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TOPOLOGIES COMPATIBLE WITH ORDER AND SEPARATION AXIOMS

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Abstract. This paper is an addendum to [1]. It deals with some types of compatibility of a topology and an order. The aim is to study conditions on a partially ordered set (P, $\leqq$) under which every topology on P, compatible in a certain sense with a given order, is Hausdorff.

Key words. Partially ordered set, topological space, compatibility of topology and order.

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In this paper we deal with three types of compatibility, so called i-compatibility for $i \in \{1, 2, 3\}$, investigated in the papers [1], [2], [3]. For definitions and notation refer [1].

Every topology which is i-compatible ($i = 1, 2, 3$) with an ordering on a given set is T_1. We shall characterize all ordered sets P, every i-compatible ($i = 2, 3$) topology of which is T_2. For $i = 1$ this question was solved in [2]:

Theorem 1. *Let P be an ordered set. The following conditions are equivalent:*
(i) *Every $\mathcal{O} \in C_1(P)$ is T_2.*
(ii) *For every two different points $a, b \in P$ there exist finite sets $M_1 \subseteq \uparrow a - \{a\}$, $M_2 \subseteq \uparrow b - \{b\}$, $N_1 \subseteq \downarrow a - \{a\}$, $N_2 \subseteq \downarrow b - \{b\}$ such that $P - \uparrow(M_1 \cup M_2) - \downarrow(N_1 \cup N_2)$ is finite.*

We are going to prove similar results for 2- and 3-compatibility. In the whole section we assume that an ordered set P is given.

Lemma 1. *If every $\mathcal{O} \in C_2(P)$ is T_2, then $N(a, b) = N(a) \cap N(b)$ is finite whenever $a, b \in P$, $a \neq b$.*

Proof. Suppose that $a, b \in P$, $a \neq b$ and $N(a, b)$ is infinite. According to 2.5 from [1] there exists $\mathcal{O} \in C_2(P)$ such that $a \in A \in \mathcal{O}$ implies $A \cap N(a, b) \neq \emptyset$. Put $\mathcal{O}_1 = \{B \subseteq P/(B \cap \{a, b\} = \emptyset)$ or $(\exists\, A \in \mathcal{O})\,(a \in A$ and $B \supseteq A \cap N(a, b))$.

It is easy to verify that $\mathcal{O}_1 \in C_2(P)$ and $\mathcal{O}_1$ does not fulfil T_2.

Theorem 2. *The following conditions are equivalent:*

(i) *Every* $\mathcal{O} \in C_2(P)$ *is* T_2.

(ii) *Every* $\mathcal{O} \in C_1(P)$ *is* T_2.

Proof. If (ii) holds, then (i) follows from the inclusion $C_2(P) \subseteq C_1(P)$. Let us assume that (i) is valid, we are going to show that the condition (ii) from theorem 1 is valid. Let $a, b \in P$, $a \neq b$. Let us put $A_1 = \uparrow a \cap \uparrow b$, $A_2 = \uparrow a \cap \downarrow b$, $A_3 = \downarrow a \cap \uparrow b$, $A_4 = \downarrow a \cap \downarrow b$, $A_5 = N(a) \cap \uparrow b$, $A_6 = N(a) \cap \downarrow b$, $A_7 = \uparrow a \cap N(b)$, $A_8 = \downarrow a \cap N(b)$, $A_9 = N(a, b)$.

For every $i = 1, 2, \ldots, 9$ we shall find $M_1^i \subseteq \uparrow a - \{a\}$, $M_2^i \subseteq \uparrow b - \{b\}$, $N_1^i \subseteq \downarrow a - \{a\}$, $N_2^i \subseteq \downarrow b - \{b\}$ such that $A_i - \uparrow(M_1^i \cup M_2^i) - \downarrow(N_1^i \cup N_2^i)$ is finite. Then $P - \uparrow\cup(M_1^i \cup M_2^i) - \downarrow\cup(N_1^i \cup N_2^i)$ will be finite. If $x \in P$, we define the topologies

$\mathcal{O}'_x = \{A \subseteq P/(x \notin A)$ or $(A \supseteq x - \uparrow M$ for suitable finite $M \subseteq \uparrow x - \{x\}\}$,
$\mathcal{O}''_x = \{A \subseteq P/(x \notin A)$ or $(A \supseteq x - \downarrow N$ for suitable finite $N \subseteq \downarrow x - \{x\}\}$.

Let us put $\mathcal{O}_1 = \mathcal{O}'_a \cap \mathcal{O}'_b$. Let x, y be different points of P, we verify (C1) for them: If $x \notin \{a, b\}$, put $A = \{x\}$. If $x = a$ (the case $x = b$ is analogous), put $A = \uparrow x - \uparrow(\{b, y\} \cap \uparrow x)$. In both cases $x \in A \in \mathcal{O}$, $y \notin \operatorname{conv} A$ hold. By the assumption $\mathcal{O}_1$ is T_2. This fact yields $A, B \in \mathcal{O}$ such that $a \in A$, $b \in B$, $A \cap B = \emptyset$. There exist finite $M_1^1 \subseteq \uparrow a - \{a\}$, $M_2^1 \subseteq \uparrow b - \{b\}$, $A \supseteq \uparrow a - \uparrow M_1^1$, $B \supseteq \uparrow b - \uparrow M_2^1$. If $N_1^1 = N_2^1 = \emptyset$, we have $A_1 - \uparrow(M_1^1 \cup M_2^1) - \downarrow(N_1^1 \cup N_2^1) \subseteq A \cap B = \emptyset$.

The sets $M_1^i, M_2^i, N_1^i, N_2^i$ for $i = 2, 3, 4$ can be constructed by analogous way using the topologies $\mathcal{O}_2 = \mathcal{O}'_a \cap \mathcal{O}''_b$,

$$\mathcal{O}_3 = \mathcal{O}''_a \cap \mathcal{O}'_b, \qquad \mathcal{O}_4 = \mathcal{O}''_a \cap \mathcal{O}''_b.$$

Now we define the topologies $\mathcal{O}'_5 = \{A \subseteq P/(a \notin A)$ or $(A \supseteq N(a) \cap (b - \{b\} - \uparrow M)$ for suitable finite $M \subseteq \uparrow b - \{b\})\}$, $\mathcal{O}_5 = \mathcal{O}'_5 \cap \mathcal{O}'_b$. Then $\mathcal{O}_5 \in C_2(P)$ follows from $\mathcal{O}'_b \in C_2(P)$. By T_2 there exist $A, B \in \mathcal{O}_5$ such that $a \in A$, $b \in B$, $A \cap B = \emptyset$. Hence $A \supseteq (\uparrow b - \{b\} - \uparrow M') \cap N(a)$, $B \supseteq \uparrow b - \uparrow M''$ for suitable $M', M'' \subseteq \uparrow b - \{b\}$. Let us put $M_1^5 = N_1^5 = N_2^5 = \emptyset$, $M_2^5 = M' \cup M''$. Clearly $A_5 - \uparrow M_2^5 \subseteq A \cap B = \emptyset$. The cases $i = 6, 7, 8$ are symmetrical with $i = 5$. According to Lemma 1 we can put $M_1^9 = M_2^9 = N_1^9 = N_2^9 = \emptyset$. This completes the proof.

For 3-compatibility we obtain a similar result:

Theorem 3. *The following conditions are equivalent:*

(i) *Every* $\mathcal{O} \in C_3(P)$ *is* T_2.

(ii) *For every two incomparable elements* $a, b \in P$ *there exist finite sets* $M_1 \subseteq \uparrow a - \{a\}$, $M_2 \subseteq \uparrow b - \{b\}$, $N_1 \subseteq \downarrow a - \{a\}$, $N_2 \subseteq \downarrow b - \{b\}$ *such that* $P - \uparrow(M_1 \cup M_2) - \downarrow(N_1 \cup N_2)$ *is finite.*

Proof. Suppose that (ii) is valid. Let $\mathcal{O} \in C_3(P)$, we are going to show that $\mathcal{O}$ is T_2. Let $a, b \in P$, $a \neq b$. If a, b are comparable, then 3-compatibility ensures the existence of disjoint neighbourhoods of a and b. Suppose that $a \parallel b$ holds and M_1, M_2, N_1, N_2 are as in (ii).

For $y \in M_1$ take $U_y \in \mathcal{O}$ such that $a \in U_y$, $U_y \cap \uparrow y = \emptyset$.
For $y \in M_2$ take $V_y \in \mathcal{O}$ such that $b \in V_y$, $V_y \cap \uparrow y = \emptyset$.
For $y \in N_1$ take $U_y \in \mathcal{O}$ such that $a \in U_y$, $U_y \cap \downarrow y = \emptyset$.
For $y \in N_2$ take $V_y \in \mathcal{O}$ such that $b \in V_y$, $V_y \cap \downarrow y = \emptyset$.

Such sets U_y, V_y exist according to the condition (C2) used for a, y or b, y respectively. Let's put $U' = \cap\{U_y/y \in M_1 \cup N_1\}$, $V' = \cap\{V_y/y \in M_2 \cup N_2\}$. The set $Z = U' \cap V' \subseteq P - \uparrow(M_1 \cup M_2) - \downarrow(N_1 \cup N_2)$ is finite. For $U = (U' - Z) \cup \cup \{a\}$, $V = (V' - Z) \cup \{b\}$ we have $U, V \in \mathcal{O}$, $U \cap V = \emptyset$. To show the converse let us suppose that (ii) does not hold for some $a, b \in P$, $a \parallel b$. Using the denotation from theorem 2 put $\mathcal{O} = \mathcal{O}'_a \cap O''_a \cap \mathcal{O}'_b \cap \mathcal{O}''_b$. If $A, B \in \mathcal{O}$, $a \in A$, $b \in B$, then for suitable finite sets $M_1 \subseteq \uparrow a - \{a\}$, $M_2 \subseteq \uparrow b - \{b\}$, $N_1 \subseteq \downarrow a - \{a\}$, $N_2 \subseteq \downarrow b - - \{b\}$ we have $A \cap B \supseteq P - \uparrow(M_1 \cup M_2) - \downarrow(N_1 \cup N_2) \neq \emptyset$. Finally we show the 3-compatibility of $\mathcal{O}$. Let $x, y \in P$, $x < y$. If $\{x, y\} \cap \{a, b\} = \emptyset$, we put $A = = \{x\}$, $B = \{y\}$. If $x \in \{a, b\}$, we put $A = P - \uparrow y$, $B = \{y\}$. (There holds $y \notin \{a, b\}$ because of $a \parallel b$.) If $y \in \{a, b\}$, we put $A = \{x\}$, $B = P - \downarrow x$. In every case the sets $A, B \in \mathcal{O}$ satisfy (C2).

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