (local) polynomials for given classes of algebras. For example, local polynomials of bounded distributive lattices are polynomials and one can characterize them as order preserving compatible functions [5]. One can prove that local polynomials of arbitrary distributive lattices are exactly order preserving compatible functions [4].

Our approach provides similar characterization of local polynomials for Kleene algebras. Of course, the order relation has to be replaced by another appropriate binary relation. It is also worth mentioning that our final results are constructive: we are able to write down explicitly the interpolating polynomials for the given compatible function.

Now we recall the definition and some terminology for Kleene algebras. For more information see e.g. [1] or [2]. A Kleene algebra is a bounded distributive lattice with an additional unary operation (complementation) denoted by $'$ and satisfying the identities:

$$(x \vee y)' = x' \wedge y' \,,\; (x \wedge x') \vee (y \vee y') = y \vee y' \,,\; 0' = 1 \,,\; x'' = x$$

and their duals. Clearly every Boolean algebra is a Kleene algebra. A typical Kleene algebra which is not Boolean is $\mathbf{3} = \{0, a, 1\}$ with $0 < a < 1$ and $a' = a$. It is known that the variety of Kleene algebras is generated by $\mathbf{3}$. Moreover, $\mathbf{3}$ and its subalgebra $\mathbf{2} = \{0, 1\}$ are the only subdirectly irreducible Kleene algebras.

For every Kleene algebra K we denote $K^\vee = \{x \vee x' | x \in K\}$. The subset $K^\wedge$ is defined dually. It is easy to check that $K^\vee$ and $K^\wedge$ are a filter and an ideal of the distributive lattice K, respectively. Obviously the complementation operation induces an antiisomorphism between them. Note that the union $K^\vee \cup K^\wedge$ is always a subalgebra of K and every congruence of the lattice $K^\vee$ extends in a natural way to a congruence of the Kleene algebra $K^\vee \cup K^\wedge$. Now, it is known [2] that the variety of Kleene algebras has the congruence extension property. Hence we have the following important lemma.

1.1 Lemma. *For every Kleene algebra K, any congruence of the lattice $K^\vee$ is a restriction of some congruence of the Kleene algebra K.*

We write $x \sqsubseteq y$ if $x \wedge s \leq y \leq x \vee s'$ for some $s \in K^\vee$ and call the binary relation $\sqsubseteq$ the *uncertainity order* for K. The main results of the present paper are the following.

(1) The function on a Kleene algebra K is a local polynomial function if and only if it preserves the congruences of K and the uncertainity order.
(2) A Kleene algebra K is affine complete if and only if the lattice $K^\vee$ is affine complete in the lattice K.
(3) A Kleene algebra K is locally affine complete if and only if the lattice $K^\vee$ is locally affine complete.

2. Necessary conditions

In [8] it was proved that a Stone algebra S is affine complete if and only if the filter $D(S)$ of its 'dense' elements is (as a lattice) affine complete in S. Our result for Kleene algebras looks very similar. The first steps toward it were done in [7] where it was proved that a Kleene algebra K with a finite filter $K^\vee$ is (locally) affine complete if and only if it is a Boolean algebra.

First we look at a canonical form for n-ary polynomials of Kleene algebras. Let $\underline{n} = \{1, \dots, n\}$ and consider a pair of subsets $\alpha_1, \alpha_2 \subseteq \underline{n}$. To every such pair $\alpha = (\alpha_1, \alpha_2)$ we assign the n-ary Kleene term

$$C_\alpha(x_1, \dots, x_n) = \Big(\bigvee_{i\in\alpha_1} x_i\Big) \vee \Big(\bigvee_{i\in\alpha_2} x_i'\Big).$$

It follows easily from the axioms of Kleene algebras that every Kleene polynomial can be represented as a meet of polynomials $k_\alpha \vee C_\alpha$ where k_α are constants from K. We refer to the polynomials $k_\alpha \vee C_\alpha$ as to *elementary polynomials* of a Kleene algebra.

2.1 Lemma. *Let K be a Kleene algebra. If K is affine complete, then $K^\vee$ and $K^\wedge$ are (as lattices) affine complete in K.*

Proof. Let $g\colon (K^\vee)^n \to K^\vee$ be a compatible function of the lattice $K^\vee$. Define a function $f\colon K^n \to K$ by

$$f(x_1, \dots, x_n) = g(x_1 \vee x_1', \dots, x_n \vee x_n') \quad \text{for all } x_1, \dots, x_n \in K.$$

It follows from Lemma 1.1 that f is a compatible function of the algebra K and its restriction to $(K^\vee)^n$ is g. Since K is affine complete, f is a polynomial function of K and hence f can be written as the meet of some elementary polynomials $k_\alpha \vee C_\alpha$. We will show that omitting in this polynomial all members of the form x_i' we get a lattice polynomial of K which coincides with the function g.

Obviously we can omit all x_i' which appear in elementary polynomials $k_\alpha \vee C_\alpha(x_1, \dots, x_n)$ with $\alpha_1 \neq \emptyset$. This follows from the fact that $x \le y$ for all $x \in K^\wedge$, $y \in K^\vee$. Now let b be the meet of all coefficients k_α for which $\alpha_1 = \emptyset$. It is easy to see that $b = f(1, \dots, 1) \in K^\vee$ implying $k_\alpha \in K^\vee$ whenever $\alpha_1 = \emptyset$. Thus the assumption that in such elementary polynomials $k_\alpha \vee C_\alpha(x_1, \dots, x_n)$ all variables take their values in $K^\vee$ implies $k_\alpha \vee C_\alpha(x_1, \dots, x_n) = k_\alpha \in K^\vee$. Hence the members of the form x_i' can be omitted again.

We have showed that if K is affine complete then $K^\vee$ is (as a lattice) affine complete in K. For $K^\wedge$ the statement can be proved dually. $\square$

The following lemma can be proved similarly as the preceding one. It is a special case of [7, Theorem 1].

2.2 Lemma. *If K is a locally affine complete Kleene algebra then the lattices $K^\vee$ and $K^\wedge$ are locally affine complete.*

As we see from Lemma 2.1, to get a description of affine complete Kleene algebras, we need to describe situations in which the lattices $K^\vee$ and $K^\wedge$ are affine complete in the lattice K. For this we use some concepts and technique developed in [11] and [8].

A filter F of a distributive lattice L is called *almost principal* if for every $x \in L$ the filter $F \cap \uparrow x = \{y \in F \mid y \geq x\}$ has a smallest element. An *almost principal ideal* of L is defined dually. A filter or an ideal of L is said to be proper if it is not equal to L. An interval of L is proper if it contains at least two elements.

2.3 Lemma. *Let D be a sublattice of a distributive lattice L. Suppose that D is affine complete in L. Then*

($\mathcal{B}$) *D does not contain a proper Boolean interval;*

($\mathcal{F}$) *for every proper almost principal filter F in D there exists $b \in L$ such that $F = D \cap \uparrow b$;*

($\mathcal{I}$) *for every proper almost principal ideal I in D there exists $c \in L$ such that $I = D \cap \downarrow c$.*

Proof. If D contains a proper Boolean interval $[a, b]$, we define a function $f\colon D \to [a, b]$ by $f(x) = ((x \vee a) \wedge b)'$ where $'$ denotes the complementation operation in $[a, b]$. This function is compatible but not isotone ($f(b) = a < b = f(a)$) and therefore cannot be represented by a lattice polynomial of L.

We have proved ($\mathcal{B}$). To prove ($\mathcal{F}$) (the proof of ($\mathcal{I}$) is similar), let F be a proper almost principal filter in D. We define a function $g\colon D \to D$ by $g(x) = \min(F \cap \uparrow x)$. We see that F is the set of all fixed points of g. It is not difficult to show that g is compatible (see [11, proof of 2.2]). By our assumption, g can be interpolated by a polynomial of L. Without loss of generality, $g(x) = x \vee b$ for some $b \in L$. We claim that $F = D \cap \uparrow b$. If $x \in F$ then $x = g(x) = x \vee b$, whence $x \geq b$, i.e. $x \in D \cap \uparrow b$. Conversely, if $x \in D \cap \uparrow b$, then $g(x) = x \vee b = x \in F$. The proof is complete. $\square$

For local affine completeness we have the folowing result.

2.4 Corollary ([4; p. 102])**.** *A distributive lattice is locally affine complete if and only if it does not contain a proper Boolean interval.* $\square$

3. On compatible functions

In what follows we always assume that the Kleene algebra K is embedded in $\mathbf{3}^I$, a power of the Kleene algebra $\mathbf{3} = \{0, a, 1\}$. We write the elements of K in the form $x = (x_i)_{i \in I}$.

3.1 Lemma. *Let $f: K^n \to K$ be compatible, $x_j, y_j \in K$, $j = 1, \ldots, n$ and $i \in I$. Then $x_{1i} = y_{1i}, \ldots, x_{ni} = y_{ni}$ implies $f(x_1, \ldots, x_n)_i = f(y_1, \ldots, y_n)_i$.*

Proof. Consider the compatibility relative to kernel congruence of the i-th projection. $\square$

This means that every compatible function f determines the coordinate functions f_i such that $f_i(x_{1i}, \ldots, x_{ni}) = f(x_1, \ldots, x_n)_i$ for all $x_1, \ldots, x_n \in K$. Obviously, the family $(f_i)_{i \in I}$ completely determines f, so we may identify f with this family.

For any $s \in K^{\vee}$ we define the subalgebra K^s of K:

$$K^s = \{x \in K | x \vee x' \geq s\}.$$

3.2 Lemma. *Let $s \in K^{\vee}$. If two n-ary compatible functions of K coincide on $\{0, s, 1\}^n$ then they coincide on $(K^s)^n$.*

Proof. Let f and g coincide on $\{0, s, 1\}^n$. We prove that

$$f(x_1, \ldots, x_n)_i = g(x_1, \ldots, x_n)_i$$

for every $x_1, \dots, x_n \in K^s$ and $i \in I$.

First we define for every x_j the element $y_j \in \{0, s, 1\}$ having the same i-component as x_j:

$$y_j = \begin{cases} 0 & \text{if} \quad x_{ji} = 0; \\ 1 & \text{if} \quad x_{ji} = 1; \\ s & \text{if} \quad x_{ji} = a\,. \end{cases}$$

Now by Lemma 3.1,

$$f(x_1, \dots, x_n)_i = f_i(x_{1i}, \dots, x_{ni}) = f_i(y_{1i}, \dots, y_{ni}) = f(y_1, \dots, y_n)_i\,,$$

$$g(x_1, \dots, x_n)_i = g_i(x_{1i}, \dots, x_{ni}) = g_i(y_{1i}, \dots, y_{ni}) = g(y_1, \dots, y_n)_i\,.$$

□

Now we define binary relations on K which turn out to be of great importance. For any $s \in K$ we put

$$x \sqsubseteq_s y \quad \Longleftrightarrow \quad x \wedge s \leq y \wedge s \;\text{ and }\; x \vee s' \geq y \vee s'$$

or equivalently

$$x \sqsubseteq_s y \quad \Longleftrightarrow \quad x \wedge s \leq y \leq x \vee s'\,.$$

3.3 Definition. *The union of all relations $\sqsubseteq_s$ where $s \in K^\vee$ is denoted by $\sqsubseteq$ and called the uncertainity order of the Kleene algebra K.*

Note that in case $K = \mathbf{3}$ the uncertainity order is the following subset of $\mathbf{3} \times \mathbf{3}$:

$$\{(0,0), (a,a), (1,1), (0,a), (1,a)\}.$$

3.4 Lemma. *The uncertainity order on K is inherited from the uncertainity order on $\mathbf{3}$, i.e. $x \sqsubseteq y$ iff $x_i \sqsubseteq y_i$ for every $i \in I$.*

Proof. Let $x \sqsubseteq y$. Then $x \wedge s \leq y \leq x \vee s'$ for some $s \in K^\vee$. For every $i \in I$ we have $x_i \wedge s_i \leq y_i \leq x_i \vee s_i'$ and $s_i \in \mathbf{3}^\vee$, hence $x_i \sqsubseteq y_i$.

Conversely, suppose that $(x_i, y_i) \in \{(0,0), (a,a), (1,1), (0,a), (1,a)\}$ for every $i \in I$. Let us set $s = (x \vee x') \wedge (y \vee y')$. Clearly $s \in K^\vee$ and it is easy to check that $x_i \wedge s_i \leq y_i \leq x_i \vee s_i'$ for every $i \in I$. □

It is easy to check that all relations $\sqsubseteq_s$ are reflexive and transitive. Further, $\sqsubseteq_s$ is a subset of $\sqsubseteq_t$ whenever $t \leq s$. Hence, $\sqsubseteq$ is a directed union of the relations $\sqsubseteq_s$ ($s \in K^\vee$). Since all the relations $\sqsubseteq_s$ are subalgebras of K^2, the same is true for $\sqsubseteq$. This implies that every local polynomial function of K preserves $\sqsubseteq$ and all relations $\sqsubseteq_s$, $s \in K$.

3.5 Lemma. *The relations $\sqsubseteq$ and $\sqsubseteq_s$ coincide on the set K^s.*

Proof. Obviously, $\sqsubseteq_s$ is contained in $\sqsubseteq$. Suppose now that $x \sqsubseteq_s$ for $x, y \in K^s$. We have to check that $x_i \wedge s_i \leq y_i \leq x_i \vee s_i'$ for every $i \in I$. This is clear if $x_i = y_i$. In the remaining cases $(x_i, y_i) = (0, a)$ and $(x_i, y_i) = (1, a)$ we have $s_i = s_i' = a$ because $a = y_i \vee y_i' \geq a$ by our assumption. □

In the next section we shall see that these properties together with compatibility characterize local polynomials of Kleene algebras.

3.6 Lemma. *If $b, c, s \in K$ and $b \sqsubseteq_s c$ then there exists a unary polynomial p of K such that $p(1) = b$ and $p(s) = c$.*

Proof. A suitable polynomial is $p(x) = (x \vee c) \wedge (c \vee b) \wedge (b \vee x')$. Indeed,

$$p(1) = (1 \vee c) \wedge (c \vee b) \wedge (b \vee 1') = 1 \wedge (c \vee b) \wedge (b \vee 0) = (c \vee b) \wedge b = b$$

and

$$p(s) = (s \vee c) \wedge (c \vee b) \wedge (b \vee s') = (s \vee c) \wedge (c \vee b) = c \vee (s \wedge b) = c\,.$$

□

3.7 Lemma. *If all unary compatible functions of K preserve the relation $\sqsubseteq_s$ for some $s \in K$ then so do all compatible functions.*

Proof. Let $x_j, y_j \in K, x_j \sqsubseteq_s y_j, j = 1, \ldots, n$, and let f be an n-ary compatible function on K such that $f(x_1 \ldots, x_n) \not\sqsubseteq_s f(y_1, \ldots, y_n)$. Using Lemma 3.6 take unary polynomials p_j such that $p_j(1) = x_j, p_j(s) = y_j$ and consider the unary compatible function $g(x) = f(p_1(x), \ldots, p_n(x))$. Obviously $1 \sqsubseteq_s s$ but $g(1) \not\sqsubseteq_s g(s)$. □

The next lemma will prove useful in a characterization of locally affine complete algebras (Theorem 4.3).

3.8 Lemma. *If all compatible functions on the lattice $K^\vee$ are order preserving then all compatible functions on Kleene algebra K preserve all relations $\sqsubseteq_s$, $s \in K^\vee$.*

Proof. Suppose there is an $s \in K^\vee$ and a compatible function f on K which does not preserve $\sqsubseteq_s$. In view of Corollary 3.5 we may assume that f is unary. We show that then there is a compatible function on the lattice $K^\vee$ which does not preserve the order relation.

Let $x, y \in K, x \sqsubseteq_s y$ but $f(x) \not\sqsubseteq_s f(y)$. Obviously then there exists $i \in I$ such that $x_i \sqsubseteq_{s_i} y_i$ but $f_i(x_i) \not\sqsubseteq_{s_i} f_i(y_i)$. This is possible only in case $s_i = a = y_i$ and $f_i(x_i) \neq f_i(y_i) \neq a$. Without loss of generality, $f_i(y_i) = 1$, otherwise consider the function $g(x) = f(x)'$.

Assume first $f_i(1) \leq a$. The function $g(z) = f(z) \vee s$ is a compatible function of the Kleene algebra K and maps $K^\vee$ into $K^\vee$. By Lemma 1.1 the restriction of g to $K^\vee$ is a compatible function of the lattice $K^\vee$. However, g does not preserve the order relation of $K^\vee$ because $g_i(1) = a$ and $g_i(a) = 1$, and hence $g(y) \not\leq g(1)$.

Now let $f_i(1) = 1$. Then we have $f_i(0) \leq a$ and considering the function $g(z) = f(z') \vee s$ we obtain a similar contradiction. □

Now we return to canonical forms of Kleene polynomials. We shall construct such polynomials in the form

$$p(x_1, \ldots, x_n) = \bigwedge_{\alpha \in P} (k_\alpha \vee C_\alpha(x_1, \ldots, x_n)) \tag{1}$$

where k_α are constants from K and P stands for the set of ordered pairs $\alpha = (\alpha_1, \alpha_2)$ of subsets of $\underline{n}$ such that $\alpha_1 \cap \alpha_2 = \emptyset$ or $\alpha_1 \cup \alpha_2 = \underline{n}$. Further, we denote $P_0 = \{\alpha \in P \mid \alpha_1 \cap \alpha_2 = \emptyset\}$, $P_n = \{\alpha \in P \mid \alpha_1 \cup \alpha_2 = \underline{n}\}$. We write $\alpha \leq beta$ for elements of P if $\alpha_1 \subseteq \beta_1$ and $\alpha_2 \subseteq \beta_2$. We keep this denotation throughout the paper.

If $f : K^n \longrightarrow K$ is any function and $\alpha \in P_0$, we denote by $f_\alpha : K \longrightarrow K$ the function defined by the rule

$$f_\alpha(y) = f(x_1, \dots, x_n), \text{ where } x_j = \begin{cases} 0 & \text{if } \; j \in \alpha_1; \\ 1 & \text{if } \; j \in \alpha_2; \\ y & \text{otherwise.} \end{cases}$$

3.9 Lemma. *Suppose that a compatible function $f : K^n \longrightarrow K$ preserves the relation $\sqsubseteq$. Then, for every $s \in K^\vee$, $\beta, \gamma \in P_0$, $\gamma \le \beta$, the following holds:*

(1) $f_\beta(s) \wedge s' \le f_\gamma(s) \le f_\beta(s) \vee s$;
(2) $f_\beta(s') \wedge s' \le f_\gamma(s') \le f_\beta(s') \vee s$.

Proof. We prove (1). If $s_i = 1$ then trivially

$$((*)) \qquad (f_\beta(s) \wedge s')_i \le f_\gamma(s)_i \le (f_\beta(s) \vee s)_i.$$

So, let $i \in I$ be such that $s_i = a$. Then also $s'_i = a$. Now the case $f_\gamma(s)_i = a$ is trivial, let us assume that $f_\gamma(s)_i \ne a$. Recall that $f_\beta(s) = f(x_1, \dots, x_n)$, where $x_k = 0$ for $k \in \beta_1$, $x_k = 1$ for $k \in \beta_2$ and $x_k = s$ otherwise. Let us define $z_1, \dots, z_n \in K$ as follows: $z_k = 0$ for $k \in \gamma 1$, $z_k = 1$ for $k \in \gamma_2$, $z_k = s'$ for $k \in \beta_1 \setminus \gamma_1$ and $z_k = s$ otherwise. Since $0 \sqsubseteq s'$ and $1 \sqsubseteq s$, we have $x_k \sqsubseteq z_k$ for every k, which implies that $f_\beta(s) \sqsubseteq f(z_1, \dots, z_n)$. By 3.1, $f(z_1, \dots, z_n)_i = f_\gamma(s)_i$, hence $f_\beta(s)_i \sqsubseteq f_\gamma(s)_i$. Since $f_\gamma(s)_i \ne a$, this is only possible if $f_\beta(s)_i = f_\gamma(s)_i$ and hence (*) holds.

The proof of (2) is similar. $\square$

3.10 Theorem. *Let $f : K^n \to K$ be a compatible function on a Kleene algebra K. Let $s \in K^\vee$ be such that f preserves $\sqsubseteq_s$. Suppose that the constants k_α ($\alpha \in P$) satisfy the following conditions:*

(i) *if $\alpha \in P_0 \cap P_n$ then $k_\alpha = f_\alpha$;*
(ii) *if $\alpha, \gamma \in P$, $\gamma_1 \cap \gamma_2 = \alpha_1 \cap \gamma_2 = \gamma_1 \cap \alpha_2 = \emptyset$ then $f_\beta(s) \le k_\alpha \vee s$ for some $\beta \in P_0$ with $\gamma \le \beta$;*
(iii) *if $\alpha \in P_0$ then $k_\delta \le f_\alpha(s) \vee s$, where $\delta = (\underline{n} \setminus \alpha_2, \underline{n} \setminus \alpha_1)$;*
(iv) *if $\alpha, \gamma \in P_0$, $\alpha \le \gamma$, then $k_\gamma \wedge s' \le f_\gamma(s) \le k_\alpha$.*

Then f coincides with the polynomial $\bigwedge_{\alpha \in P}(k_\alpha \vee C_\alpha)$ on the set $\{0, s, 1\}^n$.

Proof. Let $\overline{x} = (x_1, \dots, x_n) \in \{0, s, 1\}^n$. Without loss of generality, $x_1 = \cdots = x_k = 0$, $x_{k+1} = \cdots = x_l = s$, $x_{l+1} = \cdots = x_n = 1$.

First we prove that $f(\overline{x}) \le k_\alpha \vee C_\alpha$ for every $\alpha \in P$. If $\alpha_1 \nsubseteq \underline{l}$ or $\alpha_2 \nsubseteq \underline{n} \setminus \underline{k}$ then $C_\alpha = 1$ and the statement is trivial. Let us assume $\alpha_1 \subseteq \underline{l}$ and $\alpha_2 \subseteq \underline{n} \setminus \underline{k}$. We distinguish two cases.

1. Let $\alpha_1 \cap \alpha_2 = \emptyset$. We set $\gamma_1 = \alpha_1 \cup \underline{k}$, $\gamma_2 = \alpha_2 \cup (\underline{n} \setminus \underline{l})$. Then, by (ii), $f_\beta(s) \le k_\alpha \vee s$ for some β with $\beta_1 \supseteq \gamma_1$, $\beta_2 \supseteq \gamma_2$. By 3.8 then also $f(\overline{x}) \le k_\alpha \vee s$. Now, if $\alpha_1 \nsubseteq \underline{k}$ or $\alpha_2 \nsubseteq \underline{n} \setminus \underline{l}$ then $C_\alpha = s$ and therefore $f(\overline{x}) \le k_\alpha \vee C_\alpha$. If $\alpha_1 \subseteq \underline{k}$ and $\alpha_2 \subseteq \underline{n} \setminus \underline{l}$ then $f(\overline{x}) = f_\gamma(s) \le k_\alpha$ by (iv).

2. Let $\alpha_1 \cup \alpha_2 = \underline{n}$. Necessarily $\underline{k} \subseteq \alpha_1$, $\underline{n} \setminus \underline{l} \subseteq \alpha_2$.We set $\gamma = (\alpha_1 \setminus \alpha_2, \alpha_2 \setminus \alpha_1)$ and repeat the argument from the previous case. (If $k = l$, we use (i), otherwise $C_\alpha = s$.)

Now we prove the inverse inequality. We have to show that $f(\overline{x})_i \ge \bigwedge_{\alpha \in P}(k_\alpha \vee C_\alpha)_i$ for every $i \in I$. We discuss two cases.

1. Suppose that $s_i = 1$. We set $\gamma = (\underline{k}, \underline{n} \setminus \underline{k})$. Then $f(\overline{x})_i = f_\gamma(s)_i = (k_\gamma)_i = (k_\gamma \vee C_\gamma)_i$, because either $C_\gamma = 0$ (if $k = l$) or $C_\gamma = s'$ (if $k < l$).

2. Suppose that $s_i = a$. Then also $s'_i = a$. Let $\gamma = (\underline{k}, \underline{n} \setminus \underline{l})$, $\delta = (\underline{l}, \underline{n} \setminus \underline{k})$. We claim that $f(\overline{x})_i \geq (k_\gamma \vee C_\gamma)_i$ or $f(\overline{x})_i \geq (k_\delta \vee C_\delta)_i$. Clearly, $C_\gamma = 0$ and $C_\delta \in \{0, s \vee s'\}$, hence $(C_\delta)_i \leq a$. If $f(\overline{x})_i = 0$ then, by (iv), $(k_\gamma)_i \wedge a \leq f_\gamma(s)_i = f(\overline{x})_i$, which implies that $(k_\gamma)_i = 0$ and therefore $f(\overline{x})_i \geq (k_\gamma \vee C_\gamma)_i$. If $f(\overline{x})_i = a$ then, by (iii), $k_\delta \leq f_\gamma(s) \vee s = f(\overline{x}) \vee s$, hence $(k_\delta)_i \leq f(\overline{x})_i \vee a = a$, which implies that $f(\overline{x})_i \geq (k_\delta \vee C_\delta)_i$. □

4. Local polynomials of Kleene algebras

In this section we describe local polynomials of Kleene algebras, which leads to a characterization of local affine completeness.

4.1 Theorem. *Let f be an n-ary compatible function on a Kleene algebra K. Then the following are equivalent:*

(1) *f is a local polynomial function of K;*
(2) *f preserves the uncertainity order of K ;*
(3) *f can be interpolated by a polynomial on K^s for every $s \in K^\vee$.*

Proof. The implication (1) $\Longrightarrow$ (2) is obvious and (3) $\Longrightarrow$ (1) follows from the fact every finite subset of K is contained in some K^s , $s \in K^\vee$. Im remains to prove (2) $\Longrightarrow$ (3).

By 3.2 it suffices to interpolate f on the set $\{0, s, 1\}$. Keeping in mind the general form from Lemma 2.1 we define $p(x_1, \dots, x_n)$ as a meet of elementary polynomials $k_\alpha \vee C_\alpha$ where the coefficients k_α are calculated by the following rule:

(1) if $\alpha_1 \cup \alpha_2 = \underline{n}$ then $k_\alpha = f(x_1, \dots, x_n)$ where

$$x_j = \begin{cases} 0 & \text{if} \quad j \in \alpha_1 \setminus \alpha_2, \\ 1 & \text{if} \quad j \in \alpha_2 \setminus \alpha_1, \\ s & \text{otherwise;} \end{cases}$$

(2) if $\alpha_1 \cup \alpha_2 \neq \underline{n}$ then k_α is the join of all elements $f(x_1, \dots, x_n)$ where $x_j \in \{0, s, 1\}$ and

$$x_j = \begin{cases} 0 & \text{if} \quad j \in \alpha_1, \\ 1 & \text{if} \quad j \in \alpha_2. \end{cases}$$

Obviously, it suffices to prove that the elements k_α fulfil the conditions from 3.10. Of them, (i) is obvious. Now, let $\alpha, \gamma \in P$ satisfy the assumptions of (ii). If $\alpha_1 \cup \alpha_2 = \underline{n}$, we set $\beta = (\alpha_1 \setminus \alpha_2, \alpha_2 \setminus \alpha_1)$ and clearly $k_\alpha = f_\beta(s)$. If $\alpha_1 \cup \alpha_2 \neq \underline{n}$, we set $\beta = (\alpha_1 \cup \gamma_1, \alpha_2 \cup \gamma_2)$ and again $k_\alpha \geq f_\beta(s)$. To show (iii), it is easy to see that $\delta_1 \cup \delta_2 = \underline{n}$ and $k_\delta = f_\alpha(s)$ is the definition of k_δ. In the condition (iv), the inequality $f_\gamma(s) \leq k_\alpha$ is clear by the definition of k_α. To prove the inequality $k_\gamma \wedge s' \leq f_\gamma(s)$ we need to show that $f_\beta(s) \wedge s' \leq f_\gamma(s)$ whenever $\beta \in P$ is such that $\gamma_1 \subseteq \beta_1$ and $\gamma_2 \subseteq \beta_2$. This was proved in 3.9. □

4.2 Theorem. *Let K be a Kleene algebra. The following conditions are equivalent:*

(1) *K is locally affine complete;*
(2) *$K^\vee$ is a locally affine complete lattice;*
(3) *$K^\vee$ does not contain a proper Boolean interval.*

Proof. The conditions (2) and (3) are equivalent by 2.4. The implication (1) $\Longrightarrow$ (2) was proved in 2.2. It remains to show (2) $\Longrightarrow$ (1). (2) implies that compatible function on the lattice $K^\vee$ is order preserving. By 3.6, every compatible function on the Kleene algebra K preserves $\sqsubseteq$. By 4.1, every compatible function on K is a local polynomial. $\square$

4.3 Theorem. *Let K be a Kleene algebra such that $K^\vee$ has a smallest element. The following are equivalent:*

(1) *K is affine complete;*
(2) *$K^\vee$ is an affine complete lattice;*
(3) *$K^\vee$ does not contain a proper Boolean interval.*

Proof. Since $K^\vee$ is now a bounded lattice, (2) and (3) are equivalent by ?.?. Since every affine complete algebra is locally affine complete, the implication (1) $\Longrightarrow$ (3) comes from Theorem 4.2. To prove (3) $\Longrightarrow$ (1) observe that by 4.2 the algebra K is locally affine complete and hence every compatible function can be interpolated by a polynomial on any K^s. In our case, however, $K = K^s$ where s is the smallest element of $K^\vee$. $\square$

5. Affine completeness

In this section we prove the converse to Lemma 2.1. We start with a lemma which gives a key for using the conditions $(\mathcal{F})$ and $(\mathcal{I})$ relative to $K^\vee$ and $K^\wedge$, respectively. If Y is a subset of an ordered set X, we denote $\uparrow Y = \{x \in X \mid x \geq y \text{ for some } y \in Y\}$ (and dually for $\downarrow Y$).

5.1 Lemma. *Let $f : K \longrightarrow K$ be a local polynomial function of a Kleene algebra K. Then $K^\vee \cap \uparrow f(K^\vee)$ is an almost principal filter in $K^\vee$ and $K^\wedge \cap \downarrow f(K^\wedge)$ is an almost principal ideal in $K^\wedge$.*

Proof. We prove the first statement. Let $F = K^\vee \cap \uparrow f(K^\vee) = \{x \in K^\vee \mid f(z) \leq x \text{ for some } z \in K^\vee\}$. We claim that, for $x \in K^\vee$,

$$x \vee f(x) = \min\{y \in F \mid x \leq y\}.$$

It is clear that $x \leq x \vee f(x) \in F$. Conversely, let $x \leq y \in F$. Then $y \geq f(z)$ for some $z \in K^\vee$. By Corollary 3.8, $f(x) \leq x \vee f(z)$ and hence $x \vee f(x) \leq x \vee f(z) \leq y$.

To see that F is closed under meets, consider $x, y \in F$, $z = x \wedge y$, $t = \min\{u \in F \mid z \leq u\}$. Clearly, $z \leq t \leq x$, $t \leq y$, hence $z = t \in F$.

The other statement can be proved similarly. $\square$

5.2 Theorem. *Let K be a Kleene algebra. The following conditions are equivalent:*

(1) *K is affine complete;*
(2) *$K^\vee$ is affine complete in K;*
(3) *$K^\wedge$ is affine complete in K;*
(4) *$K^\vee$ does not contain proper Boolean intervals and for every proper almost principal filter F in $K^\vee$ there exists $b \in K$ such that $F = K^\vee \cap \uparrow b$.*
(5) *$K^\wedge$ does not contain proper Boolean intervals and for every proper almost principal ideal I in $K^\wedge$ there exists $c \in K$ such that $F = K^\wedge \cap \downarrow c$.*

Proof. The existence of the antiautomorphism $'$ for the lattice K yields that the conditions (2) and (3) and similarly the conditions (4) and (5) are equivalent. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (4) follow from Lemma 2.1 and Theorem 2.3. So we have to prove only the implication (4) $\Longrightarrow$ (1). Let K be a Kleene algebra satisfying (4) and $f : K^n \longrightarrow K$ a compatible function. Since $K^\vee$ has no Boolean intervals, by Theorem 4.3 f is a local polynomial of K and preserves $\sqsubseteq$.

Let us consider the functions f_α defined in Chapter 3. It is clear these functions are local polynomials and preserve $\sqsubseteq$. Therefore, by Lemma 5.1 the sets $K^\vee \cap \uparrow f_\alpha(K^\vee)$ and $K^\wedge \cap \downarrow f_\alpha(K^\wedge)$ are an almost principal filter and an almost principal ideal of $K^\vee$ and $K^\wedge$, respectively. Hence, by (4) and (5) we have constants $b_\alpha, c_\alpha \in K$ such that

$$K^\vee \cap \uparrow f_\alpha(K^\vee) = K^\vee \cap \uparrow b_\alpha \quad \text{and} \quad K^\wedge \cap \downarrow f_\alpha(K^\wedge) = K^\wedge \cap \downarrow c_\alpha \,.$$

Let us agree that in case $\alpha_1 \cup \alpha_2 = \underline{n}$ when the function f_α is constant, both b_α and c_α are equal to this constant value of f_α.

It is not difficult to see that $f_\alpha(s) \vee s = b_\alpha \vee s$ and $f_\alpha(s') \wedge s' = c_\alpha \wedge s'$ holds for every $s \in K^\vee$.

Consider the polynomial

$$p(x_1, \ldots, x_n) = \bigwedge_{\alpha \in P} (k_\alpha \vee C_\alpha)$$

with constants k_α defined as follows:

(1) if $\alpha_1 \cup \alpha_2 = \underline{n}$ then $k_\alpha = b_{(\alpha_1 \setminus \alpha_2, \alpha_2 \setminus \alpha_1)}$;
(2) if $\alpha_1 \cup \alpha_2 \neq \underline{n}$ then k_α is the join of c_α and all elements of the form $f(x_1, \ldots, x_n)$ where $x_j \in \{0, 1\}$ and

$$x_j = \begin{cases} 0 & \text{if} \quad j \in \alpha_1; \\ 1 & \text{if} \quad j \in \alpha_2. \end{cases}$$

We are going to prove that these constants satisfy the conditions of 3.10. (for every $s \in K^\vee$).

(i) is obvious. Suppose now that $\alpha, \gamma \in P$ satisfy the assumptions of (ii). If $\alpha \in P_n$, we set $\beta = (\alpha_1 \setminus \alpha_2, \alpha_2 \setminus \alpha_1)$, hence $k_\alpha = b_\beta$ and therefore $f_\beta(s) \leq f_\beta(s) \vee s = b_\beta \vee s = k_\alpha \vee s$. If $\alpha \notin P_n$, we can find $\beta \in P_0 \cap P_n$ such that $\alpha \leq \beta$, $\gamma \leq \beta$ and then $f_\beta(s) \leq K_\alpha$. To show (iii), it is easy to see that $\delta \in P_n$ and $k_\delta = b_\alpha \leq f_\alpha(s) \vee s$.

Finally, assume that α and γ satisfy the assumptions of (iv). The inequality $(c_\gamma \wedge s')_i \leq f_\gamma(s)_i$ is clear for $i \in I$ with $s_i = 1$. If $s_i = a$ then $f_\gamma(s)_i = f_\gamma(s')_i$ and hence $(c_\gamma \wedge s')_i \leq f_\gamma(s)_i$. This shows that $c_\gamma \wedge s' \leq f_\gamma(s)$. Further, if $\beta \in P_0 \cap P_n$, $\beta \geq \gamma$, then $f_\beta(s) \wedge s' \leq f_\gamma(s)$ by 3.9. Thus, we have proved the inequality $k_\gamma \wedge s' \leq f_\gamma(s)$. It remains to prove that $f_\gamma(s) \leq k_\alpha$. We set $\beta = (\gamma_1, \underline{n} \setminus \gamma_1)$. If, for $i \in I$, $s_i = 1$ then $f_\gamma(s)_i = f_\gamma(1)_i = (f_\beta)_i \leq (k_\alpha)_i$. Suppose now that $s_i = a$. The case $f_\gamma(s)_i = 0$ is trivial. If $f_\gamma(s)_i = 1$ then $(k_\alpha)_i \geq (f_\beta)_i = f_\gamma(1)_i = 1$ because $1 \sqsubseteq s$ and f_γ preserves $\sqsubseteq$. If $f_\gamma(s)_i = a$ then also $f_\gamma(s') = a = s'_i$ and, by 3.9, $f_\alpha(s')_i \geq f_\gamma(s')_i \wedge s'_i = a$. Since $c_\alpha \geq f_\alpha(s') \wedge s'$, we obtain that $(k_\alpha)_i \geq (c_\alpha)_i \geq a = f_\gamma(s)_i$. This completes the proof. $\square$

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