GRAPHICAL COMPOSITIONS AND WEAK CONGRUENCES

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Abstract. Graphical compositions of equivalences were introduced (independently) by B. Jónsson and H. Werner in order to determine whether a subset of $\mathrm{Eq}(X)$ (the set of all equivalences on the set X) is the set of all congruences of some algebra defined on X. Namely, a complete sublattice L of $\mathrm{Eq}(X)$ is the congruence lattice of some algebra defined on X if and only if L is closed under all graphical compositions. We generalize this result and prove that a similar characterization is possible for weak congruences (i. e. symmetric and transitive compatible relations).

Weak congruences were introduced and investigated by B. Šešelja, G. Vojvodić and A. Tepavčević in [1]-[4] and other papers. Let us recall basic concepts.

An algebra $\mathcal{A} = (A, F)$ is a set A (called the underlying set) endowed with some set F of finitary operations (called the basic operations of $\mathcal{A}$). A finitary function $f: A^n \to A$ is called a polynomial of $\mathcal{A}$, if it can be obtained from projections, constant functions and basic operations of $\mathcal{A}$ by means of compositions.

Let X be a set. A weak equivalence on X is any symmetric and transitive binary relation. We denote by $\mathrm{Eq}(X)$, $\mathrm{E}_w(X)$ and $\mathrm{Rel}(X)$ the sets of all equivalences, weak equivalences and binary relations on the set X, respectively. Let $f: X^n \to X$ be any function. We say that f preserves a relation $\rho \in \mathrm{Rel}(X)$ if $(x_1, y_1), \dots, (x_n, y_n) \in \rho$ implies $(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \in \rho$. A nullary function f (i.e. a constant $f \in X$) preserves $\rho \in \mathrm{Rel}(X)$ if $(f, f) \in \rho$. A binary relation $\rho \in \mathrm{Rel}(A)$ is called compatible with the algebra $\mathcal{A} = (A, F)$ if every $f \in F$ preserves ρ.Such a compatible relation is a (weak) congruence of $\mathcal{A}$ if it is a (weak) equivalence. It is easy to see that a relation ρ is a weak congruence of $\mathcal{A}$ if and only if it is a congruence of some subalgebra of $\mathcal{A}$. We denote bt $\mathrm{Con}(\mathcal{A})$ and $\mathrm{C}_w(\mathcal{A})$ the sets of all congruences and weak congruences of $\mathcal{A}$, respectively.

The domain of a relation $\alpha \in \mathrm{Rel}(X)$ is the set $\mathrm{dom}(\alpha) = \{x \in X \mid (x, x) \in \alpha\}$. For any $\alpha \in \mathrm{Rel}(X)$ we consider its restrictions to its domain $\alpha \restriction \mathrm{dom}(\alpha) = \alpha \cap (\mathrm{dom}(\alpha))^2 = \{(x, y) \in \alpha \mid (x, x) \in \alpha,\ (y, y) \in \alpha\}$. It is easy to see that the symmetric and transitive closure of $\alpha \restriction \mathrm{dom}(\alpha)$ is always a weak equivalence and we call it the weak equivalence generated by α. We stress that we form the closure of $\alpha \restriction \mathrm{dom}(\alpha)$ and not of α itself. In fact, the symmetric and transitive closure of any relation is a weak equivalence.

Lemma 1. *Let α be a weak equivalence on an algebra A. Then $\alpha \in \mathrm{C}_w(A)$ if and only if the following conditions hold:*

(i) $\mathrm{dom}(\alpha)$ *is a subalgebra of A;*
(ii) *every unary polynomial f of the algebra* $\mathrm{dom}(\alpha)$ *preserves α (i.e. $(x, y) \in \alpha$ implies $(f(x), f(y)) \in \alpha$).*

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Proof. It is easy to see that any weak congruence satisfies (i) and (ii). Conversely, suppose that (i) and (ii) hold for $\alpha \in \mathrm{Eq}(A)$. Let $f : A^n \longrightarrow A$ be any of the basic operations of the algebra A. Suppose that $(a_i, b_i) \in \alpha$ for $i = 1, \ldots, n$. Then clearly $a_i \in \mathrm{dom}(\alpha)$, $b_i \in \mathrm{dom}(\alpha)$ for every i. Let us consider the unary polynomial $f_1(x) = f(x, a_2, \ldots, a_n)$. Because of (ii), $(a_1, b_1) \in \alpha$ implies that $(f_1(a_1), f_1(b_1)) \in \alpha$ and therefore $(f(a_1, \ldots, a_n), f(b_1, a_2, \ldots, a_n)) \in \alpha$. Similarly we can show that $(f(b_1, \ldots, b_i, a_{i+1}, \ldots, a_n), f(b_1, \ldots, b_{i+1}, a_{i+2}, \ldots, a_n)) \in$ αholds for any $i = 0, 1, \ldots, n-1$. From the transitivity of α we infer that $(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \alpha$.

□

Lemma 2. *Let α be a compatible binary relation on an albebra A. Then the weak equivalence generated by α is also compatible. (And hence it is a weak congruence.)*

Proof. Let $\beta \in \mathrm{E}_w(A)$ be generated by α. We prove that β satisfies (i), (ii) from Lemma 1.

It is easy to see that $\mathrm{dom}(\beta) = \mathrm{dom}(\alpha)$. Let $f : A^n \longrightarrow A$ be any of the basic operations of A, let $\{a_1, \ldots, a_n\} \subseteq \mathrm{dom}(\beta)$. Then $(a_i, a_i) \in \alpha$ for every i and since α is compatible we obtain that $(f(a_1, \ldots, a_n), f(a_1, \ldots, a_n) \in \alpha$, hence $f(a_1, \ldots, a_n) \in \mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$.

To prove (ii), let f be a unary polynomial of the algebra $\mathrm{dom}(\beta)$. (That is, the constants used in f belong to $\mathrm{dom}(\beta)$.) Let $(x, y) \in \beta$. Then we have a finite sequence $x = z_0, z_1, \ldots, z_k = y$ such that, for every $i = 1, \ldots, k$, $(z_{i-1}, z_i) \in \alpha \restriction \mathrm{dom}(\alpha)$ or $(z_i, z_{i-1}) \in \alpha \restriction \mathrm{dom}(\alpha)$. Since $\mathrm{dom}(\alpha)$ is closed under f (the first part of this proof) and the relation α is compatible, it follows that $(f(z_{i-1}, f(z_i)) \in \alpha \restriction \mathrm{dom}(\alpha)$ or $(f(z_i), f(z_{i-1})) \in \alpha \restriction \mathrm{dom}(\alpha)$. Now $f(x) = f(z_0), f(z_1), \ldots, f(z_k) = f(y)$ is the sequence showing that $(f(x), f(y)) \in \beta$. □

Now we recall the definition of graphical compositions. A (undirected) graph is a pair (V, E) of sets V and E, whose elements are called vertices and edges, together with a map $\nu : E \longrightarrow \mathcal{P}_1(V) \cup \mathcal{P}_2(V)$, where $\mathcal{P}_1(V)$ and $\mathcal{P}_2(V)$ are the sets of all one element subsets and of all two element subsets of V, respectively. If $\nu(e) = \{x, y\}$, we say that e is an edge between x and y. If $\nu(e) = \{x, y\}$, we say that e is an edge between x and y. If $\nu(e) = \{x\}$, we say that e is a loop on x. Hence, we admit several edges with the same endpoints. (In [5], Werner excludes loops, but in the case of weak congruences they are useful.)

Let $G = (V, E)$ be a graph and let $\phi : E \longrightarrow \mathrm{Rel}(X)$ be a mapping. A function $f : V \longrightarrow X$ is called a ϕ-compatible labelling if, for every $e \in E$, $e = \{x, y\}$ implies that $(f(x), f(y)) \in \phi(e)$. (The case $x = y$ is included.) For every two distinguished vertices $0, 1 \in V$ we define a relation

$$S_{G,0,1}(\phi) = \{(a, b) \in X^2 \mid a = f(0), b = f(1) \text{ for some } \phi\text{-compatible labelling } f\}.$$

Thus, $S_{G,0,1}$ is a mapping $\mathrm{Rel}(X)^E \longrightarrow \mathrm{Rel}(X)$. We define a mapping

$$P_{G,0,1} : \mathrm{E}_w(X)^E \longrightarrow \mathrm{E}_w(X)$$

by the rule that $P_{G,0,1}(\phi)$ is the weak equivalence generated by $S_{G,0,1}(\phi)$. Hence, every coloured graph with two distinguished vertices determines a $|E|$-ary operation on the set $\mathrm{E}_w(X)$. Any such operation is called a graphical composition.

As an illustration, let us present two simple examples. (More examples can be found in [5].)

First, let G and ϕ be as follows.

$$\begin{tikzcd} 0 \arrow[r, dash, bend left, "\alpha"] \arrow[r, dash, bend right, "\beta"'] & 1 \end{tikzcd}$$

(In pictures like this, each edge e is labelled by $\phi(e)$.) It is easy to see that $S_{G,0,1}(\phi) = \alpha \cap \beta$. If α and β are weak equivalences, then $S_{G,0,1}(\phi)$ is also a weak equivalence and therefore $P_{G,0,1}(\phi) = S_{G,0,1}(\phi)$. Hence, this graphical composition is the usual intersection of two relations. By adding more edges between 0 and 1 we obtain a graphical composition that describes the intersection of arbitrarily many (even of infinite number) relations.

As the second example we consider the following graph.

$$\begin{tikzcd} 0 \arrow[r, dash, "\alpha"] & v \arrow[r, dash, "\beta"] & 1 \end{tikzcd}$$

Suppose that $\alpha, \beta \in \mathrm{E}_w(X)$ and denote $Y = \mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta)$. The restrictions $\alpha \restriction Y$ and $\beta \restriction Y$ are equivalences on the set Y. We claim that $P_{G,0,1}(\phi)$ is the least equivalence on Y containing $\alpha \restriction Y$ and $\beta \restriction Y$ (the join in the lattice $E(Y)$). First, it is easy to see that $S_{G,0,1}(\phi)$ is equal to the relational product

$$\alpha \cdot \beta = \{(x, y) \in X^2 \mid (x, z) \in \alpha,\ (z, y) \in \beta \text{ for some } z \in X\}.$$

Since α and β are weak equivalences, it follows that $\mathrm{dom}(P_{G,0,1}(\phi)) = \mathrm{dom}(S_{G,0,1}(\phi)) = Y$, hence $P_{G,0,1}(\phi) \in E(Y)$. Further, $\alpha \restriction Y \subseteq P_{G,0,1}(\phi)$. Indeed, if $(x, y) \in \alpha \restriction Y$, then $(y, y) \in \beta$, which shows that $(x, y) \in \alpha \cdot \beta \subseteq P_{G,0,1}(\phi)$. For similar reasons, $\beta \restriction Y \subseteq P_{G,0,1}(\phi)$. On the other hand, if θ is any weak equivalence containing both $\alpha \restriction y$ and $\beta \restriction Y$, then also $S_{G,0,1}(\phi) \restriction Y = \alpha \restriction Y \cdot \beta \restriction Y \subseteq \theta$. Since $P_{G,0,1}(\phi)$ is, by the definition, the least weak equivalence containing $S_{G,0,1}(\phi) \restriction Y$, it follows that $P_{G,0,1}(\phi) \subseteq \theta$.

If $\alpha, \beta \in E(X)$, then $Y = X$ and $P_{G,0,1}(\phi)$ is the usual join of equivalence relations. Hence, this graphical composition can be regarded as a generalization of the join operation to weak equivalences.

Lemma 3. *Let A be an algebra, let $G = (V, E)$ be a graph, $0, 1 \in V$. Suppose that $\phi : E \longrightarrow \mathrm{Rel}(A)$ is such that $\phi(e)$ is a compatible relation on $\mathcal{A}$ for every $e \in E$. Then $S_{G,0,1}(\phi)$ is also a compatible relation on $\mathcal{A}$.*

Proof. Let $(a_i, b_i) \in S_{G,0,1}(\phi)$ for $i = 1, \ldots, k$. Let $g : A^k \longrightarrow A$ be any of the basic operations of $\mathcal{A}$. For every i we have a ϕ-compatible labelling $f_i : V \longrightarrow A$ with $f_i(0) = a_i$, $f_i(1) = b_i$. Define a function $f : V \longrightarrow A$ by $f(x) = g(f_1(x), \ldots, f_k(x))$. Then $f(0) = g(a_1, \ldots, a_k)$, $f(1) = g(b_1, \ldots, b_k)$. It remains to show that f is a ϕ-compatible labelling.

Let $e \in E$, $e = \{x, y\}$. Then $(f_i(x), f_i(y)) \in \phi(e)$ for every $i = 1, \ldots, k$. Since, by the assumption, the relation $\phi(e)$ is compatible, we obtain that $(f(x), f(y)) = (g(f_1(x), \ldots, f_k(x)), g(f_1(y), \ldots, f_k(y))) \in \phi(e)$.

□

As a consequence of Lemma 2 and Lemma 3 we obtain the following assertion.

Lemma 4. *Let $\mathcal{A} = (A, F)$ be an algebra. Then $\mathrm{C}_w(\mathcal{A})$ is a subset of $\mathrm{E}_w(A)$ closed under all graphical compositions (i.e. for any graph G with two distinguished vertices, if $\phi : E \longrightarrow \mathrm{C}_w(A)$ then also $P_{G,0,1}(\phi) \in \mathrm{C}_w(\mathcal{A})$).*

Now we show that closedness under all graphical compositions is not sufficient for characterization of those subsets of $\mathrm{E}_w(X)$ that are equal to $\mathrm{C}_w(\mathcal{A})$ for some algebra $\mathcal{A}$ defined on the set X. (Similarly as in the case of usual congruence relations.)

To see this, let $\mathcal{A}$ be an algebra whose congruence lattice looks as follows.

$$\begin{array}{c} \nabla \\ | \\ \psi \\ \vdots \\ | \\ \theta_2 \\ | \\ \theta_1 \\ | \\ \Delta \end{array}$$

Such an algebra $\mathcal{A}$ certainly exists, since the lattice is algebraic. We can assume that all elements of $\mathcal{A}$ are nullary operations (constants), so that $\mathrm{C}_w(\mathcal{A}) = \mathrm{Con}(\mathcal{A})$ and our example serves both the case of usual and weak congruences. Let us consider the family $\mathcal{F} = \{\Delta, \nabla, \theta_1, \theta_2, \dots\} = \mathrm{Con}(\mathcal{A}) \setminus \{\psi\}$. This family cannot be the set of all (weak) congruences of any algebra, since it is not a complete sublattice of $\mathrm{Eq}(A)$. (It is not closed under infinite joins.) However, we claim that $\mathcal{F}$ is closed under all graphical compositions.

To see this, let $G = (V, E)$ be a graph, $0, 1 \in V$, $\phi : \ E \longrightarrow \mathcal{F}$. Without loss of generality we can assume that $\nabla \notin \phi(E)$. Indeed, if $\nabla \in \phi(E)$, then we consider the graph $G' = (V, E')$, where $E' = \{e \in E \mid \phi(e) \neq \nabla\}$, and the restriction $\phi' = \phi \restriction E'$. It is easy to see that $S_{G,0,1}(\phi) = S_{G',0,1}(\phi')$.

Thus, suppose that $\nabla \notin \phi(E)$. We distinguish two cases. First, suppose that there is a path $(e_1, \dots, e_k)$ in E connecting 0 and 1. Then $P_{G,0,1}(\phi)$ cannot be greater (in the sense of set inclusion) than the greatest relation among $\phi(e_1), \dots, \phi(e_k)$. Hence, $P_{G,0,1}(\phi)$ is equal to Δ or to some θ_i.

The second possibility is that there is no path between 0 and 1. Then it is not difficult to see that $P_{G,0,1}(\phi) = S_{G,0,1}(\phi) = A^2 = \nabla$.

We have proved that $P_{G,0,1}(\phi)$ cannot be equal to ψ, which means that $\mathcal{F}$ must be closed under all graphical compositions.

The example above suggests what we should add to graphical compositions. A family $\mathcal{F} \subseteq \mathrm{E}_w(X)$ is called up-directed if for every $\alpha, \beta \in \mathcal{F}$ there is a $\gamma \in \mathcal{F}$ with $\alpha \cup \beta \subseteq \gamma$. It is easy to see that if $\mathcal{F}$ is such an up-directed family, then the set-theretical union $\bigcup \mathcal{F}$ is a weak equivalence. Further, if all relations in $\mathcal{F}$ are compatible with some algebraic structure on X, then $\bigcup \mathcal{F}$ is also compatible (and hence a weak congruence). We obtain the following assertion.

Lemma 5. *For any algebra $\mathcal{A}$, the set $\mathrm{C}_w(\mathcal{A})$ is closed under unions of up-directed families $\mathcal{F} \subseteq \mathrm{C}_w(\mathcal{A})$.*

Now we are going to prove the converse of Lemmas 4 and 5. Let us suppose that $\mathcal{F} \subseteq \mathrm{E}_w(X)$ is closed under all graphical compositions and up-directed unions.

First notice that $\mathcal{F}$ is closed under intersections. (See the example preceding Lemma 2. In accordance with this example, the intersection of the empty family of relations is equal to the greatest relation X^2.) Hence, for every $\alpha \in \mathrm{Rel}(X)$ there is a smallest $\beta \in \mathcal{F}$ with $\alpha \subseteq \beta$. We use the denotation $\beta = \alpha^{\mathcal{F}}$.

We shall use some special graphs. Let G be the graph, whose set of vertices is X and the number of edges between vertices x and y is equal to the number of all $\alpha \in \mathcal{F}$ containing (x, y). (This applies also to loops.) Formally, the set E of edges can be expressed as $E = \{(\{x, y\}, \alpha) \mid x, y \in X,\ \alpha \in \mathcal{F} \text{ and} (x, y) \in \alpha\}$ and the map ν is defined by $\nu((\{x, y\}, \alpha)) = \{x, y\}$.

Similarly we define the graph G^n (the n-th power of G). The set of vertices of G^n will be X^n and we put an edge $(\{\overline{x}, \overline{y}\}, \alpha)$ between $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$ whenever $(x_i, y_i) \in \alpha$ for every $i = 1, \ldots, n$. Hence, $G = G^1$.

The importance of the graphs defined above lies in the following easy fact.

Lemma 6. *Let $\varphi :\ E \longrightarrow \mathcal{F}$ be defined by $\varphi((\{\overline{x}, \overline{y}\}, \alpha)) = \alpha$. A function $f : X^n \to X$ is a φ-compatible labelling on G^n if and only if f preserves all $\alpha \in \mathcal{F}$.*

Now we are ready to define an algebra on the set X whose set of all equivalences equals $\mathcal{F}$.

Lemma 7. *Let $\mathcal{G}$ be the set of all finitary operations on X that preserve every $\alpha \in \mathcal{F}$. Let $\mathcal{A}$ be the algebra with X as the underlying set and $\mathcal{G}$ as the set of basic operations. Then $\mathcal{F} = \mathrm{C}_w(\mathcal{A})$.*

Proof. By the definition, every basic operation of $\mathcal{A}$ preserves all relations in $\mathcal{F}$, hence $\mathcal{F} \subseteq \mathrm{C}_w(A)$.

To prove the other inclusion, let $\alpha \in \mathrm{C}_w(\mathcal{A})$, i.e. α is a weak equivalence on X that is preserved by all $f \in \mathcal{G}$.

Let $\beta = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ be an arbitrary finite subset (subrelation) of α. Consider the graph G^n with $n = 3k$. We distinguish the vertices
$0 = (x_1, \ldots, x_k, y_1, \ldots, y_k, x_1, \ldots, x_k)$,
$1 = (x_1, \ldots, x_k, y_1, \ldots, y_k, y_1, \ldots, y_k)$.
Let φ be defined by $\varphi((\{\overline{x}, \overline{y}\}, \alpha)) = \alpha$. By our assumption, the relation $P_{G^n,0,1}(\varphi)$ belongs to $\mathcal{F}$. It is easy to see that for every $i = 1, \ldots, n$ the i-th projection $f_i :\ X^n \to X$ (i.e. $f_i(z_1, \ldots, z_n) = z_i$) is a φ-compatible labelling. For $i = 1, \ldots, k$ we obtain that $(f_i(0), f_i(1)) = (x_i, x_i) \in S_{G^n,0,1}(\varphi)$, $(f_{i+k}(0), f_{i+k}(1)) = (y_i, y_i) \in S_{G^n,0,1}(\varphi)$, $(f_{i+2k}(0), f_{i+2k}(1)) = (x_i, y_i) \in S_{G^n,0,1}(\varphi)$. Consequently, $\beta \subseteq S_{G^n,0,1}(\varphi) \restriction \mathrm{dom}(S_{G^n,0,1}(\varphi))$, hence $\beta \subseteq P_{G^n,0,1}(\varphi)$.

Further, for every $(x, y) \in S_{G^n,0,1}(\varphi)$ there is a φ-compatible labelling $f :\ X^n \to X$ with $f(0) = x$, $f(1) = y$. By Lemma 6, $f \in \mathcal{G}$ and by our assumption, f preserves α. Since for every $i = 1, \ldots, k$ we have $(x_i, x_i) \in \alpha$, $(y_i, y_i) \in \alpha$, $(x_i, y_i) \in \alpha$, it follows that $(x, y) = (f(0), f(1)) \in \alpha$. We have shown that $S_{G^n,0,1}(\varphi) \subseteq \alpha$. Since α is a weak equivalence, we obtain that $P_{G^n,0,1}(\varphi) \subseteq \alpha$.

Hence, for every such β there is $\gamma \in \mathcal{F}$ with $\beta \subseteq \gamma \subseteq \alpha$ (namely, $\gamma = P_{G^n,0,1}(\varphi)$, where the number n and the vertices 0, 1 depend on β). Then clearly $\beta^{\mathcal{F}} \subseteq \alpha$. The family

$$\{\beta^{\mathcal{F}} \mid \beta \text{ is a finite subset of } \alpha\}$$

is an up-directed subset of $\mathcal{F}$ and its union is α. Since $\mathcal{F}$ is closed under up-directed unions, we obtain that $\alpha \in \mathcal{F}$, which was to prove. □

From Lemmas 4, 5 and 7 we obtain our main result.

Theorem 8. *A family $\mathcal{F} \subseteq \mathrm{E}_w(X)$ is the set of all weak congruences of some algebra if and only if $\mathcal{F}$ is closed under all graphical compositions and up-directed unions.*

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