AFFINE COMPLETE MEDIAN ALGEBRAS

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Abstract. In 1993, Bandelt has determined all locally affine complete median algebras as those median algebras that do not contain a nontrivial Boolean segment. In the present paper we characterize those median algebras that are not only *locally affine complete*, but even *affine complete*. The condition that forces a locally affine complete median algebra M to be affine complete is that certain subsets of M - the proper Čebyšev subsets - are finitely bounded. This condition will also be shown to be necessary for affine completeness. Hence, we obtain the following characterization: A median algebra is affine complete iff it does not contain a nontrivial Boolean segment and every proper Čebyšev set is finitely bounded.

1. Preliminaries

Let A be a universal algebra. A function $f: A^n \longrightarrow A$ is called *compatible* if, for any congruence θ of A, $(a_i, b_i) \in \theta$, $i = 1, \dots, n$, implies that $(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \in \theta$.

A polynomial function (or simply *a polynomial*) of A is any function that can be obtained by composition of the basic operations of A, the projections and the constant functions. *A local polynomial* of A is any function which can be interpolated by polynomials on all finite subsets of its domain.

Obviously, (local) polynomials are compatible functions. An algebra is called *(locally) affine complete* if the converse holds: every compatible function is a (local) polynomial.

Affine completeness has been investigated for various classes of algebras. The papers [3], [4], [7] and [9] contain some ideas that are close to our considerations. We also use results of Bandelt ([1]) about local affine completeness.

Now we recall the definition and some terminology for median algebras. For more information see also Bandelt and Hedlíková ([2]), van de Vel ([10]) and Isbell ([6]). Note that the median algebras we are dealing with in this note are called symmetric media in [6].

On any distributive lattice D we define the median polynomial by

$$m(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z).$$

This operation turns D into a median algebra. In general, a median algebra is an algebra endowed with a single ternary operation which can be embedded in (D, m)

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for some distributive lattice D. Median algebras form a variety (equational class). This variety can be defined, for example, by the following identities ([6]):

$$\begin{aligned} m(a,a,b) =& a, \\ m(a,b,c) =& m(a,c,b) = m(b,a,c) = m(b,c,a) = m(c,a,b) = m(c,b,a), \\ m(a,c,d) =& m(a, m(a,c,d), m(b,c,d)). \end{aligned}$$

(See [2] for other systems of axioms.) Every median algebra is a subdirect power of the 2-element algebra $(\underline{2}, m)$, where $\underline{2}$ is the 2-element lattice and m is its median operation. We always assume that a median algebra M is embedded in (D, m) for some distributive lattice D and lattice operations appearing in the text refer to this lattice. Since this embedding is not unique, we want to point out that our statements will be valid for any such embedding.

Let M be a median algebra. For elements $a, b, c \in M$ we say that c is *between* a and b if $c = m(a,b,c)$. A subset C of M is *convex* if $a, b \in C$ and $x \in M$ imply $m(a,b,x) \in C$. Equivalently, C is convex if, for every $a, b \in C$, C contains all elements of M that are between a and b. It is easy to see that the intersection of any number of convex sets is convex. Hence, for any subset A of M there is the smallest convex set containing A. We denote it by $\operatorname{Conv} A$. A set of the form $\operatorname{Conv}\{a,b\}$ is called *a segment*. It is not difficult to show that the segment $\operatorname{Conv}\{a,b\}$ consists of all elements that are between a and b. Any segment (or, more generally, any convex set) is a subalgebra of M. A segment is called *Boolean* if it is isomorphic to (B, m) for some Boolean lattice B.

A nonempty convex set is *prime* if its complement is also convex and nonempty. Any prime convex set C determines a congruence θ of M with the equivalence classes C and $M \setminus C$. Congruences of this form are called *split congruences*. Clearly, M/θ is isomorphic to $(\underline{2}, m)$. Using the Zorn's lemma, one can show the following.

1.1. Lemma. (Nieminen [8]) Let $A \subseteq M$ be a nonempty convex set, $x \in M \setminus A$. Then there is a prime convex set P with $A \subseteq P$, $x \notin P$.

Locally affine complete median algebras were characterized by H.-J. Bandelt as follows. A segment is nontrivial if it contains at least two elements.

1.2. Theorem([1])**.** *A median algebra M is locally affine complete if and only if it does not contain a nontrivial Boolean segment.*

Using the results of [1], we will single out those algebras that are *affine complete*.

2. Polynomials and Čebyšev sets

A term function of a universal algebra A is any function composed of the projections (i.e. variables) and the basic operations.

2.1. Lemma. *Let M be a median algebra, $\emptyset \neq I \subseteq \{1, \dots, n\}$. The formula*

$$h(x_0, x_1, \dots, x_n) = (x_0 \wedge \bigvee_{i \in I} x_i) \vee \bigwedge_{i \in I} x_i$$

defines a (median) term function $h: M^{n+1} \longrightarrow M$.

Proof. We proceed by induction on $|I|$ (the cardinality of I). If $|I| = 1$ then $h(x_0, \dots, x_n) = x_1$ is clearly a term function. Suppose that $|I| > 1$ and choose

$j \in I$ arbitrarily. Let $J = I \setminus \{j\}$. By the induction hypothesis, $g(x_0, \dots, x_n) = (x_0 \wedge \bigvee_{i \in J} x_i) \vee \bigwedge_{i \in J} x_i$ is a term function. Then $m(x_0, g(x_0, \dots, x_n), x_j)$ is also a term function and
$m(x_0, g(x_0, \dots, x_n), x_j) = (x_0 \vee (g(x_0, \dots, x_n) \wedge x_j)) \wedge (g(x_0, \dots, x_n) \vee x_j) =$
$(x_0 \vee \bigwedge_{i \in J} x_i) \wedge (x_0 \vee x_j) \wedge ((x_0 \wedge \bigvee_{i \in J} x_i) \vee x_j \vee \bigwedge_{i \in J} x_i) =$
$(x_0 \vee \bigwedge_{i \in I} x_i) \wedge (x_0 \vee x_j \vee \bigwedge_{i \in J} x_i) \wedge \bigvee_{i \in I} x_i =$
$(x_0 \vee \bigwedge_{i \in I} x_i) \wedge \bigvee_{i \in I} x_i = h(x_0, \dots, x_n)$. $\square$

The function h defined in 2.1 will be denoted by h_I. Notice that the expression is symmetrical, i.e. $h_I(x_0, \dots, x_n) = (x_0 \wedge \bigvee_{i \in I} x_i) \vee \bigwedge_{i \in I} x_i = (x_0 \vee \bigwedge_{i \in I} x_i) \wedge \bigvee_{i \in I} x_i$.

2.2. Lemma. . *Let $\mathfrak{F}$ be a family of subsets of $\{1, \dots, n\}$ such that $I \cap J \neq \emptyset$ whenever $I, J \in \mathfrak{F}$. Then the formula*

$$h_{\mathfrak{F}}(x_0, \dots, x_n) = \left(x_0 \vee (\bigvee_{I \in \mathfrak{F}} \bigwedge_{i \in I} x_i) \right) \wedge (\bigwedge_{I \in \mathfrak{F}} \bigvee_{i \in I} x_i)$$

defines a term function $M^{n+1} \longrightarrow M$.

Proof. We proceed by induction on $|\mathfrak{F}|$. If $\mathfrak{F} = \emptyset$ then $h_{\mathfrak{F}}(x_0, \dots, x_n) = x_0$, which is clearly a term function. Suppose now that $|\mathfrak{F}| > 0$ and choose $J \in \mathfrak{F}$ arbitrarily. Let $\mathfrak{G} = \mathfrak{F} \setminus \{J\}$. Then $h_{\mathfrak{G}}$ is a term function by the induction hypothesis and it suffices to show that $h_{\mathfrak{F}}(x_0, \dots, x_n) = h_J(h_{\mathfrak{G}}(x_0, \dots, x_n), x_1, \dots, x_n)$. Since J has a nonempty intersection with every $I \in \mathfrak{G}$ we have $\bigwedge_{i \in J} x_i \leq \bigwedge_{I \in \mathfrak{G}} \bigvee_{i \in I} x_i$. Using the distributivity we obtain that $h_J(h_{\mathfrak{G}}(x_0, \dots, x_n), x_1, \dots, x_n) =$
$((x_0 \vee (\bigvee_{I \in \mathfrak{G}} \bigwedge_{i \in I} x_i)) \wedge (\bigwedge_{I \in \mathfrak{G}} \bigvee_{i \in I} x_i) \wedge (\bigvee_{i \in J} x_i)) \vee (\bigwedge_{i \in J} x_i) =$
$(x_0 \vee (\bigvee_{I \in \mathfrak{F}} \bigwedge_{i \in I} x_i)) \wedge (\bigwedge_{I \in \mathfrak{G}} \bigvee_{i \in I} x_i) \wedge (\bigvee_{i \in J} x_i) = h_{\mathfrak{F}}(x_0, \dots, x_n)$. $\square$

If we set $I = \{1, \dots, n\}$, $\mathfrak{F} = \{I\}$ and replace variables $x_1, \dots, x_n$ by constants, we obtain the following consequence of 2.1.

2.3. Lemma. *Let $a_1, \dots, a_n$ be elements of a median algebra M. Then $f(x) = (x \wedge \bigwedge_{i=1}^n a_i) \vee \bigwedge_{i=1}^n a_i$ is a polynomial of M.*

2.4. Lemma. *Let M be a median algebra, $a_1, \dots, a_n \in M$. Then*

$$\operatorname{Conv}(\{a_1, \dots, a_n\}) = \{x \in M \mid \bigwedge_{i=1}^n a_i \leq x \leq \bigvee_{i=1}^n a_i\}.$$

Proof. First we show that the set $C = \{x \in M \mid \bigwedge_{i=1}^n a_i \leq x \leq \bigvee_{i=1}^n a_i\}$ is convex. Let $x, y, z \in M$, $y, z \in C$. Then $m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \leq (x \wedge \bigvee_{i=1}^n a_i) \vee (x \wedge \bigvee_{i=1}^n a_i) \vee \bigvee_{i=1}^n a_i = \bigvee_{i=1}^n a_i$ and symetrically, $m(x, y, z) \geq \bigwedge_{i=1}^n a_i$.

Now let K be a convex set, $\{a_1, \dots, a_n\} \subseteq K$. By induction on k we prove the following assertion:

$$\text{if } \bigwedge_{i=1}^k a_i \leq x \leq \bigvee_{i=1}^k a_i \text{ then } x \in K.$$

For $k = 1$ we have $a_1 \leq x \leq a_1$, hence $x = a_1 \in K$. Suppose now that $\bigwedge_{i=1}^k a_i \leq x \leq \bigvee_{i=1}^k a_i$. Let $y = (x \vee \bigwedge_{i=1}^{k-1} a_i) \wedge \bigvee_{i=1}^{k-1} a_i$. By 2.3, $y \in M$, and by the induction hypothesis $y \in K$. Since K is convex, we have $m(x, y, a_k) \in K$ and

$y \wedge a_k = (x \wedge a_k \wedge \bigvee_{i=1}^{k-1} a_i) \vee \bigwedge_{i=1}^{k} a_i$,
$y \vee a_k = (x \vee a_k \vee \bigwedge_{i=1}^{k-1} a_i) \wedge \bigvee_{i=1}^{k} a_i$,
$m(x, y, a_k) = (x \vee (y \wedge a_k)) \wedge (y \vee a_k) =$
$(x \vee \bigwedge_{i=1}^{k} a_i) \wedge (x \vee a_k \vee \bigwedge_{i=1}^{k-1} a_i) \wedge \bigvee_{i=1}^{k} a_i = (x \vee \bigwedge_{i=1}^{k} a_i) \wedge \bigvee_{i=1}^{k} a_i = x$.
Putting $k = n$ we obtain that $C \subseteq K$. $\square$

A subset C of a median algebra M is called *a Čebyšev set*, if for every $x \in M$ there exists $x_C \in C$ such that $m(x, x_C, y) = x_C$ for every $y \in C$ (i.e. x_C is between x and every element of C). Čebyšev sets have been introduced in [6]. They are connected with some maps on median algebras (called contractions or retractions) and are used in some geometrical considerations. (See also [5].)

2.5. Lemma. *Let M be a median algebra. Then $C = \mathrm{Conv}(\{a_1, \dots, a_n\})$ is a Čebyšev set for any $a_1, \dots, a_n \in M$ and $x_C = (x \vee \bigwedge_{i=1}^{n} a_i) \wedge \bigvee_{i=1}^{n} a_i$ for every $x \in M$.*

Proof. Let $x \in M$ and let x_C be defined as above. Then for any $y \in C$ we have $\bigwedge_{i=1}^{n} a_i \le y \le \bigvee_{i=1}^{n} a_i$ and hence $m(x, x_C, y) = (x \vee \bigwedge_{i=1}^{n} a_i) \wedge (x \vee y) \wedge (x_C \vee y) =$
$(x \vee \bigwedge_{i=1}^{n} a_i) \wedge (x \vee y) \wedge (x \vee y \vee \bigwedge_{i=1}^{n} a_i) \wedge (y \vee \bigvee_{i=1}^{n} a_i) =$
$(x \vee (y \wedge \bigwedge_{i=1}^{n} a_i)) \wedge (y \vee \bigvee_{i=1}^{n} a_i) = (x \vee \bigwedge_{i=1}^{n} a_i) \wedge \bigvee_{i=1}^{n} a_i = x_C$. $\square$

Čebyšev sets of the form $\mathrm{Conv}\{a_1, \dots, a_n\}$ will be called *finitely bounded.*

2.6. Lemma. *Let $f : D^n \longrightarrow D$ be a lattice polynomial of a distributive lattice D, $n \ge 1$. Then*

$$f(x_1, \dots, x_n) \le \bigvee_{i=1}^{n} f(x_i, \dots, x_i)$$

holds for every $x_1, \dots, x_n \in D$.

Proof. By the distributivity, f can be expressed in the form $f(x_1, \dots, x_n) = \bigvee_{j=1}^{k} f_j(x_1, \dots, x_n)$, where each f_j is a meet of some variables and (possibly) a constant. It suffices to show that $f_j(x_1, \dots, x_n) \le \bigvee_{i=1}^{n} f(x_i, \dots, x_i)$ for each j. But this is clear, since f_j is either constant and then $f_j(x_1, \dots, x_n) = f_j(x_1, \dots, x_1) \le f(x_1, \dots, x_1)$ or contains some variable x_i, which implies that $f_j(x_1, \dots, x_n) \le f_j(x_i, \dots, x_i) \le f(x_i, \dots, x_i)$. $\square$

Since the above statement concerns only a finite number of elements, it is obviously valid for local polynomials, too. Further, if we consider a median algebra M embedded in a distributive lattice D, then any polynomial of M is a restriction of some lattice polynomial of D. Therefore, we obtain the following consequence.

2.7. Consequence. *Let $f : M^n \longrightarrow M$ be a (local) polynomial function of a median algebra M, $n \ge 1$. Then*

$$f(x_1, \dots, x_n) \le \bigvee_{i=1}^{n} f(x_i, \dots, x_i)$$

holds for every $x_1, \dots, x_n \in M$.

We also need the following result from [1]:

2.8. Lemma([1])**.** *Every unary local polynomial function of a median algebra is idempotent.*

The above assertion can be generalized as follows.

2.9. Lemma. *Let f be a n-ary local polynomial of a median algebra M. Let $y_1, \ldots, y_n \in M$, $I \subseteq \{1, \ldots, n\}$. Then $f(y_1, \ldots, y_n) = f(z_1, \ldots, z_n)$, where*

$$z_i = \begin{cases} y_i & \text{if } i \notin I, \\ f(y_1, \ldots, y_n) & \text{if } i \in I. \end{cases}$$

Proof. We proceed by induction on the cardinality of I. If $I = \emptyset$, the statement is trivial. Suppose now that $|I| > 0$ and the statement holds for all sets $J \subseteq \{1, \ldots, n\}$ with $|J| < |I|$. Choose $j \in I$ arbitrarily. Let us define a function $h : M \longrightarrow M$ by $h(x) = f(t_1, \ldots, t_n)$, where

$$t_i = \begin{cases} y_i & \text{if } i \neq j, \\ x & \text{if } i = j. \end{cases}$$

This is clearly a unary local polynomial. By 2.8, this function is idempotent. Hence, we have $f(y_1, \ldots, y_n) = f(y'_1, \ldots, y'_n)$, where

$$y'_i = \begin{cases} y_i & \text{if } i \neq j, \\ f(y_1, \ldots, y_n) & \text{if } i = j. \end{cases}$$

Using the induction hypothesis for the set $J = I \setminus \{j\}$ we have $f(y'_1, \ldots, y'_n) = f(z'_1, \ldots, z'_n)$, where

$$z'_i = \begin{cases} y_i = z_i & \text{if } i \notin I, \\ f(y'_1, \ldots, y'_n) = z_i & \text{if } i \in J, \\ y_j = f(y_1, \ldots, y_n) = z_i & \text{if } i = j, \end{cases}$$

hence $f(y_1, \ldots, y_n) = f(z_1, \ldots, z_n)$. $\square$

3. Affine completeness - the finitely bounded case

3.1. Lemma. *Let $f, g : M^n \longrightarrow M$ be compatible functions of a median algebra M. Let $a_1, \ldots, a_k \in M$. If f and g coincide on $\{a_1, \ldots, a_k\}^n$ then they coincide on* $(\mathrm{Conv}\{a_1, \ldots, a_k\})^n$.

Proof. On the contrary, suppose that $f(x_1, \ldots, x_n) \neq g(x_1, \ldots, x_n)$ for some $x_1, \ldots, x_n \in \mathrm{Conv}\{a_1, \ldots, a_k\}$. By 1.1 there is a prime convex set P such that $f(x_1, \ldots, x_n) \in P$ and $g(x_1, \ldots, x_n) \notin P$. Let θ be the split congruence determined by P (i.e. having P and $M \setminus P$ as its equivalence classes). We claim that for every $i = 1, \ldots, n$ there is $y_i \in \{a_1, \ldots, a_k\}$ such that $(x_i, y_i) \in \theta$. This is clear if $\{a_1, \ldots, a_k\}$ has a nonempty intersection with both P and $M \setminus P$. If this condition is not satisfied, say $\{a_1, \ldots, a_k\} \subseteq P$, (the case $\{a_1, \ldots, a_k\} \subseteq M \setminus P$ is similar), then $\{x_1, \ldots, x_n\} \subseteq \mathrm{Conv}\{a_1, \ldots, a_k\} \subseteq P$. The compatibility of f and g yields that $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \theta$ and $(g(x_1, \ldots, x_n), g(y_1, \ldots, y_n)) \in \theta$. Since f and g coincide on $\{a_1, \ldots, a_n\}^n$, we have $f(y_1, \ldots, y_n) = g(y_1, \ldots, y_n)$, hence $(f(x_1, \ldots, x_n), g(x_1, \ldots, x_n)) \in \theta$, a contradiction. $\square$

3.2. Consequence. *If $f: M^n \longrightarrow M$ is a local polynomial function of a median algebra M then, for every $a_1, \dots a_k \in M$, the function f can be interpolated on $(\mathrm{Conv}\{a_1, \dots, a_k\})^n$ by a polynomial.*

Another consequence of 3.1 (together with 1.2) solves our problem in the case when M itself is finitely bounded.

3.3. Theorem. *Suppose that a median algebra M is finitely bounded. Then*

(i) *every local polynomial is a polynomial;*
(ii) *the median algebra M is affine complete if and only if it does not contain a nontrivial Boolean segment.*

We close this section with the following generalization of 3.3(i).

3.4. Lemma. *Suppose that $h: M^n \longrightarrow M$ is a local polynomial of a median algebra M such that $h(M^n) \subseteq \mathrm{Conv}\{a_1, \dots, a_k\}$ for some $a_1, \dots, a_k \in M$. Then h is a polynomial.*

Proof. Let us set $g(x) = (x \wedge \bigvee_{i=1}^k a_i) \vee \bigwedge_{i=1}^k a_i$. It is easy to see that g is an endomorphism of the median algebra M (because the formula determines an endomorphism of the distributive lattice D in which M is embedded). Therefore, g must preserve (i.e. commute with) the local polynomial h. Thus, $h(x_1, \dots, x_n) = g(h(x_1, \dots, x_n)) = h(g(x_1), \dots, g(x_n))$. The local polynomial h can be represented on the set $(\mathrm{Conv}\{a_1, \dots, a_k\})^n$ by a polynomial p, hence $h(x_1, \dots, x_n) = p(g(x_1), \dots, g(x_n))$, which means that we have represented h by a polynomial on the whole M^n. □

4. Affine completeness - the unary case

4.1. Lemma. *For any Čebyšev set C of a median algebra M, the function $f: M \longrightarrow M$ defined by $f(x) = x_C$ is compatible.*

Proof. Let θ be a congruence of M. Let $x, y \in M$, $(x, y) \in \theta$. Then $(x_C, y_C) = (m(x, x_C, y_C), m(y, x_C, y_C)) \in \theta$. □

4.2. Lemma. *Let $f: M \longrightarrow M$ be a compatible function of a median algebra M. Suppose that M does not contain a nontrivial Boolean segment. Then*

(i) *$f \circ f = f$;*
(ii) *$C = f(M)$ is a Čebyšev set;*
(iii) *$f(x) = x_C$ for every $x \in M$.*

Proof. By 1.2 f is a local polynomial and by 2.8 it is idempotent. Hence, (i) holds.

To prove (ii) and (iii) it suffices to prove that $m(x, f(x), y) = f(x)$ for every $x \in M$ and $y \in f(M)$. For a contradiction, suppose that $f(x) \neq m(x, f(x), y)$. Hence, $f(x) \notin \mathrm{Conv}\{x, y\}$. By 1.1, there is a prime convex set P such that $x, y \in P$, $f(x) \notin P$. Let θ be the congruence on M with equivalence classes P and $M \setminus P$. Then $(x, y) \in \theta$. Since $f(y) = y$ (according to (i)), we obtain that $(f(x), f(y)) = (f(x), y) \notin \theta$, which contradicts the compatibility of f. □

4.3. Lemma. *Let $a_1, \dots, a_n, b_1, \dots, b_m, c_1, \dots, c_p$ be elements of a median algebra M. Let us denote $a_* = \bigwedge_{i=1}^n a_i$, $b_* = \bigwedge_{i=1}^m b_i$, $c_* = \bigwedge_{i=1}^p c_i$. Then*

$$\bigwedge_{i,j,k} m(a_i, b_j, c_k) = (a_* \wedge b_*) \vee (a_* \wedge c_*) \vee (b_* \wedge c_*).$$

Proof. Denote $t = \bigwedge\limits_{i,j,k} m(a_i, b_j, c_k)$. We have $t \geq \bigwedge\limits_{i,j,k} (a_i \wedge b_j) = \bigwedge\limits_{i,j} (a_i \wedge b_j) = a_* \wedge b_*$ and similarly, $t \geq a_* \wedge c_*$, $t \geq b_* \wedge c_*$. Hence $t \geq (a_* \wedge b_*) \vee (a_* \wedge c_*) \vee (b_* \wedge c_*)$. Conversely, $t \leq \bigwedge\limits_{i,j} (a_i \vee b_j) = a_* \vee b_*$ and hence $t \leq (a_* \vee b_*) \wedge (a_* \vee c_*) \wedge (b_* \vee c_*) = (a_* \wedge b_*) \vee (a_* \wedge c_*) \vee (b_* \wedge c_*)$. $\square$

4.4. Lemma. *Let $f : M \longrightarrow M$ be a unary polynomial of a median algebra M. Then f is an identity mapping or $f(M)$ is a finitely bounded Čebyšev set.*

Proof. We proceed by induction on the structure of f. If f is an identity mapping or a constant, the statement is obviously true. Suppose now that $f(x) = m(f_1(x), f_2(x), f_3(x))$, where f_1, f_2, f_3 have the required property. If at least two of f_1, f_2, f_3 are identity mappings, then f is also an identity. We distinguish the remaining two cases.

I. Suppose that none of f_1, f_2, f_3 is an identity. Then $f_1(M) = \mathrm{Conv}\{a_1, \ldots, a_n\}$, $f_2(M) = \mathrm{Conv}\{b_1, \ldots, b_m\}$, $f_3(M) = \mathrm{Conv}\{c_1, \ldots, c_p\}$ for some suitable elements of M. We claim that

$$f(M) = \mathrm{Conv}\{m(a_i, b_j, c_k) \mid i = 1, \ldots, n,\ j = 1, \ldots, m,\ k = 1, \ldots, p\}.$$

Let us denote $a^* = \bigvee_{i=1}^n a_i$, $a_* = \bigwedge_{i=1}^n a_i$, $b^* = \bigvee_{i=1}^m b_i$, $b_* = \bigwedge_{i=1}^m b_i$, $c^* = \bigvee_{i=1}^p c_i$, $c_* = \bigwedge_{i=1}^p c_i$. By 4.2 and 2.5, $f_1(x) = (x \wedge a^*) \vee a_*$, $f_2(x) = (x \wedge b^*) \vee b_*$, $f_3(x) = (x \wedge c^*) \vee c_*$. By 4.3 we have $\bigwedge_{i,j,k} m(a_i, b_j, c_k) = (a_* \wedge b_*) \vee (a_* \wedge c_*) \vee (b_* \wedge c_*)$ and symetrically $\bigvee_{i,j,k} m(a_i, b_j, c_k) = (a^* \vee b^*) \wedge (a^* \vee c^*) \wedge (b^* \vee c^*)$. We have $f(x) = m(f_1(x), f_2(x), f_3(x)) = (f_1(x) \vee f_2(x)) \wedge (f_1(x) \vee f_3(x)) \wedge (f_2(x) \vee f_3(x)) = ((x \wedge (a^* \vee b^*)) \vee (a_* \vee b_*)) \wedge ((x \wedge (a^* \vee c^*)) \vee (a_* \vee c_*)) \wedge ((x \wedge (b^* \vee c^*)) \vee (b_* \vee c_*)) = (x \vee \bigwedge_{i,j,k} m(a_i, b_j, c_k)) \wedge \bigvee_{i,j,k} m(a_i, b_j, c_k)$. By 4.2, $x \in f(M)$ if and only if $x = f(x)$ if and only if $\bigwedge_{i,j,k} m(a_i, b_j, c_k) \leq x \leq \bigvee_{i,j,k} m(a_i, b_j, c_k)$.

II. Suppose that $f_1(x) = (x \wedge a^*) \vee a_*$, $f_2(x) = (x \wedge b^*) \vee b_*$, $f_3(x) = x$. We claim that $f(M) = \mathrm{Conv}\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$. We compute $f(x) = (x \vee f_1(x)) \wedge (x \vee f_2(x)) \wedge (f_1(x) \vee f_2(x)) = (x \vee a_*) \wedge (x \vee b_*) \wedge ((x \wedge a^*) \vee a_* \vee (x \wedge b^*) \vee b_*) = (x \vee (a_* \wedge b_*)) \wedge ((x \wedge (a^* \vee b^*)) \vee a_* \vee b_*) = (x \vee (a_* \wedge b_*)) \wedge (x \vee a_* \vee b_*) \wedge (a^* \vee b^*) = (x \vee (a_* \wedge b_*)) \wedge (a^* \vee b^*)$. Now it is clear that $x \in f(M)$ if and only if $x = f(x)$ if and only if $a_* \wedge b_* \leq x \leq a^* \vee b^*$ if and only if $x \in \mathrm{Conv}\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$. $\square$

A Čebyšev set $C \subseteq M$ is called proper if $C \neq M$. A median algebra M is called 1-affine complete if every unary compatible function is a polynomial.

4.5. Theorem. *A median algebra M is 1-affine complete iff*

(i) *it does not contain a nontrivial Boolean segment;*
(ii) *every proper Čebyšev set is finitely bounded.*

Proof. If M contains a nontrivial Boolean segment C then M is not even locally affine complete by 1.2. In fact, the proof in [1] exhibits a unary compatible function that is not a local polynomial (namely the function, which maps $x \in M$ into the complement of x_C in C). If a proper Čebyšev set $C \subseteq M$ is not finitely bounded then $x \mapsto x_C$ determines a compatible function that cannot be a polynomial in view of 4.4.

Conversely, let (i) and (ii) be satisfied. Let $f : M \longrightarrow M$ be a compatible function. If $f(M) = M$ then by 4.2(iii) $f(x) = x_M = x$, hence f is an identity mapping, which is a polynomial. If $f(M) \neq M$ then $f(M)$ is a proper Čebyšev set and, by our assumption, $f(M) = \mathrm{Conv}\{a_1 \ldots, a_n\}$ for some $a_1, \ldots a_n \in M$. By 4.2(iii), 2.3 and 2.5, f is a polynomial. $\square$

5. Affine completeness - the general case

Throughout this section we assume that $f : M^n \longrightarrow M$ is an n-ary compatible function of a median algebra M. Further, we suppose that M does not contain a nontrivial Boolean segment or a proper Čebyšev set which is not finitely bounded. Hence, f is a local polynomial function.

Let $\mathfrak{F}$ be the family of all sets $I \subseteq \{1, \ldots, n\}$ with the following property: if $y, x_1, \ldots, x_n \in M$ are such that $x_i = y$ for every $i \in I$, then $f(x_1, \ldots, x_n) = y$. Let us choose $u \in M$ arbitrarily. For any $I \subseteq \{1, \ldots, n\}$ we define a unary function $f_I^u : M \longrightarrow M$ by

$$f_I^u(x) = f(y_1, \ldots, y_n), \text{ where}$$

$$y_i = \begin{cases} x \text{ for } i \in I; \\ u \text{ for } i \notin I. \end{cases}$$

5.1. Lemma. *If $I \notin \mathfrak{F}$ then $f_I^u(M)$ is a finitely bounded Čebyšev set.*

Proof. It is clear that f_I^u is a compatible function and hence, by 4.2(ii), $f_I^u(M)$ must be a Čebyšev set.

Since $I \notin \mathfrak{F}$, there are $y, x_1, \ldots, x_n \in M$ such that $x_i = y$ for $i \in I$ and $f(x_1, \ldots, x_n) \neq y$. That is why the unary compatible function g defined by

$$g(x) = f(y_1, \ldots, y_n), \text{ where}$$

$$y_i = \begin{cases} x \text{ for } i \in I; \\ x_i \text{ for } i \notin I. \end{cases}$$

is not an identity function ($g(y) \neq y$), which by 4.2 means that $g(M)$ is a proper Čebyšev set. By our assumption, there are $a_1, \ldots, a_k \in M$ such that $g(M) = \mathrm{Conv}\{a_1, \ldots, a_k\}$.

By way of contradiction, assume that $f_I^u(M)$ is not finitely bounded. This is only possible if f_I^u is an identity function and M itself is not finitely bounded. Therefore, we can choose $x \in M \setminus \mathrm{Conv}(\{a_1, \ldots, a_k\} \cup \{u\} \cup \{x_i \,|\, i \notin I\})$. By 1.1 there is a prime convex set P with $x \notin P$, $\{a_1, \ldots, a_k\} \subseteq P$, $u \in P$, $\{x_i \,|\, i \notin I\} \subseteq P$. Let θ be the corresponding split congruence. Then $(x_i, u) \in \theta$ whenever $i \notin I$. The compatibility of f implies that $(g(x), f_I^u(x)) \in \theta$, which is impossible since $g(x) \in \mathrm{Conv}\{a_1, \ldots, a_k\} \subseteq A$ while $f_I^u(x) = x \notin A$. $\square$

Considering 5.1 with all possible $I \notin \mathfrak{F}$ we obtain the following assertion.

5.2. Lemma. *There are $c_1, \ldots, c_m \in M$ such that $f_I^u(x) \in \mathrm{Conv}\{c_1, \ldots, c_m\}$ for every $x \in M$, $I \subseteq \{1, \ldots, n\}$, $I \notin \mathfrak{F}$.*

5.3. Lemma. *If the median algebra M has at least two elements, then $I \cap J \neq \emptyset$ for every $I, J \in \mathfrak{F}$.*

Proof. Suppose that $I, J \in \mathfrak{F}$, $I \cap J = \emptyset$. Choose $a, b \in M$, $a \neq b$. Let $(y_1, \ldots, y_n) \in M^n$ be such that $y_i = a$ for $i \in I$ and $y_i = b$ for $i \in J$. By the definition of $\mathfrak{F}$ then $f(y_1, \ldots, y_n) = a$ and $f(y_1, \ldots, y_n) = b$, a contradiction. $\square$

5.4. Lemma. *Let $I \subseteq \{1,\dots,n\}$, $y, x_1,\dots,x_n \in M$. Let $y_1,\dots,y_n \in M$ be such that $y_i = y$ for every $i \in I$ and $y_i = x_i$ for every $i \notin I$. Then $f(x_1,\dots,x_n) \in \mathrm{Conv}(\{f(y_1,\dots,y_n)\} \cup \{x_i \mid i \in I\})$.*

Proof. For contradiction, suppose that $f(x_1,\dots,x_n) \notin \mathrm{Conv}(\{f(y_1,\dots,y_n)\} \cup \{x_i \mid i \in I\})$. Then there is a prime convex set $P \subseteq M$ such that $f(x_1,\dots,x_n) \notin P$, $f(y_1,\dots,y_n) \in P$ and $\{x_i \mid i \in I\} \subseteq P$. Let θ be the corresponding split congruence. By 2.9, $f(y_1,\dots,y_n) = f(z_1,\dots,z_n)$, where

$$z_i = \begin{cases} y_i & \text{if } i \notin I, \\ f(y_1,\dots,y_n) & \text{if } i \in I. \end{cases}$$

Clearly $(x_i, z_i) \in \theta$ for every $i = 1,2,\dots,n$, but $(f(x_1,\dots,x_n), f(z_1,\dots,z_n)) \notin \theta$, a contradiction with the compatibility of f. $\square$

Let us define a function $g:\ M^n \longrightarrow M$ by

$$g(x_1,\dots,x_n) = \left(f(x_1,\dots,x_n) \vee \bigwedge_{j=1}^{m} c_j \right) \wedge \bigvee_{j=1}^{m} c_j,$$

where c_j are the elements defined in 5.2. Since g is a composition of the compatible function f and the unary polynomial $p(x) = (x \vee \bigwedge_{j=1}^m c_j) \wedge \bigvee_{j=1}^m c_j$, g is also compatible. By 3.4, g is a polynomial function.

5.5. Lemma. *For every $x_1,\dots,x_n \in M$, the following formula holds true:*

$$(*) \qquad f(x_1,\dots,x_n) = \left(g(x_1,\dots,x_n) \vee (\bigvee_{I\in\mathfrak{F}} \bigwedge_{i\in I} x_i) \right) \wedge (\bigwedge_{I\in\mathfrak{F}} \bigvee_{i\in I} x_i).$$

Proof. If $|M| = 1$, the statement is trivial. Let us suppose that $|M| > 1$. By 5.3, any two sets from $\mathfrak{F}$ have a nonempty intersection, which implies that $\bigvee_{I\in\mathfrak{F}} \bigwedge_{i\in I} x_i \leq \bigwedge_{I\in\mathfrak{F}} \bigvee_{i\in I} x_i$. Therefore the right side of (*) is symmetrical and it suffices to prove one inequality.

First we prove that $f(x_1,\dots,x_n) \leq \bigvee_{i\in I} x_i$ holds for every $I \in \mathfrak{F}$. Choose $y \in \{x_i \mid i \in I\}$ arbitrarily. By 5.4 we have $f(x_1,\dots,x_n) \leq f(y_1,\dots,y_n) \vee \bigvee_{i\in I} x_i$, where $y_i = x_i$ for $i \notin I$ and $y_i = y$ for $i \in I$. Since $I \in \mathfrak{F}$, we have $f(y_1,\dots,y_n) = y \leq \bigvee_{i\in I} x_i$, hence $f(x_1,\dots,x_n) \leq \bigvee_{i\in I} x_i$.

It remains to show that

$$f(x_1,\dots,x_n) \leq g(x_1,\dots,x_n) \vee \bigvee_{I\in\mathfrak{F}} \bigwedge_{i\in I} x_i.$$

By the definition of g it suffices to prove that

$$f(x_1,\dots,x_n) \leq \bigvee_{j=1}^{m} c_j \vee \bigvee_{I\in\mathfrak{F}} \bigwedge_{i\in I} x_i.$$

Using the distributivity we obtain that $\bigvee_{I\in\mathfrak{F}} \bigwedge_{i\in I} x_i = \bigwedge_{J\in\mathfrak{G}} \bigvee_{i\in J} x_i$, where

$$\mathfrak{G} = \{J \subseteq \{1,\dots,n\} \mid J \cap I \neq \emptyset \text{ for every } I \in \mathfrak{F}\}.$$

Hence, we have to show that, for every $J \in \mathfrak{G}$,

$$f(x_1, \dots, x_n) \le \bigvee_{j=1}^{m} c_j \vee \bigvee_{i \in J} x_i.$$

For any such J we have $\{1, \dots, n\} \setminus J = K \notin \mathfrak{F}$. By 5.2, $f_K^u(y) \le \bigvee_{j=1}^m c_j$ for every $y \in M$. According to 2.7 and 5.4 (with J and u playing the roles of I and y), $f(x_1, \dots, x_n) \le \bigvee_{k=1}^n f(x_k, \dots, x_k) \le \bigvee_{k=1}^n (f_K^u(x_k) \vee \bigvee_{i \notin K} x_i) \le \bigvee_{j=1}^m c_j \vee \bigvee_{i \in J} x_i$. $\square$

Using the denotation from 2.2, we have obtained that

$$f(x_1, \dots, x_n) = h_{\mathfrak{F}}(g(x_1, \dots, x_n), x_1, \dots, x_n).$$

Since both $h_{\mathfrak{F}}$ and g are polynomials, f is a polynomial too.

We have proved that if a median algebra M does not contain a nontrivial Boolean segment and every proper Čebyšev set is finitely bounded, then M is affine complete. Together with 4.5 we obtain our main result.

5.6. Theorem. *A median algebra M is affine complete if and only if the following conditions are satisfied:*

(i) *M does not contain a proper boolean segment;*

(ii) *every proper Čebyšev set of M is finitely bounded.*

We conclude with some examples. Let L be any distributive lattice without the least element and m its median operation. Then it is easy to see that the median algebra (L, m) is not affine complete. Indeed, for any $x \in L$ the set $C = \{y \in L \mid y \le x\}$ is Čebyšev (with $y_C = y \wedge x$) and not finitely bounded. If there is no Boolean interval in L, we have an example of a median algebra, which is locally affine complete but not affine complete.

On the other hand, it is not easy to find an affine complete median algebra, which is not finitely bounded. (And for finitely bounded median algebras affine completeness is equivalent to local affine completeness.) As an example of such algebra we can mention the free median algebra with an infinite number of free generators. It is clear that this algebra is not finitely bounded. However, the proof of affine completeness is rather nontrivial and we omit it here.

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