AFFINE COMPLETE STONE ALGEBRAS

Miroslav Haviar and Miroslav Ploščica

Abstract. In [1] R. Beazer characterized affine complete Stone algebras having a smallest dense element. We remove this latter assumption and describe affine complete algebras in the class of all Stone algebras.

1. Introduction.

An n-ary function f on an algebra $\boldsymbol{A}$ is called *compatible* if for any congruence θ on $\boldsymbol{A}$, $a_i \equiv b_i$ (θ) $(a_i, b_i \in A)$, $i = 1, \dots, n$ yields $f(a_1, \dots, a_n) \equiv f(b_1, \dots, b_n)$ (θ). It is clear that any polynomial function of $\boldsymbol{A}$ is compatible. Following H. Werner [16], an algebra $\boldsymbol{A}$ is called *affine complete* if the polynomial functions of $\boldsymbol{A}$ are the only compatible functions.

The problem of characterizing affine complete algebras was posed by G. Grätzer in [7] (Problem 6). However, it seems hard to answer such a question in general. In fact, every algebra is a reduct of some affine complete algebra, because we can add all compatible functions to the fundamental operations. The problem was reformulated by D. Clark and H. Werner in [3] in the following form: "Characterize affine complete algebras in your favourite variety."

G. Grätzer in [5] proved that every Boolean algebra is affine complete and in [6] he characterized affine complete bounded distributive lattices as those which do not contain proper Boolean subintervals. In [3] one can find a list of particular varieties in which all affine complete members were characterized. The list includes the varieties of all sets, all vector spaces over a division ring, elementary abelian p-groups for p-prime, p-rings, abelian groups and varieties generated by a semi-primal algebra. Later on, the variety of semilattices was added to the list ([10]). The second author recently ([15]) solved the problem for the variety of all distributive lattices generalizing Grätzer's result in [6]. The method developed in [15] is also used in the present paper.

Much can be said about affine completeness in arithmetical varieties (see [12] and [13]). Varieties of affine complete algebras have been examined in [11] where it was shown that any affine complete variety is residually finite. For a survey of the most recent results concerning affine complete varieties see [14].

2. Preliminaries.

First we recall some basic facts about Stone algebras. For the general background on that topic see e.g. [8]. A *Stone algebra* is an algebra $\boldsymbol{L} = (L; \vee, \wedge, ^\star, 0, 1)$,

1991 *Mathematics Subject Classification.* 06D15, 08A40.

Key words and phrases. Stone algebra, relatively complete filter, (locally) affine complete algebra.

where $(L;\vee,\wedge,0,1)$ is a bounded distributive lattice, * is a unary operation of pseudocomplementation, i.e. for any $a\in L$,

$$a^\star=\max\{x\in L:\ x\wedge a=0\}$$

and $\boldsymbol{L}$ satisfies the identity $x^\star\vee x^{\star\star}=1$.

Two subsets of $\boldsymbol{L}$ play an important role. The subset $B(\boldsymbol{L})=\{x^\star:\ x\in L\}$ of all *closed* elements of $\boldsymbol{L}$ which is a Boolean subalgebra of $\boldsymbol{L}$, and the subset $D(\boldsymbol{L})=\{x\in L:\ x^\star=0\}$ of all *dense* elements of $\boldsymbol{L}$ which forms a filter in $\boldsymbol{L}$. It is easy to see that for any $x\in L$, the element $x\vee x^\star$ is dense. Any $x\in L$ can be expressed as a meet of closed and dense elements of $\boldsymbol{L}$ since the identity

$$x=x^{\star\star}\wedge(x\vee x^\star)$$

holds in any Stone algebra.

The class of all Stone algebras is a variety, generated by the 3-element chain $\mathbf{3}=\{\underline{0},\underline{d},\underline{1}\}$ with $\underline{0}<\underline{d}<\underline{1}$. In fact, any Stone algebra can be embedded in some power of $\mathbf{3}$.

On any Stone algebra $\boldsymbol{L}$, one can define the *Glivenko congruence* Φ by

$$x\equiv y\ (\Phi)\quad\text{iff } x^\star=y^\star.$$

It is easy to see that the factorization $\boldsymbol{L}/\Phi$ produces a Boolean algebra isomorphic to $B(\boldsymbol{L})$.

Now we recall some notions from [15]. For an element x of a lattice L, let $\uparrow x:=\{y\in L:\ x\le y\}$ and let $\downarrow x:=\{y\in L:\ y\le x\}$. A filter F of a distributive lattice L is called *relatively complete* if for any $x\in L$ there exists $\min F\cap\uparrow x=\min\{y\in F:\ y\ge x\}$. The concept of a *relatively complete ideal* is defined dually. A filter or an ideal of L is *proper* if it is not equal to the whole of L. An interval of L is said to be *proper* if it has more than one element.

In [15], affine complete distributive lattices are characterized as follows:

2.1. Theorem ([15; 2.7]). *A distributive lattice L is affine complete if and only if the following conditions are satisfied:*

(i) *L does not contain a proper Boolean interval;*
(ii) *L does not contain a proper relatively complete ideal without a largest element;*
(iii) *L does not contain a proper relatively complete filter without a smallest element.* □

It is easy to see that the condition (ii) is satisfied whenever L has a largest element. Similarly, the existence of a smallest element of L implies (iii). Thus, a bounded distributive lattice is affine complete iff it does not contain a proper Boolean interval (see [6; Theorem]).

In our investigations we shall also use some preliminary results from [15]. Because the proofs of the following two lemmas are rather short, we repeat them here to make this paper sufficiently self-content.

2.2. Lemma ([15; 2.4])**.** *Let $f : L \longrightarrow L$ be a unary compatible function on a distributive lattice L. Let $x \in L$. If there exists $y \in L$ such that $x \leq y$ and $x \leq f(y)$, then the set $\uparrow x$ is closed under f. Dually, if $x \geq y$ and $x \geq f(y)$ for some $y \in L$, then the set $\downarrow x$ is closed under f.*

Proof. Let $y \in L$ be such that $x \leq y$ and $x \leq f(y)$. Suppose to the contrary that $z \in \uparrow x$ and $f(z) \notin \uparrow x$. Then there exists a prime ideal I such that $x \notin I$ and $f(z) \in I$. Let θ be the congruence on L with the classes I and $L \setminus I$. Then $(y, z) \in \theta$ while $(f(y), f(z)) \notin \theta$, a contradiction. □

2.3. Corollary ([15; 2.5])**.** *If the set $\downarrow x$ or $\uparrow x$ contains a fixed point of f, then it is closed under f.* □

2.4. Lemma ([15; 2.6])**.** *Let $f : L \longrightarrow L$ be a compatible function on a distributive lattice L. Suppose that L does not contain a proper Boolean interval. Then (i) $f \circ f = f$; (ii) the set $\downarrow f(L) = \bigcup_{x \in L} \downarrow f(x)$ is a relatively complete ideal in L.*

Proof. (i) Take any $x \in L$. By 2.2, the sets $\uparrow(x \wedge f(x))$ and $\downarrow(x \vee f(x))$ are closed under f, hence the interval $J = [x \wedge f(x), x \vee f(x)]$ is closed under f. Obviously, the function $g = f \restriction J$ is compatible on the lattice J. Since, by 2.1, J is affine complete, we have $g(y) = (a \vee y) \wedge b$ for some $a, b \in J$ such that $a \leq b$. Now one can easily verify that $g(g(y)) = g(y)$ for any $y \in J$. Since $x \in J$ and $f(x) \in J$, we finally get $f(f(x)) = g(g(x)) = g(x) = f(x)$.

(ii) First we show that $\max(\downarrow f(L) \cap \downarrow x)$ exists for any $x \in L$ and is equal to $x \wedge f(x)$. Clearly, $x \wedge f(x) \in \downarrow f(L) \cap \downarrow x$. Now take arbitrary $y \in \downarrow f(L) \cap \downarrow x$. Since $y \in \downarrow f(L)$, the set $\uparrow y$ contains an element of $f(L)$, i.e. a fixed point of f. By 2.3, the set $\uparrow y$ is closed under f, hence $y \leq x$ implies that $y \leq f(x)$. Therefore $y \leq x \wedge f(x)$, what was to be proved. To show that $\downarrow f(L)$ is closed under joins, let $a, b \in \downarrow f(L)$. Then $a, b \leq \max(\downarrow f(L) \cap \downarrow(a \vee b)) \leq a \vee b$, hence $\max(\downarrow f(L) \cap \downarrow(a \vee b)) = a \vee b$, whence $a \vee b \in \downarrow f(L)$. □

Finally, the following result from [4] will be useful:

2.5. Theorem ([4; Theorem 4])**.** *Let L be a distributive lattice. Then every compatible function on L is order-preserving if and only if L does not contain a proper Boolean interval.*

3. Affine completeness.

R. Beazer in [1] characterized affine complete Stone algebras having a smallest dense element:

3.1. Theorem ([1; Theorem 4])**.** *Let $\boldsymbol{L}$ be a Stone algebra having a smallest dense element. Then the following are equivalent.*

(1) *$\boldsymbol{L}$ is affine complete;*
(2) *$D(\boldsymbol{L})$ is an affine complete lattice;*
(3) *no proper interval of $D(\boldsymbol{L})$ is Boolean.* □

He also asked a question whether the conditions (1) and (2) are equivalent in any Stone algebra.

In this section we generalize Beazer's result to the variety of all Stone algebras and we provide an example of a Stone algebra for which the conditions (1) and (2) in 3.1 are not equivalent.

3.2. Lemma. *Let $\boldsymbol{L}$ be a Stone algebra and $f, g : L^n \longrightarrow L$ be compatible functions on $\boldsymbol{L}$. If $f = g$ on $(\{0\} \cup D(\boldsymbol{L}))^n$ then $f = g$ on the whole L^n.*

Proof. For a contradiction, suppose that $f = g$ on $(\{0\} \cup D(\mathbf{L}))^n$ and $f(\tilde{c}) \neq g(\tilde{c})$ for some $\tilde{c} = (c_1, \dots, c_n) \in L^n$. Then there exists a Stone homomorphism $h : L \longrightarrow \mathbf{3} = \{\underline{0}, \underline{d}, \underline{1}\}$. such that $h(f(\tilde{c})) \neq h(g(\tilde{c}))$. Let θ be the kernel congruence of h, i.e. $x \equiv y \ (\theta)$ iff $h(x) = h(y)$. Further, define $\tilde{b} = (b_1, \dots, b_n) \in L^n$ by

$$b_i = \begin{cases} 0 & \text{if } h(c_i) = 0, \\ c_i \vee c_i^* & \text{otherwise.} \end{cases}$$

It is clear that $h(c_i) = h(b_i)$ for every $i = 1, \dots, n$. (If $h(c_i) = \underline{d}$ then $h(b_i) = \underline{d} \vee \underline{d}^* = \underline{d} \vee \underline{0} = \underline{d}$.) Since the functions f and g are compatible, they preserve θ and we obtain that $h(f(\tilde{c})) = h(f(\tilde{b}))$ and $h(g(\tilde{c})) = h(g(\tilde{b}))$. Clearly, $\tilde{b} \in (\{0\} \cup D(\boldsymbol{L}))^n$, which implies that $f(\tilde{b}) = g(\tilde{b})$ and hence $h(f(\tilde{c})) = h(g(\tilde{c}))$, a contradiction. □

The following generalization of affine completeness turns out to be useful in our considerations. Let $\boldsymbol{B}$ be a subalgebra of an algebra $\boldsymbol{A}$. We say that $\boldsymbol{B}$ is *affine complete in* $\boldsymbol{A}$ if for every compatible function f of $\boldsymbol{B}$ there exists a polynomial p of $\boldsymbol{A}$ such that $f = p \restriction \boldsymbol{B}$. In other words, $\boldsymbol{B}$ is affine complete in $\boldsymbol{A}$ if every compatible function on $\mathbf{B}$ is a polynomial with constants taken from $\boldsymbol{A}$. It is clear that an algebra is affine complete iff it is affine complete in itself.

Now we present the main result of our paper.

3.3. Theorem. *Let $\boldsymbol{L}$ be a Stone algebra. Then the following conditions are equivalent:*

(1) *$\boldsymbol{L}$ is affine complete;*
(2) *$D(\boldsymbol{L})$ is affine complete in the lattice L;*
(3) *($\mathcal{B}$) $D(\boldsymbol{L})$ does not contain a proper Boolean interval;*
($\mathcal{F}$) for any relatively complete filter F in $D(\boldsymbol{L})$ there exists $a \in L$ such that $F = \uparrow a \cap D(\boldsymbol{L})$.

Proof. (1) $\Longrightarrow$ (3) To show ($\mathcal{B}$), suppose that $[a, b] \subseteq D(\boldsymbol{L})$ is a proper Boolean interval. Define a function $f : D(\boldsymbol{L}) \longrightarrow [a, b]$ by $f(x) = ((x \vee a) \wedge b)'$, where $'$ means the complement in $[a, b]$. One can easily verify that f is compatible on $D(\boldsymbol{L})$. Now we define an extension $g : L \longrightarrow [a, b]$ of f by $g(x) = f(x \vee x^\star)$. It is again easy to see that g is a compatible function of the Stone algebra $\boldsymbol{L}$. By hypothesis, g is a polynomial function of $\boldsymbol{L}$, i.e. there exist $\alpha_0, \dots, \alpha_3 \in L$ such that

$$g(x) = \alpha_0 \vee (\alpha_1 \wedge x) \vee (\alpha_2 \wedge x^\star) \vee (\alpha_3 \wedge x^{\star\star})$$

for any $x \in L$ (see also [1; Lemma 1]). Hence, for any $x \in D(\boldsymbol{L})$,

$$f(x) = g(x) = \alpha_0 \vee \alpha_3 \vee (\alpha_1 \wedge x),$$

which is obviously an order-preserving function. But $f(b) = a < b = f(a)$, a contradiction.

To show ($\mathcal{F}$), suppose that F is a relatively complete filter in $D(\boldsymbol{L})$, i.e. for any $x \in D(\boldsymbol{L})$ there exists $\min F \cap \uparrow x$. Define a function $f : D(\boldsymbol{L}) \longrightarrow D(\boldsymbol{L})$ by $f(x) = \min F \cap \uparrow x$. Note that F is the set of all fixed points of f. We claim that f is compatible on $D(\boldsymbol{L})$. Let θ be a congruence of $D(\boldsymbol{L})$ and $x \equiv y \ (\theta)$, $x \geq y$. We show that $f(x) = x \vee f(y)$. Obviously, $f(x) \geq x \vee f(y)$. On the other hand, $f(y) \in F$, therefore $x \vee f(y) \in F \cap \uparrow x$, thus $f(x) \leq x \vee f(y)$. So we have

$$f(x) = x \vee f(y) \equiv y \vee f(y) = f(y) \ (\theta).$$

We again define a compatible extension $g : L \longrightarrow D(\boldsymbol{L})$ of f by $g(x) = f(x \vee x^\star)$. In the same way as above, there exist $a, b \in L$ such that $f(x) = a \vee (b \wedge x)$ for any $x \in D(\boldsymbol{L})$. Obviously, we can assume that $a \le b$. We show that $F = {\uparrow}a \cap D(\boldsymbol{L})$. If $x \in {\uparrow}a \cap D(\boldsymbol{L})$, then $b \wedge x = f(x) \in F$, i.e. $x \in F$. Conversely, if $x \in F$ then $x = f(x) = a \vee (b \wedge x)$, thus $a \le x$, whence $x \in {\uparrow}a \cap D(\boldsymbol{L})$.

(3) $\Longrightarrow$ (2) Let $g : D(\boldsymbol{L})^n \longrightarrow D(\boldsymbol{L})$ be a compatible function of the lattice $D(\boldsymbol{L})$. We show that g can be represented by a polynomial of the lattice L.

1. Let $n = 1$. By the dual assertion to 2.4(ii), ${\uparrow}g(D(\boldsymbol{L}))$ is a relatively complete filter in $D(\boldsymbol{L})$. So by $(\mathcal{F})$ there exists $a \in L$ such that ${\uparrow}g(D(\boldsymbol{L})) = {\uparrow}a \cap D(\boldsymbol{L})$. ¿From $(\mathcal{B})$ by using 2.4(i) it follows that $g(x)$ is a fixed point of g for any $x \in D(\boldsymbol{L})$. Therefore by 2.3, the set ${\uparrow}(x \wedge g(1))$ is closed under g. Further, $a \vee x \in {\uparrow}a \cap D(\boldsymbol{L}) = {\uparrow}g(D(\boldsymbol{L}))$, hence $a \vee x \ge g(y)$ for some $y \in D(\boldsymbol{L})$. Because $g(y)$ is a fixed point of g, ${\downarrow}(a \vee x)$ is closed under g. Since $x \in {\downarrow}(a \vee x) \cap {\uparrow}(x \wedge g(1))$, we have that $g(x) \in {\downarrow}(a \vee x) \cap {\uparrow}(x \wedge g(1))$. Further, $g(x) \in {\uparrow}a$ as $g(x) \in {\uparrow}g(D(\boldsymbol{L}))$, and $g(x) \in {\downarrow}g(1)$ because g is order-preserving on $D(\boldsymbol{L})$ by $(\mathcal{B})$ and 2.5. Hence

$$g(x) \in {\downarrow}((a \vee x) \wedge g(1)) \cap {\uparrow}(a \vee (x \wedge g(1)),$$

thus $g(x) = (a \vee x) \wedge g(1)$, which is a polynomial of the lattice L.

2. Now let $n > 1$. For every subset $S \subseteq \{1, \dots, n\} = \underline{n}$ we define a unary function $g_S : D(\boldsymbol{L}) \longrightarrow D(\boldsymbol{L})$ by $g_S(x) = g(y_1, \dots, y_n)$ where

$$y_i = \begin{cases} 1 & \text{if } i \in S, \\ x & \text{if } i \notin S. \end{cases}$$

Thus $g_{\underline{n}}$ is a constant function equal to $g(1, \dots, 1)$. Obviously, any g_S is a compatible function of $D(\boldsymbol{L})$, hence by case 1 there exists a constant $a_S \in L$ such that

$$g_S(x) = (a_S \vee x) \wedge g_S(1).$$

We take $a_{\underline{n}} = g(1, \dots, 1)$. Further, we show that the constants a_S can be chosen in such a way that $a_T \le a_S$ whenever $T \subseteq S$. The constant a_T can be any element of L satisfying ${\uparrow}g_T(D(\boldsymbol{L})) = {\uparrow}a_T \cap D(\boldsymbol{L})$. We claim that if a_T has this property then the element $b_T = \bigwedge_{S \supseteq T} a_S$ has it too. Clearly, ${\uparrow}g_T(D(\boldsymbol{L})) \subseteq {\uparrow}b_T \cap D(\boldsymbol{L})$. Since the function g is order-preserving by 2.5, we have ${\uparrow}g_T(D(\boldsymbol{L})) \supseteq {\uparrow}g_S(D(\boldsymbol{L})) = {\uparrow}a_S \cap D(\boldsymbol{L})$ for every $S \supseteq T$. If $x \in {\uparrow}b_T \cap D(\boldsymbol{L})$ then $x \ge b_T$ and clearly $x = \bigwedge_{S \supseteq T}(x \vee a_S)$. For every $S \supseteq T$ the element $x \vee a_S$ belongs to ${\uparrow}a_S \cap D(\boldsymbol{L})$, therefore $x \vee a_S \in {\uparrow}g_T(D(\boldsymbol{L}))$. Since ${\uparrow}g_T(D(\boldsymbol{L}))$ is a filter by (the dual to) 2.4, we obtain $x \in {\uparrow}g_T(D(\boldsymbol{L}))$. Hence, ${\uparrow}g_T(D(\boldsymbol{L})) = {\uparrow}b_T \cap D(\boldsymbol{L})$ and we can take b_T instead of a_T. So we assume that $a_T \le a_S$ whenever $T \subseteq S$.

We define a lattice polynomial $p(x_1, \dots, x_n)$ as follows:

$$p(x_1, \dots, x_n) = \bigvee_{S \subseteq \underline{n}} (a_S \wedge \bigwedge_{i \in S} x_i).$$

We prove that g and p coincide on $D(\boldsymbol{L})^n$. First we show that for any $S \subseteq \underline{n}$ the functions g_S and p_S coincide, where $p_S : D(\boldsymbol{L}) \longrightarrow D(\boldsymbol{L})$ is defined by $p_S(x) = p(y_1, \dots, y_n)$ with

$$y_i = \begin{cases} 1 & \text{if } i \in S, \\ x & \text{if } i \notin S. \end{cases}$$

But it is clear that for any $x \in D(\boldsymbol{L})$, $S \neq \underline{n}$

$$p_S(x) = \bigvee_{T \subseteq S} a_T \vee \bigvee_{T \nsubseteq S} (a_T \wedge x) = a_S \vee (a_{\underline{n}} \wedge x) = g_S(x).$$

Moreover, both $g_{\underline{n}}$ and $p_{\underline{n}}$ are constant functions equal to $a_{\underline{n}}$. Hence p_S and g_S coincide for any S.

Suppose to the contrary that there exists $\tilde{c} = (c_1, \dots, c_n) \in D(\boldsymbol{L})^n$ such that $g(\tilde{c}) \neq p(\tilde{c})$. Then there exists a lattice homomorphism $h : D(\boldsymbol{L}) \longrightarrow \mathbf{2} = \{\underline{0}, \underline{1}\}$ with $h(g(\tilde{c})) \neq h(p(\tilde{c}))$. Obviously, $h(1) = \underline{1}$ and $h(b) = \underline{0}$ for some $b \in D(\boldsymbol{L})$. Let $S = \{i \in \underline{n} : \ h(c_i) = \underline{1}\}$ and let $\tilde{b} = (b_1, \dots, b_n) \in D(\boldsymbol{L})^n$ be such that $b_i = 1$ if $i \in S$, and $b_i = b$ if $i \notin S$. Then $h(c_i) = h(b_i)$ for every $i \in \underline{n}$. The compatible functions p and g preserve the kernel congruence of h, hence $h(g(\tilde{c})) = h(g(\tilde{b}))$ and $h(p(\tilde{c})) = h(p(\tilde{b}))$. Since $g(\tilde{b}) = g_S(b) = p_S(b) = p(\tilde{b})$, we obtain that $h(g(\tilde{c})) = h(p(\tilde{c}))$, a contradiction.

(2) $\Longrightarrow$ (1) Let $f : L^n \longrightarrow L$ be a compatible function of the Stone algebra $\boldsymbol{L}$. Since $\boldsymbol{L}$ satisfies the identity $x = x^{\star\star} \wedge (x \vee x^\star)$, we can write

$$f(\tilde{x}) = f(\tilde{x})^{\star\star} \wedge (f(\tilde{x}) \vee f(\tilde{x})^\star)$$

for any $\tilde{x} = (x_1, \dots, x_n) \in L^n$. We shall show that instead of $f(\tilde{x})^{\star\star}$ and $f(\tilde{x}) \vee f(\tilde{x})^\star$ we can write in this formula some polynomials of the algebra $\boldsymbol{L}$.

Since $x_i \equiv x_i^{\star\star}\ (\Phi)$ and f preserves the Glivenko congruence Φ, we have $f(\tilde{x})^{\star\star} = f(x_1^{\star\star}, \dots, x_n^{\star\star})^{\star\star}$. Define a function $h : B(\boldsymbol{L})^n \longrightarrow B(\boldsymbol{L})$ by $h(\tilde{x}) = f(\tilde{x})^{\star\star}$. We show that h is compatible on the Boolean algebra $B(\boldsymbol{L})$. For any congruence θ_B of $B(\boldsymbol{L})$ we define a relation θ_L on L by

$$x \equiv y\ (\theta_L) \quad \text{iff} \quad x^{\star\star} \equiv y^{\star\star}\ (\theta_B).$$

Because of the identities $(x \vee y)^{\star\star} = x^{\star\star} \vee y^{\star\star}$ and $(x \wedge y)^{\star\star} = x^{\star\star} \wedge y^{\star\star}$, θ_L is a congruence of the Stone algebra $\boldsymbol{L}$ and θ_B is its restriction to $B(\boldsymbol{L})$. Hence for any $x_i, y_i \in B(\boldsymbol{L})$ $(i = 1, \dots, n)$, $x_i \equiv y_i\ (\theta_B)$ implies that $x_i \equiv y_i\ (\theta_L)$, thus $f(\tilde{x}) \equiv f(\tilde{y})\ (\theta_L)$. Therefore $h(\tilde{x}) = f(\tilde{x})^{\star\star} \equiv f(\tilde{y})^{\star\star} = h(\tilde{y})\ (\theta_B)$, which shows that h is compatible. Since any Boolean algebra is affine complete, there exists a polynomial $b(x_1, \dots, x_n)$ of $B(\boldsymbol{L})$ representing h. Hence for any $\tilde{x} = (x_1, \dots, x_n) \in L^n$ we have $f(\tilde{x})^{\star\star} = f(x_1^{\star\star}, \dots, x_n^{\star\star})^{\star\star} = h(x_1^{\star\star}, \dots, x_n^{\star\star}) = b(x_1^{\star\star}, \dots, x_n^{\star\star})$, which is a polynomial of the algebra $\boldsymbol{L}$, because the complement operation in b can be expressed by *.

To represent the function $f(\tilde{x}) \vee f(\tilde{x})^\star$ by a polynomial of $\boldsymbol{L}$, we prove that any compatible function on $\boldsymbol{L}$ whose range is a subset of $D(\boldsymbol{L})$ is a polynomial function of $\boldsymbol{L}$. Let $g : L^n \longrightarrow D(\boldsymbol{L})$ be compatible on $\boldsymbol{L}$. Note that any lattice congruence θ_D of $D(\boldsymbol{L})$ can be extended to a Stone congruence $\theta_L \cap \Phi$ of $\boldsymbol{L}$ where θ_L is any extension of θ_D to the lattice L. Therefore the function $g \restriction D(\boldsymbol{L})^n$ is compatible on the lattice $D(\boldsymbol{L})$. By hypothesis we have then a polynomial $l(\tilde{x})$ of the lattice L such that $g(\tilde{x}) = l(\tilde{x})$ for all $\tilde{x} \in D(\boldsymbol{L})^n$. To represent g by a polynomial of $\boldsymbol{L}$, we proceed by induction on the arity n of g:

1. Let $n = 1$. We show that

$$g(x) = (x^{\star\star} \wedge l(x)) \vee (x^\star \wedge g(0)).$$

By 3.2, it suffices to show that both sides are equal on $\{0\} \cup D(\boldsymbol{L})$, which is obvious.
2. Let $n > 1$. We show using 3.2 again that

$g(x_1,\dots,x_n) = (x_1^{\star\star} \wedge \cdots \wedge x_n^{\star\star} \wedge l(x_1,\dots,x_n)) \vee (x_1^{\star} \wedge x_2^{\star\star} \wedge \cdots \wedge x_n^{\star\star} \wedge g(0,x_2,\dots,x_n)) \vee \cdots \vee (x_1^{\star\star} \wedge \cdots \wedge x_{n-1}^{\star\star} \wedge x_n^{\star} \wedge g(x_1,\dots,x_{n-1},0)) \vee (x_1^{\star} \wedge x_2^{\star} \wedge x_3^{\star\star} \wedge \cdots \wedge x_n^{\star\star} \wedge g(0,0,x_3,\dots,x_n)) \vee \cdots \vee ((x_1^{\star} \wedge \cdots \wedge x_n^{\star} \wedge g(0,\dots,0))$.
Both sides are equal on $(\{0\} \cup D(\boldsymbol{L}))^n$, and by the induction hypothesis we have on the right side a polynomial of the algebra $\boldsymbol{L}$.

Hence, the function $g(\tilde{x}) = f(\tilde{x}) \vee f(\tilde{x})^\star : L^n \longrightarrow D(\boldsymbol{L})$ which is compatible on $\boldsymbol{L}$ can also be represented by a polynomial of the algebra $\boldsymbol{L}$. The proof is complete. □

Algebras of which all unary compatible functions are polynomial are called *1-affine complete* (see [4]). From the proof above it can be deduced the following statement:

3.4. Corollary. *Let $\boldsymbol{L}$ be a Stone algebra. Then $\boldsymbol{L}$ is affine complete iff $\boldsymbol{L}$ is 1-affine complete.* □

Note that if a Stone algebra $\boldsymbol{L}$ has a smallest dense element then the condition (2) in 3.3 is equivalent to (2) of 3.1 while the condition $(\mathcal{F})$ is trivially satisfied. Hence Beazer's result easily follows from 3.3.

Let us mention a weaker form of affine completeness which can be found in the literature. An algebra $\boldsymbol{A}$ is said to be *locally affine complete* if for every $n \geq 1$, every n-ary compatible function on $\boldsymbol{A}$ can be interpolated on any finite subset $F \subset A^n$ by a polynomial of $\boldsymbol{A}$ (see e.g. [10]). Clearly, every affine complete algebra is locally affine complete, and for finite algebras both concepts coincide. Locally affine complete distributive lattices have been characterized in [4]:

3.5. Theorem ([4; Corollary 1 on p. 102]). *A distributive lattice L is locally affine complete if and only if L does not contain a proper Boolean interval.* □

A characterization of locally affine complete Stone algebras has been presented in [9]:

3.6. Theorem ([9; Theorem 2]). *Let $\boldsymbol{L}$ be a Stone algebra. The following are equivalent.*

(1) *$\boldsymbol{L}$ is locally affine complete;*
(2) *$D(\boldsymbol{L})$ is locally affine complete lattice;*
(3) *no proper interval of $D(\boldsymbol{L})$ is Boolean.* □

We close this paper with several examples illustrating the results above.

3.7. Examples.

(1) Take the interval $[0,1]$ of the real numbers and put $L = [0,1]^2$. Evidently, $\boldsymbol{L}$ is a Stone algebra with $D(\boldsymbol{L}) = (0,1]^2$. Since for any $a \in L \setminus D(\boldsymbol{L})$, $F = \uparrow a \cap D(\boldsymbol{L})$ is a relatively complete filter in $D(\boldsymbol{L})$ without a smallest element, $D(\boldsymbol{L})$ is not affine complete by 2.1. On the other hand, one can verify that $\boldsymbol{L}$ satisfies the conditions $(\mathcal{B})$ and $(\mathcal{F})$ of 3.3. Hence, $\boldsymbol{L}$ is an affine complete Stone algebra, which also provides a negative answer to Beazer's question in [1].

(2) Any Stone algebra obtained by adding a new zero to an affine complete distributive lattice with unit is evidently affine complete.

(3) Consider the Stone algebra $\boldsymbol{L} = \{0\} \cup (\mathbb{R} \times \mathbb{R}) \cup \{1\}$. Let

$$F = \{(x, y) \in L : \ x \geq 0\} \cup \{1\}.$$

Obviously, F is a relatively complete filter in $D(\boldsymbol{L})$ and there is no $a \in L$ such that $F = \uparrow a \cap D(\boldsymbol{L})$. Hence $\boldsymbol{L}$ is not affine complete. On the other hand, $\boldsymbol{L}$ is locally affine complete by 3.6.

Acknowledgements. This paper was written while the first author held a Soros scholarship at Oxford University for the academic year 1993/94. He would like to express his gratitude to H.A. Priestley and the Mathematical Institute in Oxford for the hospitality during this period.

References

[1] Beazer R., *Affine complete Stone algebras*, Acta. Math. Acad. Sci. Hungar. **39** (1982), 169-174.

[2] ______, *Affine complete double Stone algebras with bounded core*, Algebra Universalis **16** (1983), 237-244.

[3] Clark D. and Werner H., *Affine completeness in semiprimal varieties*, Finite algebra and Multiple-Valued Logic (Proc. Conf. Szeged, 1979). Colloq. Math. Soc. J. Bolyai **28** (1981), Amsterdam: North-Holland, 809-823.

[4] Dorninger D. and Eigenthaler G., *On compatible and order-preserving functions on lattices*, Universal algebra and appl. **9** (1982), Banach Center Publ., Warsaw, 97-104.

[5] Grätzer G., *On Boolean functions (Notes on Lattice theory II)*, Revue de Math. Pures et appliquées **7** (1962), 693-697.

[6] ______, *Boolean functions on distributive lattices*, Acta Math. Acad. Sci. Hungar. **15** (1964), 195-201.

[7] ______, *Universal Algebra*, Van Nostrand, Toronto-London-Melbourne, 1968.

[8] ______, *Lattice Theory. First concepts and distributive lattices.*, W.H. Freeman and Co., San Francisco, Calif., 1971.

[9] Haviar M., *Affine complete algebras abstracting Kleene and Stone algebras*, Acta Math. Univ. Comenianae **2** (1993), 179-190.

[10] Kaarli K., Márki L. and Schmidt E.T., *Affine complete semilattices*, Mh. Math. **99** (1985), 297-309.

[11] Kaarli K. and Pixley A.F., *Affine complete varieties*, Algebra Universalis **24** (1987), 74-90.

[12] Pixley A.F., *Completeness in arithmetical algebras*, Algebra Universalis **2** (1972), 179-196.

[13] ______, *Characterizations of arithmetical varieties*, Algebra Universalis **9** (1979), 87-98.

[14] ______, *Functional and affine completeness and arithmetical varieties*, Algebras and Orders, NATO ASI Series, Series C **389** (1993), 317-357.

[15] Ploščica M., *Affine complete distributive lattices (submitted to Order).*

[16] Werner H., *Produckte von Kongruenzklassengeometrien universeller algebren*, Math. Z. **121** (1971), 111-140.

Department of Mathematics, M. Bel University, Tajovského 40, 975 49 Banská Bystrica, Slovakia

and

Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 040 01 Košice, Slovakia