AFFINE COMPLETIONS OF DISTRIBUTIVE LATTICES

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Abstract. For any distributive lattice L we construct its extension $\mathfrak{F}(\mathfrak{I}(L))$ with the property that every isotone compatible function on L can be interpolated by a polynomial of $\mathfrak{F}(\mathfrak{I}(L)$. Further, we characterize all extensions with this property and show that our construction is in some sense the simplest possible.

Introduction

Let A be a subalgebra of an algebra B. A function $f : A^n \longrightarrow A$ is called compatible if, for every congruence θ on A, $(x_1, y_1) \in \theta, \dots, (x_n, y_n) \in \theta$ implies $(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \in \theta$. We say that A is affine complete in B if every compatible function on A can be interpolated by a polynomial of B. Note that the converse of this property is always true, provided that we consider algebras with the Congruence extension property (like in the case of distributive lattices): every function on A, which is representable by a polynomial of B, must be compatible.

This concept is a generalization of the usual (absolute) affine completeness: an algebra A is affine complete iff it is affine complete in itself.

There are several sources of motivation for this paper. The first one is the following characterization of affine complete distributive lattices.

1.1. Theorem([8]). *A distributive lattice L is affine complete iff the following conditions hold:*

(i) *L does not contain a nontrivial Boolean interval;*
(ii) *every proper almost principal ideal in L is principal;*
(iii) *every proper almost principal filter in L is principal.*

An ideal I in L is called almost principal if $I \cap \downarrow x$ is principal for every $x \in L$. We use the denotation $\downarrow x = \{y \in L \mid y \leq x\}$ and dually for $\uparrow x$. Almost principal filters are defined analogously. An ideal (or filter) I is called proper if $I \neq L$. A Boolean interval is called nontrivial if it has at least two elements.

If the condition (i) fails then L has a unary compatible function that is not isotone (order-preserving). It is clear that such a function cannot be interpolated by any lattice polynomial.

If (i) is satisfied but (ii) or (iii) fails, the situation is different. By [2], (i) implies that every compatible function is isotone. (In [3] this was proved for bounded distributive lattices.) The reason why L is not affine complete is that it lacks some

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suitable constants. Therefore, in this case one can hope to find a lattice M such that L is affine complete in M. We call this an affine completion of L and we present a canonical way how to construct such a completion.

Our construction is motivated by the paper [4] of Grätzer and Schmidt. They considered L embedded in the ideal lattice $I(L)$ and the filter lattice $F(L)$. Their conjecture was that if (i) is satisfied then every compatible function on L can be obtained by a composition of polynomials of $I(L)$ and $F(L)$. This conjecture is true for unary functions but fails for functions of higher arities. Nevertheless, investigating the polynomials of $I(L)$ and $F(L)$ has proved fruitful in our paper.

After constructing the "canonical" affine completion, we consider the question of other possible completions. The problem is as follows. Let L be a sublattice of a distributive lattice M. Under which conditions is L affine complete in M ? Our original conjecture was that the canonical completion of L should be embeddable in any such M. This conjecture is wrong but we shall show that it is not far from the truth either.

A special case of the above problem was solved in [6] in connection with investigations of affine complete Stone algebras. If L is a Stone algebra then $D(L)$ denotes its filter of all dense elements.

1.2. Theorem([6])**.** *Let L be a Stone algebra. The following statements are equivalent:*

(1) *L is affine complete;*
(2) *$D(L)$ is affine complete (as a lattice) in L;*
(3) *the following conditions hold:*
  (B) *$D(L)$ does not contain a nontrivial Boolean interval;*
  (F) *for every almost principal filter F in $D(L)$ there is $a \in L$ such that $F = D(L) \cap \uparrow a$.*

The equivalence of (2) and (3) solves our problem in a very special case when L forms a filter in M. We shall show that, in general, (F) must be replaced by another (more complicated) condition.

Finally, we would like to mention that some ideas appearing in this paper stem also from the papers [1],[3], [5], [7] and [9].

2. Isotone compatible functions

For bounded distributive lattices, the situation is clear by Grätzer's paper [3]. A bounded distributive lattice is affine complete if and only if it does not contain a nontrivial Boolean interval. Actually, more is proved in [3]. We shall need the following assertion.

2.1. Lemma([3], Corollary 1**).** *Let L be a bounded distributive lattice. Then every isotone compatible function on L is a polynomial of L.*

Now we shall show some basic properties of compatible functions on (possibly) unbounded distributive lattices.

2.2. Lemma. *Let $f : L \longrightarrow L$ be a unary isotone compatible function on a distributive lattice L. Then*

(i) *f is idempotent*
(ii) *the set $\{x \in L \mid x \leq f(y)$ for some $y \in L\}$ is an almost principal ideal.*

Proof. Let $x \in L$. We claim that $f(x) \leq x \vee f(x \wedge f(x))$. For contradiction, suppose that $f(x) \not\leq x \vee f(x \wedge f(x))$. Then there is a prime ideal P in L such that $x \vee f(x \wedge f(x)) \in P$ and $f(x) \notin P$. Let θ be the congruence on L having P and $L \setminus P$ as its equivalence classes. Then $(x, x \wedge f(x)) \in \theta$ (since both x and $x \wedge f(x)$ belong to P), but $(f(x), f(x \wedge f(x))) \notin \theta$, a contradiction with the compatibility of f.

Hence, $f(x) \leq x \vee f(x \wedge f(x))$. Since f is isotone, we obtain that $f(x) \leq x \vee f^2(x)$, where $f^2(x)$ means $f(f(x))$. Symmetrically we can prove that $f(x) \geq x \wedge f^2(x)$.

By a similar method one can prove that $x \wedge f(x) \leq f^2(x) \leq x \vee f(x)$. (It is a consequence of [8], Lemma 2.4.) We obtain that $x \wedge f(x) \leq x \wedge f^2(x) \leq x \wedge f(x)$, hence $x \wedge f(x) = x \wedge f^2(x)$, and similarly, $x \vee f(x) = x \vee f^2(x)$. Therefore, $f^2(x) = f^2(x) \vee (x \wedge f(x)) = (f^2(x) \vee x) \wedge (f^2(x) \vee f(x)) = (x \vee f(x)) \wedge (f^2(x) \vee f(x)) = (x \wedge f^2(x)) \vee f(x) = (x \wedge f(x)) \vee f(x) = f(x)$.

The proof of (ii) is the same as the proof of 2.6(iii) in [8]. $\square$

2.3. Lemma. *Let $f : L^n \longrightarrow L$ be an isotone compatible function on a distributive lattice L. Let $M \subseteq \{1, \dots, n\}$. Then for every $x_1, \dots, x_n \in L$ the equality $f(x_1, \dots, x_n) = f(z_1, \dots, z_n)$ holds, where*

$$z_i = \begin{cases} x_i & \text{if} \quad i \notin M, \\ f(x_1, \dots, x_n) & \text{if} \quad i \in M. \end{cases}$$

Proof. We proceed by induction on the cardinality of M. If $M = \emptyset$, the statement is trivial. Suppose now that $|M| \geq 1$ and choose $j \in M$. Let $N = M \setminus \{j\}$. Let us consider the unary function $g : L \longrightarrow L$ defined by $g(y) = f(y_1, \dots, y_n)$, where

$$y_i = \begin{cases} x_i & \text{if} \quad i \notin M, \\ f(x_1, \dots, x_n) & \text{if} \quad i \in N, \\ y & \text{if} \quad i = j. \end{cases}$$

It is clear that g is isotone and compatible. By 2.2(i), it is idempotent, hence $g(x_j) = g(g(x_j))$. Since $g(x_j) = f(t_1, \dots, t_n)$, where

$$t_i = \begin{cases} x_i & \text{if} \quad i \notin N, \\ f(x_1, \dots, x_n) & \text{if} \quad i \in N, \end{cases}$$

the induction hypothesis yields that $g(x_j) = f(x_1, \dots, x_n)$. Hence, $f(x_1, \dots, x_n) = g(f(x_1, \dots, x_n))$ and, obviously, $g(f(x_1, \dots, x_n)) = f(z_1, \dots, z_n)$. $\square$

2.4. Lemma. *Let $f : L^n \longrightarrow L$ be an isotone compatible function. Let $M \subseteq \{1, \dots, n\}$. For every $x_1, \dots, x_n, y_1, \dots, y_n \in L$ the following holds true:*

$$f(x_1, \dots, x_n) \leq f(z_1, \dots, z_n) \vee \bigvee_{i \in M} x_i,$$

$$f(x_1, \dots, x_n) \geq f(z_1, \dots, z_n) \wedge \bigwedge_{i \in M} x_i,$$

where

$$z_i = \begin{cases} x_i & \text{if} \quad i \notin M, \\ y_i & \text{if} \quad i \in M. \end{cases}$$

Proof. We prove the first inequality. For contradiction, assume that $f(x_1,\dots,x_n)\not\leq f(z_1,\dots,z_n)\vee\bigvee_{i\in M}x_i$. Then there is a prime ideal P such that $f(x_1,\dots,x_n)\notin P$ and $f(z_1,\dots,z_n)\vee\bigvee_{i\in M}x_i\in P$. Let θ be the congruence on L whose equivalence classes are P and $L\setminus P$.

Since $f(z_1,\dots,z_n)\in P$ and $x_i\in P$ for every $i\in M$, we have $(f(z_1,\dots,z_n),x_i)\in\theta$ for every $i\in M$. The compatibility of f yields that

$$(f(x_1,\dots,x_n),f(u_1,\dots,u_n))\in\theta,$$

where

$$u_i=\begin{cases}x_i\ (=z_i) & \text{if}\quad i\notin M,\\ f(z_1,\dots,z_n) & \text{if}\quad i\in M.\end{cases}$$

By 2.3, $f(u_1,\dots,u_n)=f(z_1,\dots,z_n)$ and hence

$$(f(x_1,\dots,x_n),f(z_1,\dots,z_n))\in\theta,$$

a contradiction. $\square$

3. The canonical affine completion

If I is an almost principal ideal in a distributive lattice L, then for every $x\in L$ the ideal $I\cap\downarrow x$ has a greatest element. We denote this element by x_I. Similarly, if F is an almost principal filter, then $F\cap\uparrow x=\uparrow x^F$. It is easy to prove (see [4]) that the set $\mathfrak{I}(L)$ of all almost principal ideals of L forms a sublattice of the lattice $I(L)$ of all ideals of L. We have the canonical embedding $x\mapsto\downarrow x$ of L into $\mathfrak{I}(L)$.

Similarly, we consider the lattice $\mathfrak{F}(L)$ of all almost principal filters of L. Note that $\mathfrak{F}(L)$ is ordered by the inverse inclusion, i.e. $F_1\leq F_2$ iff $F_1\supseteq F_2$. The canonical embedding $L\longrightarrow\mathfrak{F}(L)$ is given by $x\mapsto\uparrow x$.

The lattice $\mathfrak{I}(L)$ always has a greatest element, since L itself is regarded as an almost principal ideal. Similarly, L is the least element of $\mathfrak{F}(L)$.

3.1. Lemma. *Every distributive lattice L is an ideal in $\mathfrak{I}(L)$. (More precisely, the set $\{\downarrow x\mid x\in L\}$ is an ideal in $\mathfrak{I}(L)$.) Similarly, L is a filter in $\mathfrak{F}(L)$.*

Proof. It is clear that the set $\{\downarrow x\mid x\in L\}$ is closed under joins. Further, let $I\in\mathfrak{I}(L)$, $I\subseteq\downarrow x$ for some $x\in L$. Then $I\cap\downarrow x$ is a principal ideal, but $I\cap\downarrow x=I$, hence $I\in\{\downarrow x\mid x\in L\}$. The proof for filters is similar. $\square$

3.2. Lemma. *For every distributive lattice L, every $I,J\in\mathfrak{I}(L)$ and every $x,y\in L$, the following equalities hold:*

$$x_I\wedge y_J=(x\wedge y)_{I\wedge J},\quad x_I\vee x_J=x_{I\vee J},\quad x_I\vee y_I=(x\vee y)_I.$$

Proof. I. Clearly, $x\wedge y\geq x_I\wedge y_J\in I\wedge J$. Further, if $t\in(I\wedge J)\cap\downarrow(x\wedge y)$, then $x\geq t\in I$ and $y\geq t\in J$. Hence, $t\leq x_I$ and $t\leq y_J$ and therefore $t\leq x_I\wedge y_J$.

II. Clearly, $x_I\vee x_J\in(I\vee J)\cap\downarrow x$. Further, if t is any element of $(I\vee J)\cap\downarrow x$ then $t\leq t_1\vee t_2$ for some $t_1\in I$, $t_2\in J$ and hence $t\leq x\wedge(t_1\vee t_2)=(x\wedge t_1)\vee(x\wedge t_2)\leq x_I\vee x_J$.

III. Clearly, $x_I\vee y_I\in I\cap\downarrow(x\vee y)$. Further, if t is any element of $I\cap\downarrow(x\vee y)$ then $t=t\wedge(x\vee y)=(t\wedge x)\vee(t\wedge y)\leq x_I\vee y_I$. $\square$

Now we are going to prove that every isotone compatible function on L can be extended to an isotone compatible function on $\mathfrak{I}(L)$. So, let us assume that $f:\ L^n \longrightarrow L$ is compatible. We wish to extend it to a function $\overline{f}:\ \mathfrak{I}(L)^n \longrightarrow \mathfrak{I}(L)$. Let $I_1, \dots, I_n \in \mathfrak{I}(L)$. For any $M \subseteq \{1, \dots, n\}$ we set

$$\begin{aligned} I(M) = \big\{ y \in L \mid y \leq f(x_1, \dots, x_n) \vee \bigvee_{i \in M} x_i \text{ for some } x_1, \dots, x_n \in L \text{ such} \\ \text{that } x_i \in I_i \text{ for all } i \in M \big\}. \end{aligned}$$

3.3. Lemma. *I(M) is an almost principal ideal in L.*

Proof. We prove that, for every $y \in L$,

$$y_{I(M)} = \bigvee_{i \in M} y_{I_i} \vee (f(z_1, \dots, z_n) \wedge y),$$

where

$$z_i = \begin{cases} y & \text{if} \quad i \notin M, \\ y_{I_i} & \text{if} \quad i \in M. \end{cases}$$

It is clear that such $y_{I(M)}$ belongs to $I(M)$ and $y_{I(M)} \leq y$. We need to show that $y_{I(M)}$ is the greatest element of L having these properties.

So, let $u \in I(M)$, $u \leq y$. Hence,

$$u \leq \bigvee_{i \in M} x_i \vee f(x_1, \dots, x_n)$$

for suitable $x_1, \dots, x_n \in L$ such that $x_i \in I_i$ whenever $i \in M$. Then

$$u \leq \bigvee_{i \in M} (x_i \wedge y) \vee (f(x_1, \dots, x_n) \wedge y).$$

Obviously, for $i \in M$ we have $x_i \wedge y \leq y_{I_i} \leq y_{I(M)}$. It remains to show that $f(x_1, \dots, x_n) \wedge y \leq y_{I(M)}$. According to 2.4, $f(t_1, \dots, t_n) \geq f(x_1, \dots, x_n) \wedge y$, where

$$t_i = \begin{cases} y & \text{if} \quad i \notin M, \\ x_i & \text{if} \quad i \in M. \end{cases}$$

Hence, $f(x_1, \dots, x_n) \wedge y \leq f(t_1, \dots, t_n) \wedge y$ and, by 2.4 again, $f(t_1, \dots, t_n) \leq f(z_1, \dots, z_n) \vee \bigvee_{i \in M} x_i$. We obtain that

$$\begin{aligned} f(x_1, \dots, x_n) \wedge y &\leq \big(f(z_1, \dots, z_n) \vee \bigvee_{i \in M} x_i\big) \wedge y \\ &\leq \big(f(z_1, \dots, z_n) \wedge y\big) \vee \bigvee_{i \in M} y_{I_i} = y_{I(M)}. \end{aligned}$$

Thus, we have proved that $I(M) \cap \downarrow y$ is a principal ideal for every $y \in L$. This implies that $I(M)$ is closed under joins. Hence, $I(M)$ is an almost principal ideal. □

Let us define

$$\overline{f}(I_1, \dots, I_n) = \bigcap \big\{ I(M) \mid M \subseteq \{1, \dots, n\} \big\}.$$

Since all $I(M)$'s are almost principal ideals, $\overline{f}(I_1, \dots, I_n)$ is an almost principal ideal too. Hence, we have defined a function

$$\overline{f}:\ \mathfrak{I}(L)^n \longrightarrow \mathfrak{I}(L).$$

It is clear that $\overline{f}$ is isotone.

3.4. Lemma. $\overline{f}$ *is a compatible function on* $\mathfrak{I}(L)$.

Proof. Let θ be a congruence on $\mathfrak{I}(L)$, let $I_1, \dots, I_n, J_1, \dots, J_n \in \mathfrak{I}(L)$ and $(I_1, J_1) \in \theta, \dots, (I_n, J_n) \in \theta$. We have to prove that $(\overline{f}(I_1, \dots, I_n), \overline{f}(J_1, \dots, J_n)) \in \theta$. We can restrict ourselves to the case $I_1 \subseteq J_1, \dots, I_n \subseteq J_n$. (Indeed, we could consider the ideals $K_i = I_i \cap J_i$ and prove that $(\overline{f}(K_1, \dots, K_n), \overline{f}(I_1, \dots, I_n)) \in \theta$ and $(\overline{f}(K_1, \dots, K_n), \overline{f}(J_1, \dots, J_n)) \in \theta$.

Let $I(M)$, $M \subseteq \{1, \dots, n\}$ be the ideals defined above for $I_1, \dots, I_n$. Let $J(M)$ be similar ideals, defined by using $J_1, \dots, J_n$ instead of $I_1, \dots, I_n$. We claim that

$$J(M) = I(M) \vee \bigvee_{i \in M} J_i.$$

Obviously, $I(M) \subseteq J(M)$ and also $\bigvee_{i \in M} J_i \subseteq J(M)$. To prove the other inclusion, let $t \in J(M)$, hence

$$t \leq f(x_1, \dots, x_n) \vee \bigvee_{i \in M} x_i,$$

where $x_1, \dots, x_n \in L$ are such that $x_i \in J_i$ whenever $i \in M$. Choose arbitrary elements $y_i \in I_i$ $(i \in M)$. By 2.4, $f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n) \vee \bigvee_{i \in M} x_i$, where $y_i = x_i$ for $i \notin M$. Hence,

$$t \leq f(y_1, \dots, y_n) \vee \bigvee_{i \in M} x_i.$$

Since

$$f(y_1, \dots, y_n) \leq f(y_1, \dots, y_n) \vee \bigvee_{i \in M} y_i \in I(M)$$

and

$$\bigvee_{i \in M} x_i \in \bigvee_{i \in M} J_i,$$

we obtain that $t \in I(M) \vee \bigvee_{i \in M} J_i$.

Thus, we have $J(M) = I(M) \vee \bigvee_{i \in M} J_i$. Since $(I_i, J_i) \in \theta$ for every i, we have

$$(I(M), J(M)) = (I(M) \vee \bigvee_{i \in M} I_i,\ I(M) \vee \bigvee_{i \in M} J_i) \in \theta,$$

which implies that

$$(\overline{f}(I_1, \dots, I_n), \overline{f}(J_1, \dots, J_n)) = (\bigwedge_{M \subseteq \underline{n}} I(M), \ \bigwedge_{M \subseteq \underline{n}} J(M)) \in \theta.$$

(Here and in the sequel $\underline{n}$ stands for $\{1, \dots, n\}$.) $\square$

3.5. Lemma. *For every* $x_1, \dots, x_n \in L$*, the following holds:*

$$\overline{f}(\downarrow x_1, \dots, \downarrow x_n) = \downarrow f(x_1, \dots, x_n).$$

(In other words: $\overline{f}$ *can be regarded as an extension of* f*.)*

Proof. Obviously, $f(x_1,\dots,x_n)\in I(M)$ for every $M\subseteq\{1,\dots,n\}$, hence $\downarrow f(x_1,\dots,x_n)\subseteq \overline{f}(\downarrow x_1,\dots,\downarrow x_n)$. (It is understood that, in the definition of $I(M)$, the ideals $\downarrow x_1,\dots,\downarrow x_n$ play the role of $I_1,\dots,I_n$.)

Suppose now that $y\in\bigcap_{M\subseteq\underline{n}} I(M)=\overline{f}(\downarrow x_1,\dots,\downarrow x_n)$. We show that for every $M\subseteq\{1,\dots,n\}$ there exist $y_1,\dots,y_n\in L$ such that $y\le f(y_1,\dots,y_n)$ and $y_i=x_i$ for every $i\in M$.

We proceed by induction on the cardinality of M. Since $y\in I(\emptyset)$, we have $y\le f(y_1,\dots,y_n)$ for some $y_1,\dots,y_n\in L$, which means that the assertion holds for $|M|=0$.

Suppose now that $|M|>0$. From $y\in I(M)$ we get that

$$y\le f(z_1,\dots,z_n)\vee\bigvee_{i\in M} z_i,$$

where $z_i\le x_i$ for every $i\in M$. For every $k\in M$ we consider the set $M_k=M\setminus\{k\}$. By the induction hypothesis for M_k we have $y\le f(y_1^k,\dots,y_n^k)$, where $y_1^k,\dots,y_n^k\in L$ are such that $y_i^k=x_i$ whenever $i\in M_k$. Let us denote

$$w=\bigwedge_{k\in M} f(y_1^k,\dots,y_n^k).$$

Further, let $t_1,\dots,t_n\in L$ be defined by the rule

$$t_i=\begin{cases} x_i & \text{if} \quad i\in M,\\ z_i\vee\bigvee_{k\in M} y_i^k & \text{if} \quad i\notin M.\end{cases}$$

By 2.4 and the isotonicity of f, for every $k\in M$ we get $z_k\wedge w\le z_k\wedge f(y_1^k,\dots,y_n^k)\le f(y_1^k,\dots,y_{k-1}^k,z_k,y_{k+1}^k,\dots,y_n^k)\le f(t_1,\dots,t_n)$ and also $f(z_1,\dots,z_n)\le f(t_1,\dots,t_n)$. Since $y\le w$, we obtain

$$y\le (f(z_1,\dots,z_n)\vee\bigvee_{k\in M} z_k)\wedge w=(f(z_1,\dots,z_n)\wedge w)\vee\bigvee_{k\in M}(z_k\wedge w)\le f(t_1,\dots,t_n).$$

This completes the induction. If we set $M=\{1,\dots,n\}$, we obtain that $y\le f(x_1,\dots,x_n)$, which was to prove. □

Of course, assertions analogous to 3.3–3.5 hold also for almost principal filters. Having this in mind, we can prove the main theorem of this section.

3.6. Theorem. *Let L be a distributive lattice. For a function $f:\ L^n\longrightarrow L$, the following statements are equivalent:*

(1) *f is isotone and compatible;*
(2) *f can be interpolated by a polynomial of $\mathfrak{F}(\mathfrak{I}(L))$.*

Proof. Obviously, every function on L that can be interpolated by a polynomial of $\mathfrak{F}(\mathfrak{I}(L))$ is isotone and compatible. (Distributive lattices have the Congruence extension property.) Conversely, let $f:\ L^n\longrightarrow L$ be an isotone compatible function. By 3.4 and 3.5 we can extend it to an isotone compatible function $\overline{f}:\ \mathfrak{F}(\mathfrak{I}(L))^n\longrightarrow\mathfrak{F}(\mathfrak{I}(L))$. The lattice $\mathfrak{F}(\mathfrak{I}(L))$ is bounded. By 2.1, $\overline{f}$ is a polynomial function of $\mathfrak{F}(\mathfrak{I}(L))$, which is the desired interpolation of f. □

If L does not contain a nontrivial Boolean interval, then every compatible function on L is isotone and hence can be interpolated by a polynomial of $\mathfrak{F}(\mathfrak{I}(L))$.

3.7. Theorem. *Suppose that a distributive lattice L does not contain a nontrivial Boolean interval. Then L is affine complete in $\mathfrak{F}(\mathfrak{I}(L))$.* $\square$

Let us remark that if L does not contain a nontrivial Boolean interval, then $\mathfrak{F}(\mathfrak{I}(L))$ itself is affine complete. Indeed, it is bounded and the following assertion holds.

3.8. Lemma. *If L does not contain a nontrivial Boolean interval, then $\mathfrak{I}(L)$ and $\mathfrak{F}(L)$ do not contain such interval either.*

Proof. Let $[I, J]$ be a Boolean interval in $\mathfrak{I}(L)$, $I \subsetneq J$. Let us choose $x \in J \setminus I$. Consider the interval $[\downarrow x \cap I, \downarrow x]$. This interval is isomorphic to the nontrivial Boolean interval $[I, \downarrow x \vee I]$. On the other hand, by 3.1 it is isomorphic to the interval $[x_I, x]$ of the lattice L. Hence, L contains a nontrivial Boolean interval. $\square$

Hence, we can regard $\mathfrak{F}(\mathfrak{I}(L))$ as an "affine completion" of L. Because of the symmetry, $\mathfrak{I}(\mathfrak{F}(L))$ is another affine completion. Rest of this section is devoted to the proof that these two completions are canonically isomorphic.

So, let L be a distributive lattice. If $\mathcal{U} \in \mathfrak{F}(\mathfrak{I}(L))$, (i.e. $\mathcal{U}$ is a family of almost principal ideals that forms an almost principal filter in $\mathfrak{I}(L)$) and $F \in \mathfrak{F}(L)$, then we denote by $F(\mathcal{U})$ the filter in L generated by the set $\{x_I \mid x \in F,\ I \in \mathcal{U}\}$. Since the set $\{x_I \mid x \in F,\ I \in \mathcal{U}\}$ is (by 3.2) closed under meets, we have

$$F(\mathcal{U}) = \{t \in L \mid t \geq x_I \text{ for some } x \in F,\ I \in \mathcal{U}\}.$$

3.9. Lemma. *$F(\mathcal{U}) \in \mathfrak{F}(L)$.*

Proof. Since $\mathcal{U}$ is almost principal, for every $x \in L$ there exists an ideal $(\downarrow x)^{\mathcal{U}} \in \mathcal{U}$. We claim that $x^{F(\mathcal{U})} = (x^F)_J$, where $J = (\downarrow x)^{\mathcal{U}}$.

Since $\downarrow x \subseteq J$, we have $x \in J$. Since $x \leq x^F$, it follows that $x \leq (x^F)_J$. From the definition we have $(x^F)_J \in F(\mathcal{U})$.

It remains to show that $(x^F)_J$ is the least element in $\uparrow x \cap F(\mathcal{U})$. Let $y \in F(\mathcal{U})$, $x \leq y$. Then $y \geq z_I$ for some $z \in F$, $I \in \mathcal{U}$. Obviously, $J \subseteq I \vee \downarrow x$, hence
$(x^F)_J \leq (x^F)_{I \vee \downarrow x} = (x^F)_I \vee (x^F)_{\downarrow x} = (x^F)_I \vee x$.
Further $x^F \leq z \vee x$, hence $(x^F)_I \leq (z \vee x)_I = z_I \vee x_I$. We obtain that $(x^F)_J \leq z_I \vee x_I \vee x = z_I \vee x \leq y$. $\square$

Let us denote

$$\varphi(\mathcal{U}) = \{F \in \mathfrak{F}(L) \mid F \cap I \neq \emptyset \text{ for every } I \in \mathcal{U}\}.$$

3.10. Lemma. *$\varphi(\mathcal{U})$ is an almost principal ideal in $\mathfrak{F}(L)$.*

Proof. It is easy to see that $\varphi(\mathcal{U})$ is an ideal in $\mathfrak{F}(L)$. (Recall that $\mathfrak{F}(L)$ is ordered by the inverse inclusion.) Let $F \in \mathfrak{F}(L)$. We claim that $F_{\varphi(\mathcal{U})} = F(\mathcal{U})$.

Let $I \in \mathcal{U}$. Choose $x \in F$ arbitrarily. Then $x_I \in F(\mathcal{U}) \cap I$. We have proved that $F(\mathcal{U}) \in \varphi(\mathcal{U})$.

Let $x \in F$. Choose $I \in \mathcal{U}$ arbitrarily. Then $x \geq x_I \in F(\mathcal{U})$, hence $x \in F(\mathcal{U})$. We have proved that $F \subseteq F(\mathcal{U})$.

Now, let $F \subseteq G \in \varphi(\mathcal{U})$. We need to show that $F(\mathcal{U}) \subseteq G$. It suffices to prove that $x_I \in G$ for every $x \in F$, $I \in \mathcal{U}$. Since $x \in F$, obviously $x \in G$. Since $G \in \varphi(\mathcal{U})$, there exists $z \in G \cap I$. Then $x \wedge z \in G$ and since $x \wedge z \leq x_I$, we obtain that $x_I \in G$.

3.11. Theorem. *The lattices $\mathfrak{F}(\mathfrak{I}(L))$ and $\mathfrak{I}(\mathfrak{F}(L))$ are isomorphic for every distributive lattice L.*

Proof. Let $\varphi :\ \mathfrak{F}(\mathfrak{I}(L)) \longrightarrow \mathfrak{I}(\mathfrak{F}(L))$ be the mapping defined in 3.10. Similarly we define $\psi :\ \mathfrak{I}(\mathfrak{F}(L)) \longrightarrow \mathfrak{F}(\mathfrak{I}(L))$ by

$$\psi(\mathcal{K}) = \{I \in \mathfrak{I}(L) \mid I \cap F \neq \emptyset \text{ for every } F \in \mathcal{K}\}.$$

It is clear that both φ and ψ are order-preserving. It remains to prove that they are inverse to each other. Because of the symmetry, it suffices to show that $\psi(\varphi(\mathcal{U})) = \mathcal{U}$ for every $\mathcal{U} \in \mathfrak{F}(\mathfrak{I}(L))$.

It is obvious that $\mathcal{U} \subseteq \psi(\varphi(\mathcal{U}))$. Conversely, suppose that an almost principal ideal I has a nonempty intersection with every $F \in \varphi(\mathcal{U})$. For contradiction, suppose that $I \notin \mathcal{U}$. Then $I \subsetneq I^{\mathcal{U}}$ and we can choose $x \in I^{\mathcal{U}} \setminus I$. For the filter $G = \uparrow x$ consider the filter $G(\mathcal{U})$ generated by $\{x_K \mid K \in \mathcal{U}\}$. By 3.9, $G(\mathcal{U}) \in \varphi(\mathcal{U})$, hence $G(\mathcal{U}) \cap I \neq \emptyset$. It follows that $x_J \in I$ for some $J \in \mathcal{U}$. Clearly, $I^{\mathcal{U}} \subseteq I \vee J$, hence $x \in I \vee J$, which implies that $x = x_{I \vee J} = x_I \vee x_J$. Since $x_J \in I$, we have $x_J \leq x_I$, hence $x = x_I$, which means that $x \in I$, a contradiction. $\square$

We close this section with some examples.

(1) If L is a bounded distributive lattice then $L \cong \mathfrak{F}(\mathfrak{I}(L))$.

(2) If L is an affine complete distributive lattice withou the least and the greatest element, then $\mathfrak{F}(\mathfrak{I}(L)) \cong L \cup \{0, 1\}$.

(3) Let L be the "open square" lattice

$$L = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1,\ 0 < y < 1\},$$

where $\mathbb{R}$ is the chain of reals. Then, up to isomorphism,

$$\mathfrak{F}(\mathfrak{I}(L)) = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}.$$

(4) Let L be the set of all sequences of real numbers, which contain only finitely many nonzero members. This set, under the pointwise ordering, is a distributive lattice. Then, up to isomorphism, $\mathfrak{I}(L)$ is the set of all sequences of $\mathbb{R} \cup \{\infty\}$ with only finitely many negative members. Every such sequence represents the ideal of all elements of L that lie below it. Similarly, $\mathfrak{F}(\mathfrak{I}(L))$ is the set of all sequences of $\mathbb{R} \cup \{\infty, -\infty\}$.

4. Other affine completions

In this section we investigate general conditions characterizing the situation when a distributive lattice L is affine complete in a distributive lattice M. Our basic technique lies in analyzing binary compatible functions.

First, one denotation. If $f :\ L \longrightarrow L$ is any function on a distributive lattice L, then $I(f) = \{x \in L \mid x \leq f(y) \text{ for some } y \in L\}$.

In the next assertion we use the fact that every unary isotone compatible function $f :\ L \longrightarrow L$ is a lattice endomorphism. Indeed, any such function is representable by some unary polynomial of $\mathfrak{F}(\mathfrak{I}(L))$ and it is easy to see that unary polynomials are endomorphisms.

4.1. Lemma. *Let $f : L^2 \longrightarrow L$ be an isotone compatible function on a distributive lattice L. For every $x \in L$ we define a unary function $f_x : L \longrightarrow L$ by $f_x(y) = f(x,y)$. Then*

(i) *$I(f_{x\wedge y}) = I(f_x) \cap I(f_y)$ for every $x, y \in L$;*
(ii) *the family $\{I(f_x) \mid x \in L\}$ generates an almost principal filter in $\mathfrak{I}(L)$.*

Proof. We know from 2.2(ii) that $I(f_x) \in \mathfrak{I}(L)$. Denote by $\mathcal{U}$ the filter in $\mathfrak{I}(L)$ generated by $\{I(f_x) \mid x \in L\}$.

(i) The inclusion $I(f_{x\wedge y}) \subseteq I(f_x) \cap I(f_y)$ follows from the isotonicity of f. Conversely, suppose that $t \le f(x,u)$ and $t \le f(y,v)$ for some $t, u, v \in L$. Define a compatible function $g : L \longrightarrow L$ by $g(z) = f(z, u \vee v)$. Since unary compatible functions are lattice homomorphisms, we obtain that $t \le f(x,u) \wedge f(y,v) \le g(x) \wedge g(y) = g(x \wedge y) = f(x \wedge y, u \vee v)$, hence $t \in I(f_{x\wedge y})$.

(ii) Because of (i), for any $J \in \mathfrak{I}(L)$ we have

$$J \in \mathcal{U} \quad \text{iff} \quad J \supseteq I(f_x) \quad \text{for some} \quad x \in L.$$

To prove that $\mathcal{U}$ is almost principal, consider arbitrary $I \in \mathfrak{I}(L)$. Choose $x \in I$ arbitrarily. We claim that $I^{\mathcal{U}} = I \vee I(f_x)$. Obviously, $I \subseteq I \vee I(f_x) \in \mathcal{U}$. Now, let $J \in \mathcal{U}$, $I \subseteq J$. Thus, $I(f_u) \subseteq J$ for some $u \in L$. By 2.4, $f(x,z) \le x \vee f(u,z)$ for every $z \in L$. Since $x \in I \subseteq J$ and $f(u,z) \in I(f_u) \subseteq J$, we have $f(x,z) \in J$. This shows that $I(f_x) \subseteq J$, hence $I \vee I(f_x) \subseteq J$. $\square$

Now we formulate "the condition of substitutes". Let L be a sublattice of a distributive lattice M.

(CS)

For every $\mathcal{U} \in \mathfrak{F}(\mathfrak{I}(L))$, $\mathcal{U} \ne \{L\}, \mathfrak{I}(L)$, there exists $c \in M$ such that

$y_I = y \wedge (c \vee x)$ for every $x, y \in L$, where $I = (\downarrow x)^{\mathcal{U}}$.

4.2. Theorem. *Suppose that every binary isotone compatible function on L can be interpolated by a polynomial of M. Then (CS) holds.*

Proof. Consider the function

$$f(x,y) = y_I, \quad \text{where } I = (\downarrow x)^{\mathcal{U}}.$$

This function is isotone and compatible, because it is representable by a polynomial of $\mathfrak{F}(\mathfrak{I}(L))$ (under the canonical embedding $L \longrightarrow \mathfrak{F}(\mathfrak{I}(L))$), namely $f(x,y) = y \wedge (\mathcal{U} \vee x)$.

By our assumption, f is representable by a polynomial of M. Let $\overline{M}$ be the lattice which arises from M by adding a new 0 and 1. (If M possesses the greatest and the least element, this step is not necessary.) Hence, f can be expressed in the canonical form

$$f(x,y) = a \wedge (b \vee y) \wedge (c \vee x) \wedge (d \vee x \vee y),$$

where $a, b, c, d \in \overline{M}$, $a \ge b \ge d$, $a \ge c \ge d$.

It is easy to see that if $y \le x$ then $y \in \downarrow x \subseteq (\downarrow x)^{\mathcal{U}}$, hence $f(x,y) = y$. In particular, $f(x,x) = x$ for every $x \in L$. Clearly, $x = f(x,x) = a \wedge (d \vee x)$. Thus, $a \ge x \ge d$ for every $x \in L$. Then $d \vee x \vee y = x \vee y$, $a \ge x \vee y$ and we have

$$f(x,y) = (b \vee y) \wedge (c \vee x) \wedge (x \vee y).$$

Since $y \le x \vee y$, we have $y = f(x \vee y, y) = (b \vee y) \wedge (c \vee x \vee y) \wedge (x \vee y \vee y) = (b \vee y) \wedge (x \vee y)$, hence $f(x, y) = y \wedge (c \vee x)$. It remains to show that $c \in M$ (i.e. $c \ne 0, 1$).

Suppose that $c = 0$. Then $f(x, y) = x \wedge y$. We claim that $\downarrow x \in \mathcal{U}$ for every $x \in L$. Indeed, $y \in (\downarrow x)^{\mathcal{U}} = I$ implies that $y = y_I = f(x, y) = x \wedge y$, hence $y \le x$, which shows that $(\downarrow x)^{\mathcal{U}} = \downarrow x$ and therefore $\downarrow x \in \mathcal{U}$. Thus, $\mathcal{U} = \mathfrak{I}(L)$, which is excluded.

Suppose that $c = 1$. Then $f(x, y) = y$ for every $x, y \in L$. This is only possible if $(\downarrow x)^{\mathcal{U}} = L$ for every $x \in L$. Thus, if $I \in \mathcal{U}$ then choosing an arbitrary $x \in I$ we find that $L = (\downarrow x)^{\mathcal{U}} \subseteq I$, hence $I = L$. We have shown that $\mathcal{U} = \{L\}$, which is an excluded case too. □

Now we are going to prove the converse. In the sequel we keep the following assumptions:

(1) L is a sublattice of a distributive lattice M;
(2) the condition (CS) is satisfied;
(3) $f : L^n \longrightarrow L$ is an isotone compatible function ($n \ge 1$).

Our aim is to represent f by a polynomial of M. Without loss of generality we can assume that M contains 0 and 1. (If a polynomial contains 0 or 1, then it always can be simplified so that these constants disappear.)

For every $S \subseteq \underline{n} = \{1, \dots, n\}$ we define a binary function $f^S : L^2 \longrightarrow L$ by

$$f^S(x, y) = f(x_1, \dots, x_n), \quad \text{where}$$

$$x_i = \begin{cases} y & \text{if} \quad i \in S; \\ x & \text{if} \quad i \notin S. \end{cases}$$

Obviously, each f^S is a compatible function. For every $x \in L$ we define an unary function f^S_x by $f^S_x(y) = f^S(x, y)$. By 4.1, the family $\{I(f^S_x) \mid x \in L\}$ generates some $\mathcal{U} \in \mathfrak{F}(\mathfrak{I}(L))$. (Of course, $\mathcal{U}$ depends on S.) According to (CS), there is $a_S \in M$ such that $y \wedge (x \vee a_S) = y_I$ for every $x, y \in L$, $I = (\downarrow x)^{\mathcal{U}}$. (If the case $\mathcal{U} = \mathfrak{I}(L)$ or $\mathcal{U} = \{L\}$ occurs, then (CS) does not apply, but the constants $a_S = 0$ or $a_S = 1$ have the required property.)

In general, the condition (CS) does not determine the elements a_S uniquely. We shall need a special choice of those elements, which is ensured by the following lemma.

4.3. Lemma. *The elements a_S, $S \subseteq \underline{n}$, can be chosen in such a way that*

(1) $a_S \le a_T$ *whenever* $S \subseteq T$;
(2) $a_\emptyset \le f(x_1, \dots, x_n) \le a_{\underline{n}}$ *for every* $x_1, \dots, x_n \in L$.

Proof. Let a_S, $S \subseteq \underline{n}$ be arbitrary elements of M satisfying the condition that $y_I = y \wedge (x \vee a_S)$ for every $x, y \in L$, $I = (\downarrow x)^{\mathcal{S}}$, where the filter $\mathcal{S}$ is generated by the family $\{I(f^S_u) \mid u \in L\}$.

I. If (1) is not fulfilled, we set $b_T = \bigwedge_{S \supseteq T} a_S$ for every $T \subseteq \underline{n}$. The elements b_T clearly satisfy (1) and it suffices to show that $y \wedge (x \vee a_T) = y \wedge (x \vee b_T)$ for every $x, y \in L$, $T \subseteq \underline{n}$.

Consider arbitrary $S \supseteq T$. Let $\mathcal{S}$ be as above and let $\mathcal{T} \in \mathfrak{F}(\mathfrak{I}(L))$ be generated by the family $\{I(f^T_u) \mid u \in L\}$. For every $v \in L$ we have $f^S_u(u \vee v) \ge f^T_u(v)$, because f is isotone. Hence, $I(f^S_u) \supseteq I(f^T_u)$, and consequently, $\mathcal{S} \subseteq \mathcal{T}$. Then for $I = (\downarrow x)^{\mathcal{S}}$, $J = (\downarrow x)^{\mathcal{T}}$ we have $I \supseteq J$ and hence $y_I \ge y_J$, or equivalently,

$y \wedge (x \vee a_S) \geq y \wedge (x \vee a_T)$. Further,
$y \wedge (x \vee b_T) = y \wedge (x \vee \bigwedge_{S \supseteq T} a_S) = \bigwedge_{S \supseteq T}(y \wedge (x \vee a_S)) = y \wedge (x \vee a_T)$.

II. We denote $f(L^n) = \{f(x_1, \dots, x_n) \mid x_1, \dots, x_n \in L\}$. Clearly, $f_u^{\emptyset}$ is for every $u \in L$ a constant function equal to $f(u, \dots, u)$. Thus, if $S = \emptyset$, then the corresponding filter $\mathcal{S}$ is
$\mathcal{S} = \{I \in \mathfrak{I}(L) \mid f(u, \dots, u) \in I$ for some $u \in L\}$, or equivalently
$\mathcal{S} = \{I \in \mathfrak{I}(L) \mid I \cap f(L^n) \neq \emptyset\}$. (Indeed, if $f(u_1, \dots, u_n) \in I$ then $f(u, \dots, u) \in I$, where $u = \bigwedge_{i=1}^n u_i$.)

We claim that, for any $t \in L$, $t \wedge a_\emptyset$ is a lower bound of $f(L^n)$. Indeed, if $w \in f(L^n)$ then $\downarrow w \in \mathcal{S}$, hence $(\downarrow w)^{\mathcal{S}} = \downarrow w$ and therefore $w \geq t \wedge w = t_{\downarrow w} = t \wedge (w \vee a_\emptyset) \geq t \wedge a_\emptyset$.

Choose $t \in f(L^n)$ arbitrarily and set $b_\emptyset = t \wedge a_\emptyset$. Then clearly $b_\emptyset \leq f(x_1, \dots, x_n)$ for every $x_1, \dots, x_n \in L$. It remains to prove that $y \wedge (x \vee a_\emptyset) = y \wedge (x \vee b_\emptyset)$ for every $x, y \in L$. Since $y \wedge a_\emptyset$ is a lower bound of $f(L^n)$, we have $y \wedge a_\emptyset \leq t$ and hence $y \wedge (x \vee a_\emptyset) = (y \wedge x) \vee (y \wedge a_\emptyset) = (y \wedge x) \vee (y \wedge a_\emptyset \wedge t) = y \wedge (x \vee (a_\emptyset \wedge t)) = y \wedge (x \vee b_\emptyset)$.

III. Clearly, $f_u^{\underline{n}}(y) = f(y, \dots, y)$ for every $u, y \in L$, i.e. the function $f_u^{\underline{n}}$ does not depend on u. If $S = \underline{n}$ then the corresponding filter is $\mathcal{S} = \{I \in \mathfrak{I}(L) \mid I(f_u^{\underline{n}}) \subseteq I\} = \{I \in \mathfrak{I}(L) \mid f(L^n) \subseteq I\}$.

We claim that, for every $t \in L$, $t \vee a_{\underline{n}}$ is an upper bound of $f(L^n)$. Indeed, if $w \in f(L^n)$ then $w \in I = (\downarrow t)^{\mathcal{S}} \supseteq f(L^n)$, hence $w = w_I = w \wedge (t \vee a_{\underline{n}}) \leq t \vee a_{\underline{n}}$.

Choose $t \in f(L^n)$ arbitrarily and set $b_{\underline{n}} = t \vee a_{\underline{n}}$. It remains to show that $y \wedge (x \vee a_{\underline{n}}) = y \wedge (x \vee b_{\underline{n}})$ for every $x, y \in L$. Since $x \vee a_{\underline{n}}$ is an upper bound of $f(L^n)$, we have $t \leq x \vee a_{\underline{n}}$ and therefore $y \wedge (x \vee a_{\underline{n}}) = y \wedge (x \vee a_{\underline{n}} \vee t) = y \wedge (x \vee b_{\underline{n}})$.

4.4. Lemma. *For every $\emptyset \neq S \subsetneq \underline{n}$ and every $x_1, \dots, x_n \in L$ the following holds true:*

$$a_S \wedge \bigwedge_{i \in S} x_i \leq f(x_1, \dots, x_n) \leq a_S \vee \bigvee_{i \notin S} x_i.$$

Proof. Denote by $\mathcal{S}$ the filter in $\mathfrak{I}(L)$ generated by the family $\{I(f_u^S) \mid u \in L\}$.

I. Consider the ideal $K = I(f_x^S)$, where $x = \bigwedge_{i \notin S} x_i$. Let $y = \bigwedge_{i \in S} x_i$. Clearly, $y_K \leq f^S(x, u)$ for some $u \in L$. ¿From 2.4 and the isotonicity of f we infer that $y_K \leq y \wedge f^S(x, u) \leq f(x_1, \dots, x_n)$. Choose arbitrary $t \in K$. Then $K \supseteq I = (\downarrow t)^{\mathcal{S}}$, because $\downarrow t \subseteq K \in \mathcal{S}$. By the definition of a_S we have $f(x_1, \dots, x_n) \geq y_K \geq y_I = y \wedge (t \vee a_S) \geq y \wedge a_S$.

II. Denote $w = \bigvee_{i \in S} x_i$, $z = \bigvee_{i \notin S} x_i$. Consider the ideal $J = (\downarrow z)^{\mathcal{S}}$. Clearly, $J \supseteq I(f_u^S)$ for some $u \in L$, hence $f^S(u, w) \in J$ and $(f^S(u, w))_J = f^S(u, w)$. By 2.4 and the isotonicity of f we have $f(x_1, \dots, x_n) \leq f^S(u, w) \vee z = (f^S(u, w))_J \vee z = (f^S(u, w) \wedge (z \vee a_S)) \vee z \leq z \vee a_S$. $\square$

¿From 4.4 and 4.3(ii) we deduce that

$$\bigvee_{S \subseteq \underline{n}} (a_S \wedge \bigwedge_{i \in S} x_i) \leq f(x_1, \dots, x_n) \leq \bigwedge_{S \subseteq \underline{n}} (a_S \vee \bigvee_{i \notin S} x_i).$$

It is not difficult to see that, due to distributivity and 4.3(i),

$$\bigvee_{S \subseteq \underline{n}} (a_S \wedge \bigwedge_{i \in S} x_i) = \bigwedge_{S \subseteq \underline{n}} (a_S \vee \bigvee_{i \notin S} x_i).$$

(It can be proved by induction on n.) Hence,

$$f(x_1, \dots, x_n) = \bigvee_{S \subseteq \underline{n}} (a_S \wedge \bigwedge_{i \in S} x_i),$$

which means that we have represented f by a polynomial of M. As a consequence we have obtained the main results of this section.

4.5. Theorem. *Let L be a sublattice of a distributive lattice M. The following statements are equivalent:*

(1) *L and M satisfy (CS);*
(2) *every isotone compatible binary function on L can be interpolated by a polynomial of M.*
(3) *every isotone compatible function on L can be interpolated by a polynomial of M.*

□

4.6. Theorem. *Let L be a sublattice of a distributive lattice M. Suppose that L does not contain a nontrivial Boolean interval. The following statements are equivalent:*

(1) *L is affine complete in M;*
(2) *L and M satisfy (CS);*
(3) *every compatible binary function on L can be interpolated by a polynomial of M.*

□

The condition (CS) says that M in some sense contains the canonical completion $\mathfrak{F}(\mathfrak{I}(L))$. Every element of $\mathfrak{F}(\mathfrak{I}(L))$ (possibly except the greatest and the least element) has some "substitute" in M. However, we cannot claim that $\mathfrak{F}(\mathfrak{I}(L))$ is a sublattice of M, which is shown by the following example.

Let $L = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1,\ 0 < y < 1\}$ be the open square lattice, which was discussed at the end of the previous chapter. Let M be the sublattice of $\mathbb{R}^2 \times \{0, 1\}$ given by

$$M = \{(x, y, 0) \mid 0 \leq x < 1,\ 0 \leq y < 1\} \cup \{(x, y, 1) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}.$$

We can regard L as a sublattice of M, identifying $(x, y) \in L$ with $(x, y, 0) \in M$. Then (CS) is satisfied, i.e. L is affine complete in M. However, there is no lattice embedding $\mathfrak{F}(\mathfrak{I}(L)) \longrightarrow M$ that preserves the elements of L.

The above lattice L also provides a counterexample to other conjectures.

In [4], an I-polynomial of a distributive lattice L is defined as a function on L that can be interpolated by a polynomial of the ideal lattice $I(L)$. Similarly, D-polynomials are functions representable by polynomials of the filter (dual ideal) lattice $F(L)$. The conjecture in [4] says that every isotone compatible function is an ID-polynomial (a composition of I-polynomials and D-polynomials). In fact, it is not difficult to prove that every I-polynomial (D-polynomial) is representable by a polynomial of $\mathfrak{I}(L)$ (of $\mathfrak{F}(L)$). Now, consider the open square lattice L embedded in the lattice

$$Q = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\} \setminus \{(0, 1)\}.$$

Then both
$$\mathfrak{I}(L) = \{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1,\ 0 < y \leq 1\}$$
and $\mathfrak{F}(L)$ are embeddable in Q, which means that every ID-polynomial can be interpolated by a polynomial of Q. However, L is not affine complete in Q and the function
$$f(x, y) = y \wedge (x \vee (0, 1))$$
is isotone compatible on L and cannot be represented by a polynomial of Q. This contradicts the conjecture in [4].

The above example also shows that in 4.5 and 4.6 unary compatible functions cannot be used instead of binary ones. Indeed, every unary compatible function on L is an ID-polynomial ([4]) and hence representable by a polynomial of Q.

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