INJECTIVE ORDERED TOPOLOGICAL SPACES

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Abstract. We investigate injective objects in the category of ordered topological spaces and show their close connection to continuous lattices.

Injective objects have been investigated for various categories. Many references can be found in [3]. Our basic reference for notions not explained here is [2].

Let $\mathcal{M}$ be some class of morphisms of a category $\mathcal{C}$. An object X of $\mathcal{C}$ is called $\mathcal{M}$-injective if, for every $f \in \mathcal{M}$, $f: A \to B$, and every morphism $g: A \to X$ there exists a morphism $h: B \to X$ with $hf = g$. Usually, $\mathcal{M}$ is chosen to be the class of all embeddings (or, in abstract categories, monomorphisms). Then, in the above definition, A is a subobject of B and the condition just requires that every morphism $A \to X$ extend to a morphism $B \to X$.

Let $\mathcal{T}$ denote the category of all topological T_0-spaces and continuous maps. We always assume that a topological space (X, Ω) is given by its underlying set X and a family Ω of open sets. For any such space we define the specialization order relation $\leq$ on the set X by the rule

$$x \leq y \quad \text{iff} \quad (x \in A \in \Omega \text{ implies } y \in A).$$

Let $\mathcal{J}$ be the class of all embeddings of T_0-spaces. The following well-known result of D. Scott serves as a motivation for our investigation. Recall that A is a retract of B if there are morphisms $f: A \to B$, $g: B \to A$ with $gf = \mathrm{id}_A$.

Theorem 1 (See [2], pp.121-124)**.** *Let (X, Ω) be a T_0-space and $\leq$ its specialization order. The following conditions are equivalent:*

(1) (X, Ω) *is a $\mathcal{J}$-injective object in $\mathcal{T}$;*
(2) $(X, \leq)$ *is a continuos lattice and Ω is its Scott topology;*
(3) (X, Ω) *is a retract of some power of the space (S, σ), where $S = \{0, 1\}$ and $\sigma = \{\emptyset, \{1\}, S\}$.*

□

The condition (2) shows that injective T_0-spaces are ordered in a natural way. Thus, one can expect that they will play an important role in characterizing injective objects in a suitable category of ordered topological spaces. The aim of the present paper is to confirm this conjecture.

Let us denote by $\mathcal{U}$ the category of all ordered T_0-spaces $(X, \Omega, \leq)$ (where Ω is a T_0-topology and $\leq$ is a partial order relation on X, without any requirements

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on compatibility) and continuous isotone maps. It is not difficult to prove that $\mathcal{U}$ has no nontrivial injectives with respect to the class of all embeddings. For our purposes, the class of all embeddings must be restricted.

Let Ω be a topology on a partially ordered set $(X, \leq)$. For $x \in X$, $A \subseteq X$ we set $\uparrow x = \{y \in X \mid x \leq y\}$, $\downarrow x = \{y \in X \mid x \geq y\}$, $\uparrow A = \bigcup_{x \in A} \uparrow x$, $\downarrow A = \bigcup_{x \in A} \downarrow x$, $\uparrow \Omega = \{A \in \Omega \mid A = \uparrow A\}$, $\downarrow \Omega = \{A \in \Omega \mid A = \downarrow A\}$. For any $Y \subseteq X$ we consider the ordered space $(Y, \Omega_Y, \leq_Y)$, where Ω_Y and $\leq_Y$ are the relativizations of Ω and $\leq$, respectively. We say that $(Y, \Omega_Y, \leq_Y)$ is an order subspace of $(X, \Omega, \leq)$ if the following conditions are satisfied:

$$\uparrow \Omega_Y = \{Y \cap A \mid A \in \uparrow \Omega\};$$

$$\downarrow \Omega_Y = \{Y \cap A \mid A \in \downarrow \Omega\}.$$

The notion of an order subspace was introduced by H. A. Priestley [5] and S. D. McCartan [4]. Let $\mathcal{M}$ denote the class of all embeddings $f : Z \to X$ of ordered T_0-spaces, for which $f(Z)$ is an order subspace of X. The $\mathcal{M}$-injective objects in $\mathcal{U}$ will be called injective ordered spaces and we shall give their description. First we summarize some general facts about injective objects, which are easy to prove.

Lemma 2. *Let $\mathcal{M}$ be a class of morphisms of a category $\mathcal{C}$. Then*

(1) *every product of $\mathcal{M}$-injective objects is $\mathcal{M}$-injective;*
(2) *every retract of a $\mathcal{M}$-injective object is $\mathcal{M}$-injective;*
(3) *if X is a $\mathcal{M}$-injective object and $f \in \mathcal{M}$, $f : X \to Y$, then X is a retract of Y.*

□

Let S_1 and S_2 be the ordered spaces defined on the underlying set $\{0, 1\}$ with the natural order $(0 < 1)$ and topologies $\sigma_1 = \{\emptyset, \{1\}, S\}$ and $\sigma_2 = \{\emptyset, \{0\}, S\}$, respectively.

Lemma 3. *S_1 and S_2 are injective ordered spaces.*

Proof. We prove the statement for S_1. Suppose that $f \in \mathcal{M}$, $f : (Y, \Omega, \leq) \to (Z, \Omega', \leq')$ and let $g : Y \to S_1$ be a continuous isotone map. Then $g^{-1}(\{1\}) \in \uparrow \Omega$, hence $f(g^{-1}(\{1\})) = f(Y) \cap A$ for a suitable $A \in \uparrow \Omega'$. Let $h : Z \to S_1$ be the characteristic function of A. It is easy to see that h is a continuous isotone map and $hf = g$. □

Lemmas 2 and 3 provide us a rich family of injective ordered spaces. In the sequel we shall prove the converse: every injective ordered space can be constructed from S_1 and S_2 by means of products and retracts.

Lemma 4. *Let $(X, \Omega, \leq)$ be an injective ordered space. Then (X, Ω) is a $\mathcal{J}$-injective object in $\mathcal{T}$.*

Proof. Suppose that $f \in \mathcal{J}$, $f : (Y, \Omega_Y) \to (Z, \Omega_Z)$ and let $g : Y \to X$ be continuous. If we order Y and Z as antichains (no two elements are comparable) then $f \in \mathcal{M}$ and g turns out to be isotone. The injectivity of $(X, \Omega, \leq)$ yields a continuous (and isotone) map $h : Z \to X$ with $hf = g$. □

Lemma 5. *Let $(X, \Omega, \leq)$ be an injective ordered space. Then $\uparrow\Omega \cup \downarrow\Omega$ forms a subbase of the topology Ω.*

Proof. Choose $A \in \Omega$, $x \in A$. We are going to find $W \in \uparrow\Omega$, $V \in \downarrow\Omega$ such that $x \in W \cap V \subseteq A$. Let us denote $\mathcal{I}_x = \{M \in \uparrow\Omega \mid x \in M\}$, $\mathcal{D}_x = \{M \in \downarrow\Omega \mid x \in M\}$. We set $Z = X \cup \{a, u\}$, where $a, u \notin X$, $a \neq u$. It is easy to verify that

$$\begin{aligned}\Omega' =& \{B \subseteq Z \mid (B \cap X \in \Omega) \text{ and (if } u \in B \text{ then } B \supseteq W \cap V \\ & \text{for suitable } W \in \mathcal{I}_x,\ V \in \mathcal{D}_x)\}\end{aligned}$$

is a T_0-topology on Z and Ω is its relativization to X. Further, we define a partial order $\leq'$ on Z by the rule

$$\begin{aligned}p \leq' q \text{ iff } & (p, q \in X,\ p \leq q) \text{ or } (p \in \{a, u\},\ q \in X,\ x \leq q) \text{ or} \\ & (p = a,\ q = u) \text{ or } p = q.\end{aligned}$$

Let Ω'' and $\leq''$ be the relativizations of Ω' and $\leq'$ to the set $Y = X \cup \{a\}$. Now we check that the embedding $i : (Y, \Omega'', \leq'') \to (Z, \Omega', \leq')$ belongs to $\mathcal{M}$.

I. Let $M \in \uparrow\Omega''$. Then $M = Y \cap M'$ for some $M' \in \Omega'$. Since $Y \in \Omega'$, we obtain that $M \in \Omega'$. Let us set

$$N = \begin{cases} M \cup \{u\} & \text{if } a \in M \\ M & \text{otherwise.} \end{cases}$$

If $a \in M$ then $x \in M \supseteq \{y \in Y \mid a \leq'' y\}$, hence $N = M \cup \{u\} \supseteq (X \cap M) \cap X$, where $X \in \mathcal{D}_x$, $X \cap M \in \mathcal{I}_x$. In both cases we obtain that $N \in \uparrow\Omega'$, $M = N \cap Y$.

II.Let $M \in \downarrow\Omega''$. Again we have $M \in \Omega'$. Let us set

$$N = \begin{cases} M \cup \{u\} & \text{if } u \leq'' q \text{ for some } q \in M \\ M & \text{otherwise.} \end{cases}$$

We obtain that $N \in \downarrow\Omega'$, $M = N \cap Y$.

Now we can make use of the injectivity of $(X, \Omega, \leq)$. Let us define $f : Y \to X$ by the rule $f(a) = x$ and $f(p) = p$ for all $p \in X$. Clearly, f is continuous and isotone. Thus, there exists a continuous isotone map $g : Z \to X$ with $g(y) = f(y)$ for all $y \in Y$. We obtain that $x = g(a) \leq g(u) \leq g(x) = x$. The continuity of g yields $g^{-1}(A) \in \Omega'$. Since $u \in g^{-1}(A)$, the definition of Ω' gives $W \in \mathcal{I}_x$, $V \in \mathcal{D}_x$ with $g^{-1}(A) \supseteq W \cap V$, hence $W \cap V = g(W \cap V) \subseteq A$. □

Corollary 6. *Let $(X, \Omega, \leq)$ be an injective ordered space and let $x, y \in X$. If there exists $A \in \Omega$ with $x \in A$, $y \notin A$, then there exists also $B \in \uparrow\Omega \cup \downarrow\Omega$ with $x \in B$, $y \notin B$.* □

Lemma 7. *Let $(X, \Omega, \leq)$ be an injective ordered space and let $x, y \in X$. Suppose that neither $W \in \uparrow\Omega$ with $x \in W$, $y \notin W$ nor $V \in \downarrow\Omega$ with $y \in V$, $x \notin V$ exists. Then $x \leq y$.*

Proof. Let $Z = X \cup \{u, v, w\}$, where $u, v, w \notin X$. It is easy to verify that

$$\begin{aligned}\Omega' =& \{A \subseteq Z \mid (A \cap X \in \Omega) \text{ and } (x \in A \text{ implies } \{u, w\} \subseteq A) \\ & \text{and } (v \in A \text{ implies } x \in A)\}\end{aligned}$$

is a T_0-topology on Z. Consider the partial order $\leq'$ on Z given by the rule

$$p \leq' q \text{ iff } (p, q \in X,\ p \leq q) \text{ or } (p \in \{u, v, w\},\ y \leq q \in X) \text{ or } (p = v,\ q = u) \text{ or } (p = w,\ q \in \{u, v\}) \text{ or } p = q.$$

The inclusion $i: X \to Z$ is obviously an embedding and we check that $i \in \mathcal{M}$.

I. Let $M \in \uparrow \Omega$. If $x \notin M$ then $M \in \Omega'$. If $x \in M$ then $y \in M$ by our assumptions. Let us set $N = M \cup \{u, v, w\}$. We have $N \in \uparrow \Omega'$, $N \cap X = M$.

II. Let $M \in \downarrow \Omega$. If $x \in M$ then for $N = M \cup \{u, v, w\}$ we have $N \in \downarrow \Omega'$, $N \cap X = M$. If $x \notin M$ then $y \notin M$ by our assumptions. We can set $N = M$.

Now the injectivity implies that there exists a continuous isotone map $g: Z \to X$ such that $gi = \mathrm{id}_X$, i.e. $g(p) = p$ for all $p \in X$. By way of contradiction, assume that $x \not\leq y$. Since g is isotone, we have $g(v) \leq g(y) = y$, hence $g(v) \neq x$. The existence of $W \in \Omega$ with $g(v) \in W$, $x \notin W$ implies that $g^{-1}(W) \notin \Omega'$ and contradicts the continuity of g. Since Ω is T_0, there must be $W \in \Omega$ with $x \in W$, $g(v) \notin W$. By Corollary 6 we can assume that $W \in \uparrow \Omega \cup \downarrow \Omega$. From $W \in \downarrow \Omega$ we get that $g(v) \leq g(u) \notin W$. Further, $W \in \uparrow \Omega$ implies that $g(w) \notin W$. In both cases we have $x \in g^{-1}(W)$, $\{u, w\} \not\subseteq g^{-1}(W)$, hence $g^{-1}(W) \notin \Omega'$, which contradicts the continuity of g. $\square$

Lemma 8. *Every injective ordered space $(X, \Omega, \leq)$ is isomorphic to an order subspace of a product of powers of the spaces S_1 and S_2.*

Proof. Let us denote $I = \{1\} \times \uparrow \Omega$, $J = \{0\} \times \downarrow \Omega$. Let P be the Cartesian product

$$P = \prod_{i \in I} S_1 \times \prod_{j \in J} S_2$$

endowed with the usual product topology Ω_P and the pointwise order $\leq_P$. The topology Ω_P has a subbase $\mathcal{B} = \{B_i \mid i \in I \cup J\}$, where $B_i = \prod A_j$ with $A_j = \{0, 1\}$ for $i \neq j$ and $A_i = \{n\}$ for $i = (n, A)$. We define a function $f: X \to P$ by the rule

$$f(x)(i) = \begin{cases} 0 \text{ if } (i = (0, A),\ x \in A) \text{ or } (i = (1, A),\ x \notin A); \\ 1 \text{ if } (i = (0, A),\ x \notin A) \text{ or } (i = (1, A),\ x \in A). \end{cases}$$

By Corollary 6, the map f is one-to-one. For $i = (n, A)$ we have $f^{-1}(B_i) = A \in \uparrow \Omega \cup \downarrow \Omega$, hence f is continuous. Further, for every $A \in \uparrow \Omega \cup \downarrow \Omega$ we have $f(A) = f(X) \cap B_i$, where $i = (n, A)$. (If $A \in \uparrow \Omega \cap \downarrow \Omega$, we can choose $n \in \{0, 1\}$ arbitrarily.) Since $\uparrow \Omega \cup \downarrow \Omega$ is a subbase of Ω, we have proved that f is a topological homeomorphism $X \to f(X)$. Now we show that $x \leq y$ iff $f(x) \leq_P f(y)$. First, let $x \leq y$. For any $A \in \uparrow \Omega$, $B \in \downarrow \Omega$ we have the implications $(x \in A \ \Rightarrow\ y \in A)$ and $(x \notin B \ \Rightarrow\ y \notin B)$. That is why $f(x)(i) = 0$ or $f(y)(i) = 1$ holds for all i, hence $f(x) \leq_P f(y)$. Conversely, assume that $f(x) \leq_P f(y)$. By the definition of f, neither $W \in \uparrow \Omega$ with $x \in W$, $y \notin W$ nor $V \in \downarrow \Omega$ with $x \notin V$, $y \in V$ exists. Now Lemma 7 yields $x \leq y$.

It remains to show that $f \in \mathcal{M}$. But this is clear, because for $W \in \uparrow \Omega$, $V \in \downarrow \Omega$ we have $f(W) = f(X) \cap B_i$, $f(V) = f(X) \cap B_j$, where $i = (1, W)$, $j = (0, V)$, $B_i \in \uparrow \Omega_P$, $B_j \in \downarrow \Omega_P$. $\square$

Putting together Lemmas 2, 3 and 8 we obtain the following result.

Theorem 9. *An ordered T_0-space is injective if and only if it is a retract of a product of powers of the spaces S_1 and S_2.* $\square$

Finally, we characterize injective ordered spaces in terms of continuous lattices.

Lemma 10. *Let $(X_1, \Omega_1, \leq_1)$ and $(X_2, \Omega_2, \leq_2)$ be ordered T_0-spaces satisfying the following conditions:*

(1) *$(X_1, \leq_1)$ and $(X_2, \leq_2)$ contain least elements (we denote them by o_1, o_2);*
(2) *$x \leq_1 y$ iff ($x \in A \in \Omega_1$ implies $y \in A$);*
(3) *$x \leq_2 y$ iff ($y \in A \in \Omega_2$ implies $x \in A$).*

Further, let us denote $(X, \Omega, \leq) = (X_1, \Omega_1, \leq_1) \times (X_x, \Omega_2, \leq_2)$ and let $f : X \to X$ be a continuous isotone map with $ff = f$. Then there exist continuos isotone maps $f_1 : X_1 \to X_1$, $f_2 : X_2 \to X_2$ such that $f_1 f_1 = f_1$, $f_2 f_2 = f_2$ and $f(X) = f_1(X_1) \times f_2(X_2)$.

Proof. Denote by p_1 and p_2 the projections of X onto X_1 and X_2, respectively. Let $e_1 : X_1 \to X$, $e_2 : X_2 \to X$ be defined by the rules $e_1(x) = (x, o_2)$, $e_2(x) = (o_1, x)$. Obviously, the functions p_1, p_2, e_1, e_2 are continuous and isotone. Let us set $f_1 = p_1 f e_1$, $f_2 = p_2 f e_2$. Thus, f_1 and f_2 are also continuous and isotone.

Let $\sqsubset$ be the specialization order on X, i.e.

$$a \sqsubset b \quad \text{iff} \quad (a \in A \in \Omega \;\Rightarrow\; b \in A).$$

Since Ω is the product topology, we have $(x_1, x_2) \sqsubset (y_1, y_2)$ if and only if $x_1 \leq_1 y_1$ and $y_2 \leq_2 x_2$. Note that the continuous function f must preserve the relation $\sqsubset$ (i.e. $a \sqsubset b$ implies $f(a) \sqsubset f(b)$).

Now we show the equality $p_1(f(x, o_2)) = p_1(f(x, y))$ for every $x \in X_1$, $y \in X_2$. From $x \leq_1 x$, $o_2 \leq_2 y$ it follows that $(x, y) \sqsubset (x, o_2)$, hence $f(x, y) \sqsubset f(x, o_2)$. From this we get $p_1(f(x, y)) \leq_1 p_1(f(x, o_2))$. On the other hand, $(x, o_2) \leq (x, y)$, hence $p_1(f(x, o_2)) \leq_1 p_1(f(x, y))$.

Analogously, $p_2(f(o_1, x)) = p_2(f(x, y))$ holds for every $x \in X_1$, $y \in X_2$.

Now we obtain:
$f_1 f_1(x) = p_1(f(p_1(f(x, o_2)), o_2)) = p_1(f(p_1(f(x, o_2)), p_2(f(x, o_2)))) =$
$p_1(f(f(x, o_2))) = p_1(f(x, o_2)) = f_1(x)$,
hence $f_1 f_1 = f_1$. The proof of $f_2 f_2 = f_2$ is analogous.

Finally we show that $f(X) = f_1(X_1) \times f_2(X_2)$. If $(x, y) \in f(X)$, then $f(x, y) = (x, y)$ and $f_1(x) = p_1(f(x, o_2)) = p_1(f(x, y)) = p_1(x, y) = x$, hence $x \in f_1(X_1)$ and similarly $y \in f_2(X_2)$. Conversely, if $(x, y) \in f_1(X_1) \times f_2(X_2)$ then $p_1(f(x, y)) = p_1(f(x, o_2)) = f_1(x) = x$, $p_2(f(x, y)) = p_2(f(o_1, y)) = f_2(y) = y$, hence $f(x, y) = (x, y) \in f(X)$. $\square$

If we set $(X_1, \Omega_1, \leq_1) = \prod_{i \in I} S_1$, $(X_2, \Omega_2, \leq_2) = \prod_{j \in J} S_2$, then the conditions (1)-(3) in Lemma 10 are satisfied. It is easy to see that an object Y is a retract of X in the category $\mathcal{U}$ if and only if it is isomorphic to $f(X)$ for some continuous isotone map $f : X \to X$ with $ff = f$. Thus, we obtain the following assertion.

Corollary 11. *An ordered T_0-space is injective if and only if it is isomorphic to a product of retracts of the spaces $\prod_{i \in I} S_1$ and $\prod_{j \in J} S_2$ for suitable index sets I, J.* $\square$

Theorem 12. *An ordered T_0-space is injective if and only if it is isomorphic to a product of a continuous lattice with its Scott topology and a dually continuous lattice with its dual Scott topology.*

Proof. We prove that retracts of spaces $\prod_{i\in I} S_1$ are exactly continuous lattices with their Scott topology.

Let $(R, \Omega, \leq)$ be a retract of $\prod_{i\in I} S_1$. By Lemma 4, (R, Ω) is $\mathcal{J}$-injective in $\mathcal{T}$. Thus, Ω is the Scott topology of the continuous lattice $(R, \sqsubset)$, where $\sqsubset$ is the specialization order associated with Ω. This means that the relations $\leq$ and $\sqsubset$ coincide.

Conversely, let $(R, \leq)$ be a continuous lattice and Ω its Scott topology. Then (R, Ω) is $\mathcal{J}$-injective in $\mathcal{T}$. By Theorem 1, there exists a set I and continuous maps $f : R \to \prod_{i\in I} S_1$, $g : \prod_{i\in I} S_1 \to R$ with $gf = \mathrm{id}_R$. Both R and $\prod_{i\in I} S_1$ are ordered by the specialization order, so the continuous maps f and g are isotone. This shows that $(R, \Omega, \leq)$ is a retract of $\prod_{i\in I} S_1$.

Similarly we can prove that retracts of spaces $\prod_{i\in I} S_2$ are exactly dually continuous lattices endowed with their dual Scott topology. Our assertion now follows from Corollary 11. □

Finally, let us mention the link of our results to order varieties of D. Duffus and I. Rival. In [1], an order variety was defined as a class of ordered sets closed under products ans retracts. The class of injective ordered spaces is closed under these operators (and is generated by the 2-element set $\{S_1, S_2\}$). Hence, it can play an important role in a classification of ordered spaces based on the concept of the order variety.

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