EVAPORATION SCHEMES IN CONGRUENCE LATTICES OF MAJORITY ALGEBRAS

MIROSLAV PLOŠČICA

Abstract. We investigate the problem whether every distributive algebraic lattice is isomorphic to the congruence lattice of a majority algebra. We try the method used in Wehrung's solution of Dilworth's Congruence Lattice Problem. For this purpose we introduce the concept of a strong evaporation scheme in distributive semilattices. The main result says that the semilattices of compact congruences of majority algebras can have strong evaporation schemes of any cardinality. The situation for majority algebras strongly contrasts with the one for lattices, as the latter do not have strong evaporation schemes of cardinality greater than $\aleph_1$. This leaves the original problem open, but we believe that our results and methods are a significant step towards its solution. We also find a few general results, especially we prove that distributive semilattices that are lattices can contain strong evaporation schemes of cardinality at most $\aleph_1$.

1. Introduction

Congruence lattices are one of the central concepts in universal algebra. One of the most important (and hardest) problems is to determine, for a given class $\mathcal{K}$ of algebras, which lattices are isomorphic to congruence lattices of algebras in $\mathcal{K}$. We refer to [17], [18] and [19] (parts of [5]), for a survey of results and methods most relevant for our present paper.

The congruences of any algebra A form an algebraic lattice $\operatorname{Con} A$. The compact (finitely generated) congruences form a $(0, \vee)$-subsemilattice of $\operatorname{Con} A$, denoted by $\operatorname{Con}_c A$. The semilattice $\operatorname{Con}_c A$ determines $\operatorname{Con} A$ uniquely, because $\operatorname{Con} A$ is isomorphic to the ideal lattice of $\operatorname{Con}_c A$.

A $(0, \vee)$-semilattice S is called distributive if, for every $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in S$, the inequality $\boldsymbol{z} \leq \boldsymbol{x} \vee \boldsymbol{y}$ implies the existence of $\boldsymbol{u}, \boldsymbol{v} \in S$ with $\boldsymbol{u} \leq \boldsymbol{x}$, $\boldsymbol{v} \leq \boldsymbol{y}$, and $\boldsymbol{z} = \boldsymbol{u} \vee \boldsymbol{v}$. It is well known that the semilattice S is distributive if and only if its ideal lattice is distributive. A lattice is distributive according to the above definition if and only if it is distributive in the usual sense.

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Now we can define the central concept of our paper. Recall that $[\Omega]^k$ denotes the family of all k-element subsets of a set Ω, while $[\Omega]^{<\omega}$ stands for the family of all finite subsets of Ω. Let S be a distributive $(0,\vee)$-semilattice and $\boldsymbol{e} \in S$. *A decomposition system* at $\boldsymbol{e}$ is a family $\mathcal{F} = \{(\boldsymbol{a}_0^\alpha, \boldsymbol{a}_1^\alpha) \mid \alpha \in \Omega\}$ such that $\boldsymbol{a}_0^\alpha \vee \boldsymbol{a}_1^\alpha = \boldsymbol{e}$ for every $\alpha \in \Omega$. The pairs $(\boldsymbol{a}_0^\alpha, \boldsymbol{a}_1^\alpha)$ for distinct α need not be distinct.

Definition 1.1. Let S be a distributive $(0,\vee)$-semilattice and let $\mathcal{F} = \{(\boldsymbol{a}_0^\alpha, \boldsymbol{a}_1^\alpha) \mid \alpha \in \Omega\}$ be a decomposition system at $\boldsymbol{e} \in S$. Let $\operatorname{supp} : S \to [\Omega]^{<\omega}$ be a function. We say that the pair $(\mathcal{F}, \operatorname{supp})$ is a strong evaporation scheme (SES) at $\boldsymbol{e}$ if, for all distinct $\xi_1, \dots, \xi_n, \eta_1 \dots, \eta_m, \delta \in \Omega$ $(m, n \geq 1)$, all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in S$, and $i \in \{0,1\}$, the conditions

(i) $\xi_1, \dots, , \xi_n \notin \operatorname{supp}(\boldsymbol{y})$, $\eta_1 \dots \eta_m \notin \operatorname{supp}(\boldsymbol{x})$, $\delta \notin \operatorname{supp}(\boldsymbol{z})$;

(ii) $\boldsymbol{x} \leq \boldsymbol{a}_0^\delta$, $\boldsymbol{y} \leq \boldsymbol{a}_1^\delta$, $\boldsymbol{x} \leq \boldsymbol{a}_i^{\xi_1} \vee \dots \vee \boldsymbol{a}_i^{\xi_n}$, $\boldsymbol{y} \leq \boldsymbol{a}_i^{\eta_1} \vee \dots \vee \boldsymbol{a}_i^{\eta_m}$;

(iii) $\boldsymbol{z} \leq \boldsymbol{x} \vee \boldsymbol{y}$

imply

(iv) $\boldsymbol{z} = \boldsymbol{0}$.

The denotation supp in 1.1 stands for *support*. We refer to the set $\operatorname{supp}(\boldsymbol{x})$ as the support of $\boldsymbol{x}$. The cardinality of Ω will be referred to as the cardinality of $(\mathcal{F}, \operatorname{supp})$.

The idea of evaporation has first appeared in Wehrung's solution of the Congruence Lattice Problem (CLP), although he has not defined the concept explicitly. The main parts of his proof are reformulated in the following two assertions.

Theorem 1.2. ([16], [12]) *For every lattice L, the semilattice* $\operatorname{Con}_c L$ *does not have any SES of cardinality at least $\aleph_2$, at any $\boldsymbol{e} \in \operatorname{Con}_c L$, $\boldsymbol{e} \neq \boldsymbol{0}$.*

Theorem 1.3. ([16]) *For every cardinality κ there exists a distributive semilattice S of cardinality κ having a SES of cardinality κ at its largest element* $\boldsymbol{1}$.

These theorems imply that there exist distributive semilattices not isomorphic to $\operatorname{Con}_c L$ for any lattice L. Equivalently, there are distributive algebraic lattices not isomorphic to the congruence lattice of any lattice. This solves a longstanding problem referred to as CLP.

Theorem 1.2 has been first proved by Wehrung with $|\Omega| \geq \aleph_{\omega+1}$. This has been later improved to $\aleph_2$ by Růžička ([12]). The cardinality $\aleph_2$ in Theorem 1.2 and other results might look surprising, but it is partially explained by Gillibert's result ([3]) saying that, under certain conditions, the critical point between two varieties of algebras cannot exceed $\aleph_2$.

The cardinality bound $\aleph_2$ is optimal. Every distributive semilattice of cardinality at most $\aleph_1$, including the one constructed in Theorem 1.3, is isomorphic to $\mathrm{Con}_c\, L$ for some lattice L. (See [6].) Hence, we have the following theorem.

Theorem 1.4. *There exists a lattice L ($|L| = \aleph_1$) with compact top congruence $\mathbf{1} \in \mathrm{Con}_c\, L$ such that $\mathrm{Con}_c\, L$ has a SES of cardinality $\aleph_1$ at $\mathbf{1}$.*

A weaker version of our main concept, called *evaporation scheme* (ES) has been used in [8] to find more examples of algebraic distributive lattices that are not isomorphic to the congruence lattices of lattices. One of the results of [8] concerns *majority algebras*, that is algebras $(M; m)$, where m is a ternary operation satisfying the majority law

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x.$$

It is proved ([8], Theorem 4.4) that semilattices of compact congruences of majority algebras can contain arbitrarily large ESs. Another result of [8] strengthens Theorem 1.2 by proving that semilattices of compact congruences of lattices do not have ESs of cardinality at least $\aleph_2$. Hence, [8] establishes that the class of congruence lattices of majority algebras is strictly larger that the class of congruence lattices of lattices.

In our Theorem 3.4, using SESs instead of ESs, we strengthen the first of the above results about majority algebras. This, together with Theorem 1.2, provides an alternative proof of the distinction between congruence lattices of majority algebras and lattices.

Evaporation schemes are connected with various refinement properties in distributive semilattices or, more generally, commutative monoids, see for instance [1], [14], [15], [10], or other papers.

After the negative solution of CLP, it is natural to ask whether there is any congruence-distributive variety $\mathcal{V}$ such that every distributive algebraic lattice is isomorphic to $\mathrm{Con}\, A$ for some $A \in \mathcal{V}$. The most obvious candidate is the variety of majority algebras, as the majority law ensures that the variety is congruence-distributive. Hence, we have the following problem.

Problem 1.5. (See [19], Problem 9.3.) *Is every distributive algebraic lattice isomorphic to the congruence lattice of some majority algebra?*

This paper is an attempt to solve the above problem. We use Wehrung's approach and investigate an analogue of Theorem 1.2 for majority algebras.

Unfortunately, we find that congruence lattices of majority algebras can contain SESs of any size, which leaves the above problem open. Nevertheless, we believe the result is worthwhile for the following reasons.

(1) The result and its proof can still be helpful in solving Problem 1.5. For instance, one could try to further strengthen the concept of a SES.
(2) Our result strengthens a result from [8] and provides a new proof (a new construction) for Theorem 1.3.
(3) We have found a nontrivial piece of information, which leads to a better understanding of congruence lattices of majority algebras. This infomation may be used, for instance, to distinguish congruence lattices of majority algebras from congruence lattices of other kind of algebras.

In our investigation we also found several properties of evaporation schemes in distributive semilattices in general. These results are comprised in the next section. They are not needed for the majority algebras result, but we consider them interesting in their own right. One of them strengthens Theorem 1.4. Generally, we believe that (strong) evaporation schemes are worth further investigation.

2. General results

In the definition of a decomposition system $\mathcal{F}$, the pairs $(\boldsymbol{a}_0^\alpha, \boldsymbol{a}_1^\alpha)$ need not be distinct for distinct $\alpha \in \Omega$. This way a semilattice S can contain a SES larger that the cardinality of S. Indeed, for any decomposition system with finite Ω we can define $\operatorname{supp}(\boldsymbol{x}) = \Omega$ for every $\boldsymbol{x} \in S$. Then $(\mathcal{F}, \operatorname{supp})$ is clearly a SES, as the condition 1.1(i) cannot be satisfied. Now we prove that this cannot happen for infinite Ω. (Except for the trivial case $\boldsymbol{e} = \boldsymbol{0}$.)

Theorem 2.1. *Let* $(\mathcal{F}, \operatorname{supp})$ *with* $\mathcal{F} = \{(\boldsymbol{a}_0^\alpha, \boldsymbol{a}_1^\alpha) \mid \alpha \in \Omega\}$ *be a decomposition system at* $\boldsymbol{e} \neq \boldsymbol{0}$ *in a distributive* $(0, \vee)$*-semilattice* S*. Let* $(\mathcal{F}, \operatorname{supp})$ *be a SES at* $\boldsymbol{e}$*. If* Ω *is infinite, then* $|\Omega| \leq |S|$*.*

Proof. For contradiction, suppose that $|S| < |\Omega|$. For every $\boldsymbol{x} \in S$, the set $\operatorname{supp}(\boldsymbol{x})$ is finite, so

$$|\bigcup\{\operatorname{supp}(\boldsymbol{x}) \mid \boldsymbol{x} \in S\}| < |\Omega|.$$

Let $\Omega_0 = \Omega \setminus \bigcup\{\operatorname{supp}(\boldsymbol{x}) \mid \boldsymbol{x} \in S\}$. Choose distinct $\xi, \eta, \delta \in \Omega_0$ arbitrarily. Since $\boldsymbol{a}_0^\delta \vee \boldsymbol{a}_1^\delta = \boldsymbol{e} \neq \boldsymbol{0}$, we have $\boldsymbol{a}_0^\delta \neq \boldsymbol{0}$ or $\boldsymbol{a}_1^\delta \neq \boldsymbol{0}$. Without loss of generality we may assume $\boldsymbol{a}_0^\delta \neq \boldsymbol{0}$. Since $\boldsymbol{a}_0^\delta \leq \boldsymbol{e} = \boldsymbol{a}_0^\xi \vee \boldsymbol{a}_1^\xi$, the distributivity implies the existence of $\boldsymbol{u}, \boldsymbol{v} \in S$ with $\boldsymbol{a}_0^\delta = \boldsymbol{u} \vee \boldsymbol{v}$, $\boldsymbol{u} \leq \boldsymbol{a}_0^\xi$, and $\boldsymbol{v} \leq \boldsymbol{a}_1^\xi$. We have

$\boldsymbol{u} \neq \mathbf{0}$ or $\boldsymbol{v} \neq \mathbf{0}$. If $\boldsymbol{u} \neq \mathbf{0}$, we set $\boldsymbol{x} = \boldsymbol{z} = \boldsymbol{u}$, $\boldsymbol{y} = \mathbf{0}$, $i = 0$. If $\boldsymbol{v} \neq \mathbf{0}$, we set $\boldsymbol{x} = \boldsymbol{z} = \boldsymbol{v}$, $\boldsymbol{y} = \mathbf{0}$, $i = 1$. In both cases the conditions 1.1(i)-(iii) are satisfied, while 1.1(iv) is not, which is a contradiction. □

The cardinality bound $|\Omega| \leq |S|$ is optimal by Theorem 1.3. An interesting question is whether large SESs can occur in *lattices* S. We show that cardinalities greater than $\aleph_1$ are not possible. In the proof we use the following Kuratowski's principle of infinite combinatorics.

Theorem 2.2. *([7]) Let Ω be a set of cardinality at least $\aleph_2$. Let Φ be a map $[\Omega]^2 \to [\Omega]^{<\omega}$. Then there are distinct $\alpha, \beta, \gamma \in \Omega$ such that $\alpha \notin \Phi(\beta, \gamma)$, $\beta \notin \Phi(\alpha, \gamma)$, and $\gamma \notin \Phi(\alpha, \beta)$.*

Conversely, for any set Ω with $|\Omega| \leq \aleph_1$ there exists a map $\Phi\colon\ [\Omega]^2 \to [\Omega]^{<\omega}$ such that for every distinct $\alpha, \beta, \gamma \in \Omega$, at least one of the conditions $\alpha \in \Phi(\beta, \gamma)$, $\beta \in \Phi(\alpha, \gamma)$, and $\gamma \in \Phi(\alpha, \beta)$ holds.

A set $\{\alpha, \beta, \gamma\}$ satisfying $\alpha \notin \Phi(\beta, \gamma)$, $\beta \notin \Phi(\alpha, \gamma)$, and $\gamma \notin \Phi(\alpha, \beta)$ is usually called *free* with respect to Φ. The above principle is a special case of a much more general theorem.

Theorem 2.3. *Let $(\mathcal{F}, \operatorname{supp})$ with $\mathcal{F} = \{(\boldsymbol{a}_0^\alpha, \boldsymbol{a}_1^\alpha) \mid \alpha \in \Omega\}$ be a decomposition system at $\boldsymbol{e} \neq 0$ in a distributive lattice S with 0. Let $(\mathcal{F}, \operatorname{supp})$ be a SES at $\boldsymbol{e}$. Then $|\Omega| \leq \aleph_1$.*

Proof. For contradiction, suppose that $|\Omega| \geq \aleph_2$. For every $\alpha, \beta \in \Omega$, $\alpha \neq \beta$ we define

$$\Phi(\alpha, \beta) = \operatorname{supp}(\boldsymbol{a}_0^\alpha \wedge \boldsymbol{a}_0^\beta) \cup \operatorname{supp}(\boldsymbol{a}_0^\alpha \wedge \boldsymbol{a}_1^\beta) \cup \operatorname{supp}(\boldsymbol{a}_1^\alpha \wedge \boldsymbol{a}_0^\beta) \cup \operatorname{supp}(\boldsymbol{a}_1^\alpha \wedge \boldsymbol{a}_1^\beta).$$

This defines a function $\Phi\colon\ [\Omega]^2 \to [\Omega]^{<\omega}$. By Kuratowski's principle, there are distinct $\alpha, \beta, \gamma \in \Omega$ such that $\alpha \notin \Phi(\beta, \gamma)$, $\beta \notin \Phi(\alpha, \gamma)$, and $\gamma \notin \Phi(\alpha, \beta)$. Now we claim that at least one of the elements $\boldsymbol{a}_i^\alpha \wedge \boldsymbol{a}_i^\beta$, $\boldsymbol{a}_i^\alpha \wedge \boldsymbol{a}_i^\gamma$, $\boldsymbol{a}_i^\beta \wedge \boldsymbol{a}_i^\gamma$ $(i = 0, 1)$ is nonzero. Indeed, $\boldsymbol{a}_i^\alpha \wedge \boldsymbol{a}_i^\beta = 0$ together with $\boldsymbol{a}_i^\beta \leq \boldsymbol{a}_0^\alpha \vee \boldsymbol{a}_1^\alpha$ implies

$$\boldsymbol{a}_i^\beta = (\boldsymbol{a}_i^\alpha \wedge \boldsymbol{a}_i^\beta) \vee (\boldsymbol{a}_{1-i}^\alpha \wedge \boldsymbol{a}_i^\beta) = \boldsymbol{a}_{1-i}^\alpha \wedge \boldsymbol{a}_i^\beta,$$

hence $\boldsymbol{a}_i^\beta \leq \boldsymbol{a}_{1-i}^\alpha$. If all the six elements above were equal to 0, we would obtain

$$\boldsymbol{a}_0^\beta \leq \boldsymbol{a}_1^\alpha \leq \boldsymbol{a}_0^\gamma \leq \boldsymbol{a}_1^\beta \leq \boldsymbol{a}_0^\alpha \leq \boldsymbol{a}_1^\gamma \leq \boldsymbol{a}_0^\beta,$$

so all the elements are equal to zero, which contradicts the assumption $\boldsymbol{a}_0^\alpha \vee \boldsymbol{a}_1^\alpha = \boldsymbol{e} \neq 0$.

Now assume, for instance, that $\boldsymbol{a}_0^\alpha \wedge \boldsymbol{a}_0^\beta \neq 0$. (Other cases are similar.) We set $\boldsymbol{x} = \boldsymbol{a}_0^\alpha \wedge \boldsymbol{a}_0^\gamma$, $\boldsymbol{y} = \boldsymbol{a}_0^\beta \wedge \boldsymbol{a}_1^\gamma$, $\boldsymbol{z} = \boldsymbol{a}_0^\alpha \wedge \boldsymbol{a}_0^\beta$, $\xi = \alpha$, $\eta = \beta$, $\delta = \gamma$, and

$i = 0$. It is easy to check that the conditions 1.1(i)-(iii) are fulfilled, while 1.1(iv) is not. (For instance, $\xi = \alpha \notin \Phi(\beta, \gamma) \supseteq \mathrm{supp}(\boldsymbol{y})$.) Hence, we have a contradiction. □

The above theorem could be also deduced from earlier results. According to Schmidt's theorem ([13]), every distributive lattice with 0 is isomorphic to $\mathrm{Con}_c\, L$ for some lattice L. And by Theorem 1.2, $\mathrm{Con}_c\, L$ cannot contain a SES of cardinality $\aleph_2$, which implies Theorem 2.3. However, both 1.2 and Schmidt's theorem are much more complicated than 2.3, so we prefer the direct proof above.

Now we prove the converse to Theorem 2.3 by constructing a distributive lattice having a SES of cardinality $\aleph_1$. We are inspired by the construction of free distributive semilattices in [9], which was used by Wehrung in his proof of Theorem 1.3.

Let Ω be a set of cardinality $\aleph_1$. Let $S(\Omega)$ be the free bounded distributive lattice generated by elements $\boldsymbol{a}_0^\alpha$ and $\boldsymbol{a}_1^\alpha$, $\alpha \in \Omega$ satisfying the condition $\boldsymbol{a}_0^\alpha \vee \boldsymbol{a}_1^\alpha = \mathbf{1}$. For every $X \subseteq \Omega$ let $S(X)$ be the bounded sublattice of $S(\Omega)$ generated by all $\boldsymbol{a}_0^\alpha$ and $\boldsymbol{a}_1^\alpha$, $\alpha \in X$.

Lemma 2.4. *Let $X, Y \subseteq \Omega$, $\boldsymbol{z} \in S(X)$, $\boldsymbol{x} \in S(Y)$, $i \in \{0, 1\}$, and let $\boldsymbol{z} \le \boldsymbol{x} \vee \boldsymbol{a}_i^{\alpha_1} \vee \cdots \vee \boldsymbol{a}_i^{\alpha_n}$ for some $\alpha_1, \ldots, \alpha_n \in \Omega \setminus Y$. Then $\boldsymbol{z} \le \boldsymbol{x} \vee \bigvee\{\boldsymbol{a}_i^{\alpha_k} \mid \alpha_k \in X\}$.*

Proof. Let $h : S(\Omega) \to S(\Omega)$ be the lattice homomorphism determined uniquely by

$$h(\boldsymbol{a}_i^\alpha) = \begin{cases} \boldsymbol{a}_i^\alpha & \text{if } \alpha \in X \cup Y \\ \mathbf{0} & \text{if } \alpha \notin X \cup Y \end{cases},$$

$$h(\boldsymbol{a}_{1-i}^\alpha) = \begin{cases} \boldsymbol{a}_{1-i}^\alpha & \text{if } \alpha \in X \cup Y \\ \mathbf{1} & \text{if } \alpha \notin X \cup Y \end{cases}.$$

Such a homomorphism exists by the defining property of $S(\Omega)$. Then $h(\boldsymbol{x}) = \boldsymbol{x}$, $h(\boldsymbol{z}) = \boldsymbol{z}$, so we obtain that

$$\boldsymbol{z} = h(\boldsymbol{z}) \le \boldsymbol{x} \vee h(\boldsymbol{a}_i^{\alpha_1}) \vee \ldots h(\boldsymbol{a}_i^{\alpha_n}) = \boldsymbol{x} \vee \bigvee\{\boldsymbol{a}_i^{\alpha_k} \mid \alpha_k \in X\}.$$

□

As a special case for $\boldsymbol{x} = \mathbf{0}$, $Y = \emptyset$ we get the following consequence.

Lemma 2.5. *Let $X \subseteq \Omega$, $\boldsymbol{z} \in S(X)$, $i \in \{0, 1\}$, and let $\boldsymbol{z} \le \boldsymbol{a}_i^{\alpha_1} \vee \cdots \vee \boldsymbol{a}_i^{\alpha_n}$ for some $\alpha_1, \ldots, \alpha_n \in \Omega$. Then $\boldsymbol{z} \le \bigvee\{\boldsymbol{a}_i^{\alpha_k} \mid \alpha_k \in X\}$.*

By Kuratowski's principle, there exists $\Phi : [\Omega]^2 \to [\Omega]^{<\omega}$ such that $\alpha \in \Phi(\beta, \gamma)$ or $\beta \in \Phi(\alpha, \gamma)$ or $\gamma \in \Phi(\alpha, \beta)$ holds for every distinct $\alpha, \beta, \gamma \in \Omega$.

For every $\boldsymbol{x} \in S(\Omega)$ we can choose a finite set $A_{\boldsymbol{x}} \subset \Omega$ such that $\boldsymbol{x} \in S(A_{\boldsymbol{x}})$. Let us define

$$\operatorname{supp}(\boldsymbol{x}) = A_{\boldsymbol{x}} \cup \bigcup \{\Phi(\alpha, \beta) \mid \alpha, \beta \in A_{\boldsymbol{x}},\ \alpha \neq \beta\}.$$

Clearly, $\operatorname{supp}: \ S(\Omega) \to [\Omega]^{<\omega}$.

Theorem 2.6. *The decomposition system* $\mathcal{F} = \{(\boldsymbol{a}_0^\alpha, \boldsymbol{a}_1^\alpha) \mid \alpha \in \Omega\}$ *together with the support function defined above form a strong evaporation scheme at* $\mathbf{1} \in S(\Omega)$.

Proof. Let $\xi_1, \ldots, \xi_n, \eta_1 \ldots, \eta_m, \delta \in \Omega$, $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in S$, and $i \in \{0, 1\}$ satisfy 1.1 (i)-(iii).

If $\boldsymbol{z} \leq \boldsymbol{y}$, then $\boldsymbol{z} \leq \boldsymbol{y} \leq \boldsymbol{a}_1^\delta$. Since $\delta \notin \operatorname{supp}(\boldsymbol{z}) \supseteq A_{\boldsymbol{z}}$, Lemma 2.5 applied to the inequality $\boldsymbol{z} \leq \boldsymbol{a}_1^\delta$ implies that $\boldsymbol{z} = \mathbf{0}$. Similarly, the assumption $\boldsymbol{z} \leq \boldsymbol{x}$ leads to $\boldsymbol{z} = \mathbf{0}$.

Suppose now that $\boldsymbol{z} \not\leq \boldsymbol{x}$, $\boldsymbol{z} \not\leq \boldsymbol{y}$. We show that this case leads to a contradiction. Since $\boldsymbol{z} \leq \boldsymbol{x} \vee \boldsymbol{y}$, we have $\boldsymbol{x}, \boldsymbol{y} \neq \mathbf{0}$. By Lemma 2.5 we can assume that $\{\xi_1, \ldots, \xi_n\} \subseteq A_{\boldsymbol{x}}$, $\{\eta_1, \ldots, \eta_m\} \subseteq A_{\boldsymbol{y}}$. Similarly, the inequalities $\boldsymbol{x} \leq \boldsymbol{a}_0^\delta$ and $\boldsymbol{y} \leq \boldsymbol{a}_1^\delta$ imply that $\delta \in A_{\boldsymbol{x}} \cap A_{\boldsymbol{y}}$. Now, $\xi_k \in \Phi(\eta_j, \delta)$ would imply $\xi_k \in \operatorname{supp}(\boldsymbol{y})$, a contradiction. Hence, $\xi_k \notin \Phi(\eta_j, \delta)$. For similar reasons, $\eta_j \notin \Phi(\xi_k, \delta)$. The remaining possibility is that $\delta \in \Phi(\xi_k, \eta_j)$ for every k, j.

The conditions (ii) and (iii) imply $\boldsymbol{z} \leq \boldsymbol{x} \vee \boldsymbol{a}_i^{\eta_1} \vee \cdots \vee \boldsymbol{a}_i^{\eta_m}$. Since $\boldsymbol{z} \not\leq \boldsymbol{x}$, Lemma 2.4 implies that there exists $\eta_j \in A_{\boldsymbol{z}}$. Similarly, from $\boldsymbol{z} \leq \boldsymbol{y} \vee \boldsymbol{a}_i^{\xi_1} \vee \cdots \vee \boldsymbol{a}_i^{\xi_m}$ we obtain that $\xi_k \in A_{\boldsymbol{z}}$ for some ξ_k. But then $\delta \in \Phi(\xi_k, \eta_j) \subseteq \operatorname{supp}(\boldsymbol{z})$, a contradiction. □

So, there exists a bounded distributive lattice having a SES at $\mathbf{1}$ with $|\Omega| = \aleph_1$. Since every such lattice is isomorphic to $\operatorname{Con}_c L$ for some lattice L ([13]), our result strengthens Theorem 1.4.

3. Congruence lattices of majority algebras

We consider bounded majority algebras $(M; m, 0, 1)$, which means that besides the majority law we have constants 0, 1 satisfying

$$m(x, 0, 1) = m(x, 1, 0) = m(0, x, 1) = m(0, 1, x) =$$

$$= m(1, 0, x) = m(1, x, 0) = x$$

for every $x \in M$.

The boundedness ensures that the largest congruence $\mathbf{1}$ on M is generated by the pair $(0, 1)$ and therefore compact. Indeed, if $(0, 1) \in \tau \in \operatorname{Con} M$, then $(x, 0) = (m(0, x, 1), m(0, x, 0)) \in \tau$ for every $x \in M$.

Every bounded lattice $(L; \vee, \wedge, 0, 1)$ gives rise to the majority algebra $(L; m, 0, 1)$, where

$$m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z).$$

Since $x \vee y = m(x, y, 1)$ and $x \wedge y = m(x, y, 0)$, both algebras are term equivalent, and hence have the same congruences. On the other hand, the class of majority algebras is more general, and there are majority algebras that do not arise from lattices.

Bounded majority algebras form a variety, so there exist free algebras. Let $F(\Omega)$ denote the free bounded majority algebra with Ω as the set of free generators. For $X \subseteq \Omega$ let $F(X)$ be the subalgebra of $F(\Omega)$ generated by X. It is easy to see that $F(X)$ is freely generated by X.

For every $\alpha \in \Omega$ denote $\boldsymbol{a}_0^\alpha = \theta(0, \alpha)$, $\boldsymbol{a}_1^\alpha = \theta(1, \alpha)$. That is, $\boldsymbol{a}_0^\alpha$ and $\boldsymbol{a}_1^\alpha$ are the smallest congruences containing the pairs $(0, \alpha)$ and $(1, \alpha)$, respectively. Clearly, $\boldsymbol{a}_0^\alpha \vee \boldsymbol{a}_1^\alpha = \mathbf{1}$. Thus, $\mathcal{F} = \{(\boldsymbol{a}_0^\alpha, \boldsymbol{a}_1^\alpha) \mid \alpha \in \Omega\}$ is a decomposition system at $\mathbf{1} \in \operatorname{Con}_c F(\Omega)$.

Further, every $\boldsymbol{x} \in \operatorname{Con}_c F(\Omega)$ is finitely generated, so there exists a finite subset $\operatorname{supp}(\boldsymbol{x}) \subseteq \Omega$ such that $\boldsymbol{x}$ is generated by its restriction to $F(\operatorname{supp}(\boldsymbol{x}))$. (We choose any such finite set, without any minimality requirements.) This defines a function $\operatorname{supp}: S = \operatorname{Con}_c F(\Omega) \to [\Omega]^{<\omega}$. The essential property of the support is described in the following easy lemma.

Lemma 3.1. *Let $\boldsymbol{a} \in \operatorname{Con}_c F(\Omega)$, $\tau \in \operatorname{Con} F(\Omega)$, and $\operatorname{supp}(\boldsymbol{a}) \subseteq Y \subseteq \Omega$. Then $\boldsymbol{a} \leq \tau$ if and only if $\boldsymbol{a} \restriction F(Y) \leq \tau \restriction F(Y)$.*

The following two easy assertions are true not just for majority algebras, but in general. If A is an algebra, $\tau \in \operatorname{Con} A$ and $x \in A$, then x/τ denotes the τ-equivalence class containing x. Also recall that for every homomorphism $h: A \to B$ the relation $\operatorname{Ker}(h) = \{(x, y) \in A \mid h(x) = h(y)\}$ (the kernel of h) is a congruence on A.

Lemma 3.2. *Let $\tau \in \operatorname{Con} F(X)$, $X \subseteq \Omega$. Let $\overline{\tau}$ be the congruence on $F(\Omega)$ generated by τ. Then $\overline{\tau} \restriction F(X) = \tau$.*

Proof. Choose a homomorphism $g: F(\Omega) \to F(X)/\tau$ satisfying $g(x) = x/\tau$ for every $x \in X$. (That is, choose $g(x)$ for $x \in \Omega \setminus X$ arbitrarily.) Then $\tau \leq \operatorname{Ker}(g)$ and $\operatorname{Ker}(g) \in \operatorname{Con} F(\Omega)$, so $\overline{\tau} \leq \operatorname{Ker}(g)$. Since $\operatorname{Ker}(g) \restriction F(X) = \tau$, we obtain $\overline{\tau} \restriction F(X) \leq \tau$. The converse inequality is obvious. □

Lemma 3.3. *Let $X, Y \subseteq \Omega$. If $F(X \cap Y) \neq \emptyset$, then $F(X) \cap F(Y) = F(X \cap Y)$.*

Proof. Let $u \in F(X) \cap F(Y)$. Then $u = s(x_1, \ldots, x_n) = t(y_1, \ldots, y_m)$ for some terms s, t and $x_1, \ldots, x_n \in X$, $y_1, \ldots, y_m \in Y$. Choose $z \in F(X \cap Y)$ and consider the homomorphism $h:\ F(\Omega) \to F(\Omega)$ given by

$$h(\alpha) = \begin{cases} z & \text{if } \alpha \in X \setminus Y \\ \alpha & \text{if } \ \alpha \in \Omega \setminus (X \setminus Y). \end{cases}$$

Then

$$u = t(y_1, \ldots, y_m) = h(t(y_1, \ldots, y_m)) = h(s(x_1, \ldots, x_n)) =$$

$$s(h(x_1), \ldots, h(x_n)) \in F(X \cap Y),$$

since all $h(x_i)$ belong to $F(X \cap Y)$. □

In bounded majority algebras, the condition $F(X \cap Y) \neq \emptyset$ is always satisfied, as $F(\emptyset) = \{0, 1\}$.

Now we prove the main result of this section.

Theorem 3.4. *For any* Ω*, the pair* $(\mathcal{F}, \operatorname{supp})$ *is a strong evaporation scheme at* $1 \in \operatorname{Con}_c F(\Omega)$.

Proof. Let $\xi_1, \ldots, \xi_n, \eta_1 \ldots, \eta_m, \delta \in \Omega$, $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in S$, and $i \in \{0, 1\}$ satisfy 1.1(i)-(iii). Suppose $i = 0$. (The proof for $i = 1$ is similar.) Now we define several majority algebras. Let

$$W_0 = F(\Omega \setminus \{\delta, \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m\}),$$

$$W_1 = F(\Omega \setminus \{\delta, \eta_1, \ldots, \eta_m\}),$$

$$W_2 = F(\Omega \setminus \{\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m\}),$$

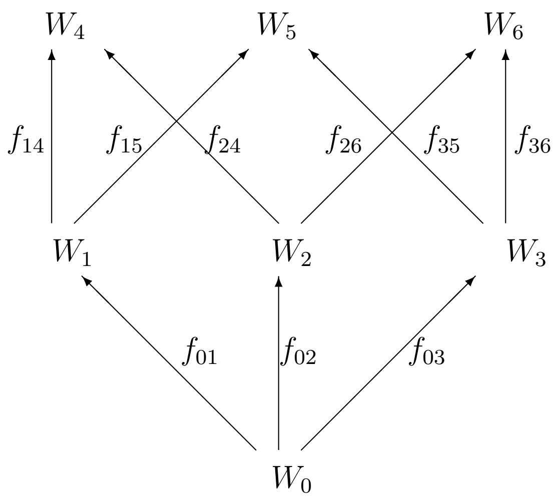
$$W_3 = F(\Omega \setminus \{\delta, \xi_1, \ldots, \xi_n\}),$$

$$W_4 = W_1 \times W_2,$$

$$W_5 = F(\Omega \setminus \{\delta\}),$$

$$W_6 = W_3 \times W_2.$$

Consider the diagram



with homomorphisms defined as follows. The maps f_{01}, f_{02}, f_{03}, f_{15}, and f_{35} are set inclusions. Further,

$$f_{14}(x) = \begin{cases} (x,0) & \text{if } x \in \{\xi_1,\dots,\xi_n\} \\ (x,x) & \text{for every} \quad x \in \Omega \setminus \{\delta,\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_m\} \end{cases}$$

$$f_{24}(x) = \begin{cases} (0,\delta) & \text{if } x = \delta \\ (x,x) & \text{for every} \quad x \in \Omega \setminus \{\delta,\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_m\} \end{cases}$$

$$f_{36}(x) = \begin{cases} (x,0) & \text{if } x \in \{\eta_1,\dots,\eta_m\} \\ (x,x) & \text{for every} \quad x \in \Omega \setminus \{\delta,\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_m\} \end{cases}$$

$$f_{26}(x) = \begin{cases} (1,\delta) & \text{if } x = \delta \\ (x,x) & \text{for every} \quad x \in \Omega \setminus \{\delta,\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_m\}. \end{cases}$$

Since W_1, W_2, and W_3 are free algebras, the above rules define the homomorphisms f_{14}, f_{24}, f_{26}, and f_{36} uniquely. It is easy to see that the diagram is commutative. (It is enough to check it on the free generators of W_0.)

Lemma 3.3 implies that $W_1 \cap W_2 = W_1 \cap W_3 = W_2 \cap W_3 = W_0$. Further, since f_{15} and f_{35} are set inclusions, $f_{15}(x) = f_{35}(y)$ implies $x = y \in W_0$. Now we claim that, similarly, $f_{14}(x) = f_{24}(y)$ implies $x = y \in W_0$. Clearly, $\operatorname{rng} f_{14} \subseteq W_1 \times W_0$ and $\operatorname{rng} f_{24} \subseteq W_0 \times W_2$. Then $f_{14}(x) = (x, z_1)$ and $f_{24}(y) = (z_2, y)$ for some $z_1, z_2 \in W_0$. The equality $f_{14}(x) = f_{24}(y)$ implies that $x = z_2 \in W_0$ and $y = z_1 \in W_0$. Since the homomorphism $f_{14}f_{01} = f_{24}f_{02}$ is injective, we have $x = y$. Similarly one can prove that $f_{36}(x) = f_{26}(y)$ implies $x = y \in W_0$.

Now let W be the colimit of our diagram in the category of sets. That means, W is the disjoint union of W_k, $k = 0,\dots,6$ factored by the smallest equivalence relation ρ containing all pairs of the form $(x, f_{kl}(x))$, $x \in W_k$. Equivalently, consider the disjoint union of W_4, W_5, and W_6 factored by

$\rho = \Delta \cup \rho_1 \cup \rho_2 \cup \rho_3$, where Δ is the diagonal (all pairs of the form (x, x)) and

$$\rho_1 = \{(f_{14}(x), f_{15}(x)) \mid x \in W_1\} \cup \{(f_{15}(x), f_{14}(x)) \mid x \in W_1\},$$

$$\rho_2 = \{(f_{24}(x), f_{26}(x)) \mid x \in W_2\} \cup \{(f_{26}(x), f_{24}(x)) \mid x \in W_2\},$$

$$\rho_3 = \{(f_{35}(x), f_{36}(x)) \mid x \in W_3\} \cup \{(f_{36}(x), f_{35}(x)) \mid x \in W_3\}.$$

The consideration in the previous paragraph show the transitivity of this relation ρ. For instance, if $(a, b) = (f_{14}(x), f_{15}(x))$ and $(b, c) = (f_{35}(y), f_{36}(y))$, then $f_{15}(x) = f_{35}(y)$, so $x = y \in W_0$ and then, by the commutativity of the diagram, $(a, c) = (f_{24}(x), f_{26}(x)) \in \rho$. As another case we consider $(a, b) = (f_{14}(x), f_{15}(x))$ and $(b, c) = (f_{15}(y), f_{14}(y))$. Since f_{15} is injective, we have $x = y$, so $a = c$ and hence $(a, c) \in \rho$. Other cases are similar.

Notice that ρ restricted to any of W_4, W_5, and W_6 is trivial. So, identifying x with x/ρ, these sets can be considered as subsets of W. In this sense, $W_1 = W_4 \cap W_5$, $W_2 = W_4 \cap W_6$, and $W_3 = W_5 \cap W_6$.

This enables us to define the majority operation m on W by the following rules.

a) If $a, b, c \in W_j$ for some $j \in \{4, 5, 6\}$, then $m(a, b, c)$ is evaluated in W_j.
b) If two of a, b, c are equal, then $m(a, b, c)$ is determined by the majority rule.
c) $m(a, b, c) = 0$ in all remaining cases.

It is easy to see that W is defined correctly.

Let

$$h: \ F(\Omega) \to W$$

be the homomorphism determined by the rule $h(\alpha) = \alpha$ for every $\alpha \in \Omega$. More precisely, $h(\xi_j) = f_{14}(\xi_j)/\rho = f_{15}(\xi_j)/\rho$, $h(\eta_j) = f_{35}(\eta_j)/\rho = f_{36}(\eta_j)/\rho$, $h(\delta) = f_{24}(\delta)/\rho = f_{26}(\delta)/\rho$, and $h(\alpha) = f_{14}(\alpha)/\rho = f_{15}(\alpha)/\rho = f_{26}(\alpha)/\rho$ for $\alpha \in \Omega \setminus \{\delta, \xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m\}$.

We claim that $\boldsymbol{x} \leq \operatorname{Ker}(h)$. By Lemma 3.1 it suffices to prove that

$$\boldsymbol{x} \restriction F(X) \leq \operatorname{Ker}(h) \restriction F(X),$$

where $X = \Omega \setminus \{\eta_1, \dots, \eta_m\}$.

Let p_1 and p_2 be the projections of W_4 onto W_1 and W_2, respectively. Since h maps $F(X)$ into W_4, the restrictions $p_1 h \restriction F(X)$ and $p_2 h \restriction F(X)$ are well-defined. Clearly,

$$\operatorname{Ker}(p_1 h \restriction F(X)) \cap \operatorname{Ker}(p_2 h \restriction F(X)) = \operatorname{Ker}(h \restriction F(X)).$$

Let τ be the congruence on $F(X)$ generated by the pair $(0,\delta)$. By Lemma 3.2, $\tau = \boldsymbol{a}_0^\delta \restriction F(X)$. Since $p_1h(\delta) = 0$, we have $(0,\delta) \in \operatorname{Ker}(p_1h \restriction F(X))$, so $\boldsymbol{a}_0^\delta \restriction F(X) = \tau \leq \operatorname{Ker}(p_1h \restriction F(X))$. Similarly, let σ be the congruence on $F(X)$ generated by all pairs $(0,\xi_j)$. Since $p_2h(\xi_j) = 0$ for every j, we have $\boldsymbol{a}_0^{\xi_1} \vee \dots \vee \boldsymbol{a}_0^{\xi_n} \restriction F(X) = \sigma \leq \operatorname{Ker}(p_2h \restriction F(X))$. By the assumption (ii) we obtain that

$$\begin{aligned}\boldsymbol{x} \restriction F(X) &\leq \boldsymbol{a}_0^\delta \wedge (\boldsymbol{a}_0^{\xi_1} \vee \dots \vee \boldsymbol{a}_0^{\xi_n}) \restriction F(X)\\ &\leq \operatorname{Ker}(p_1h \restriction F(X)) \cap \operatorname{Ker}(p_2h \restriction F(X)) = \operatorname{Ker}(h \restriction F(X)).\end{aligned}$$

So, $\boldsymbol{x} \leq \operatorname{Ker}(h)$. By the same way we show that $\boldsymbol{y} \leq \operatorname{Ker}(h)$. It suffices to prove that

$$\boldsymbol{y} \restriction F(Y) \leq \operatorname{Ker}(h \restriction F(Y)),$$

where $Y = \Omega \setminus \{\xi_1, \dots, \xi_n\}$.

Let p_3 and p_4 be the projections of W_6 onto W_3 and W_2, respectively. Let τ' be the congruence on $F(Y)$ generated by the pair $(1,\delta)$. By Lemma 3.2, $\tau' = \boldsymbol{a}_1^\delta \restriction F(Y)$. Since $p_3h(\delta) = 1$, we have $(1,\delta) \in \operatorname{Ker}(p_3h \restriction F(Y))$, so $\boldsymbol{a}_1^\delta \restriction F(Y) = \tau' \leq \operatorname{Ker}(p_3h \restriction F(Y))$. Similarly, let σ' be the congruence on $F(Y)$ generated by all pairs $(0,\eta_j)$. Since $p_4h(\eta_j) = 0$ for every j, we have $\boldsymbol{a}_0^{\eta_1} \vee \dots \vee \boldsymbol{a}_0^{\eta_m} \restriction F(Y) = \sigma' \leq \operatorname{Ker}(p_4h \restriction F(Y))$. By the assumption (ii) we obtain that

$$\begin{aligned}\boldsymbol{y} \restriction F(Y) &\leq \boldsymbol{a}_1^\delta \wedge (\boldsymbol{a}_0^{\eta_1} \vee \dots \vee \boldsymbol{a}_0^{\eta_m}) \restriction F(Y)\\ &\leq \operatorname{Ker}(p_3h \restriction F(Y)) \cap \operatorname{Ker}(p_4h \restriction F(Y)) = \operatorname{Ker}(h \restriction F(Y)).\end{aligned}$$

So, we have $\boldsymbol{x} \leq \operatorname{Ker}(h)$, $\boldsymbol{y} \leq \operatorname{Ker}(h)$. By the assumption (iii), $\boldsymbol{z} \leq \boldsymbol{x} \vee \boldsymbol{y} \leq \operatorname{Ker}(h)$. Then

$$\boldsymbol{z} \restriction F(Z) \leq \operatorname{Ker}(h \restriction F(Z)),$$

where $Z = \Omega \setminus \{\delta\}$. However, $h \restriction F(Z)$ is injective, it maps $F(Z)$ identically onto W_5. So, $\boldsymbol{z} \restriction F(Z) = \mathbf{0}$. Since $\operatorname{supp}(\boldsymbol{z}) \subseteq Z$, we obtain $\boldsymbol{z} = \mathbf{0}$. □

Theorem 1.2 says that the above proof cannot work for lattices instead of majority algebras. However, the only place where the difference between lattices and majority algebras matters, is the definition of the algebra W. This step is impossible to do with lattices. Indeed, we would have $\xi_i \wedge \delta = 0$ in W_4 and $\eta_j \leq \delta$ in W_6. In lattices, this implies $\xi_i \wedge \eta_j = 0$, which is not true in W_5. So, W cannot be a lattice.

Theorem 3.4 means that Problem 1.5 remains open. The situation is now analogous to the situation with CLP after the result from [10]. One can try to achieve a negative solution by a further strengthening of the SES concept. On the other hand, the proof of 3.4 suggests that the method of diagram lifting (see [11], [2], [4], [19]) can be used to attempt a positive solution.

If $|\Omega| = \kappa \geq \aleph_0$, then $|F(\Omega)| = |\operatorname{Con}_c F(\Omega)| = \kappa$. So, Theorem 3.4 provides a new proof of Theorem 1.3.

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Faculty of Natural Sciences, Šafárik's University, Jesenná 5, 04154 Košice, Slovakia

E-mail address: miroslav.ploscica@upjs.sk