UNIFORM REFINEMENTS IN DISTRIBUTIVE SEMILATTICES

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Abstract. This is a contribution to the well known problem whether every algebraic distributive lattice can be represented as the congruence lattice of some lattice. We present two constructions of distributive semilattices without a special refinement property, called WURP. Failure of WURP means that these semilattices are difficult to represent as semilattices of compact congruences of lattices. In fact, we do not know if these semilattices have such representation. Our construction is motivated by a recent paper of F. Wehrung, who found a similar example using ordered vector spaces.

1. Introduction

A distributive semilattice is a (join)-semilattice S with 0 satisfying the following condition. Whenever $x, y, z \in S$ and $x + y \geq z$, then there are elements $x' \leq x$, $y' \leq y$ such that $x' + y' = z$. Distributivity of semilattices can be also defined in a more symmetric way. Whenever $x_0, x_1, y_0, y_1 \in S$ are such that $x_0 + x_1 = y_0 + y_1$, then there are elements $z_{ij} \in S$, $i, j = 0, 1$, such that $z_{i0} + z_{i1} = x_i$, $z_{0j} + z_{1j} = y_j$, $i, j = 0, 1$.

Distributive semilattices are exactly semilattices of compact elements of distributive algebraic lattices. So the classical Congruence Lattice Problem whether every algebraic distributive lattice can be represented as the congruence lattice of a lattice can be reformulated as the problem whether every distributive semilattice is isomorphic to the semilattice of compact congruences of a lattice.

In [3] F. Wehrung formulated a so called Uniform Refinement Property (URP) for distributive semilattices and constructed a distributive semilattice that fails this property. In a subsequent paper [4] he proved that compact congruence semilattices of a large class of lattices satisfy (URP). This class of "congruence splitting" lattices contains all sectionally complemented lattices, relatively complemented lattices, atomistic lattices, and is closed under direct limits.

On the other hand, the authors of the present paper together with F. Wehrung proved in [2] that compact congruence semilattice of the free lattice with at least $\aleph_2$ free generators in any non-distributive variety of lattices does not satisfy even weaker condition, which is called the Weak Uniform Refinement Property (WURP). We say that a distributive semilattice S satisfies (WURP) at an element $e \in S$ if, for every system a_i^α, $\alpha \in \Omega$, $i = 0, 1$ of elements of S such that $a_0^\alpha + a_1^\alpha = e$ for every $\alpha \in \Omega$, there are elements $c^{\alpha\beta} \in S$, $\alpha, \beta \in \Omega$ satisfying the following conditions:

(1) $c^{\alpha\beta} \leq a_0^\alpha, a_1^\beta$,
(2) $c^{\alpha\beta} + a_1^\alpha + a_0^\beta = e$,
(3) $c^{\alpha\gamma} \leq c^{\alpha\beta} + c^{\beta\gamma}$.

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Hence the failure of (WURP) at some point of a distributive semilattice S is not an obstacle for representing this semilattice as the compact congruence semilattice of a lattice.

In the present paper we present two constructions of distributive semilattices that fail (WURP). Both constructions consist in a "free distributive extension" of any semilattice with 0. The constructions differ according to the definition of distributivity we use. The proof that our "free distributive extensions" fail (WURP) depend on the following Kuratowski's characterization of $\aleph_k$ that appeared in [1]. The discovery that Kuratowski's theorem can be used to prove that certain distributive semilattices fail (URP) or (WURP) is due to F. Wehrung [3].

In what follows $[S]^n$ denotes the set of all n-element subsets of a set S and $[S]^{<\omega}$ denotes the set of all finite subsets of S.

[. *Kuratowski's Theorem] Let n be a non negative integer, let S be a set of cardinality at least $\aleph_n$ and let $f : [S]^n \to [S]^{<\omega}$ be any mapping. Then there exists $U \in [S]^{n+1}$ such that $x \notin f(U \setminus \{x\})$ for every $x \in U$.*

2. Free distributive extensions of semilattices

Let $(L, +)$ be a (join-) semilattice with 0. Let us denote

$$C(L) = \{(x, y, z) \in L^3 \mid x + y \geq z\}.$$

A finite set $R \subseteq C(L)$ is called reduced if the following conditions hold:

(1) R contains exactly one triple of the form (x, x, x); this element x will be denoted by t_R;
(2) if $(x, y, z) \in R$ and $(y, x, z) \in R$ then $x = y = z$;
(3) if $(x, y, z) \in R \setminus \{t_R, t_R, t_R\}$ then $x \nleq t_R$, $y \nleq t_R$, $z \nleq t_R$.

Let $\mathcal{R}(L)$ be the family of all reduced sets. We define an order relation on $\mathcal{R}(L)$ by

$$R \leq S \quad \text{iff, for every} \quad (x, y, z) \in R \setminus S,\ x \leq t_S \quad \text{or} \quad z \leq t_S.$$

Lemma 2.1. *$(\mathcal{R}(L), \leq)$ is a semilattice. The supremum $R + S$ can be computed by the following algorithm.*

(i) *Set $T_0 = R \cup S$. If T_0 contains two different elements (x, y, z) and (y, x, z), we replace this pair of elements by a single element (z, z, z). After all such replacements we obtain a set T_1.*
(ii) *Let $(x_1, x_1, x_1), \dots, (x_n, x_n, x_n)$ be all elements of the form (x, x, x) in T_1. We replace all these elements by the single element $(\Sigma x_i, \Sigma x_i, \Sigma x_i)$ and denote the resulting set by T_2.*
(iii) *If there is $(x, y, z) \in T_2$ such that $y \leq t_{T_2}$, we replace the elements (x, y, z), $(t_{T_2}, t_{T_2}, t_{T_2})$ by the element $(z + t_{T_2}, z + t_{T_2}, z + t_{T_2})$. (Of course, this changes the value of t_{T_2}.) We repeat this procedure until no such situation occurs. Let T_3 be the resulting set.*
(iv) *$R + S$ is obtained from T_3 by deleting all elements (x, y, z) with $x \leq t_{T_3}$ or $z \leq t_{T_3}$.*

Proof. I. First we show that $R + S \in \mathcal{R}(L)$. The condition (1) is ensured by the step (ii) and remains valid after performing steps (iii) and (iv). The condition (2) is satisfied because of the step (i). The condition $b \nleq t_{R+S}$ for every $(a, b, c) \in R + S$ holds because of the step (iii) and the conditions $a \nleq t_{R+S}$, $c \nleq t_{R+S}$ are ensured by the step (iv) of our algorithm.

II. Now we show that $R \le R+S$. (The proof that $S \le R+S$ is similar.) Because of (ii) we have $t_R + t_S \le t_{R+S}$. Suppose that $(a,b,c) \in R \setminus (R+S)$. Since $(a,b,c) \in R \cup S$, the triple (a,b,c) must have been deleted in some step of the algorithm. If this happened in the step (i) then (c,c,c) was one of the summands in the step (ii) and therefore $c \le t_{R+S}$. If this happened in the step (ii) then (a,b,c) is of the form (x,x,x) and $x \le t_{R+S}$. If this happened in the step (iii) then $c + t_{T_2} \le t_{R+S}$, hence $c \le t_{R+S}$. If this happened in the step (iv), the case is trivial.

III. Let $Q \in \mathcal{R}(L)$, $R \le Q$, $S \le Q$. We show that $R+S \le Q$. Let $(x,y,z) \in (R+S) \setminus Q$. If $(x,y,z) \in R \cup S$ then $x \le t_Q$ or $z \le t_Q$, because $R \le Q$, $S \le Q$. The remaining case is that (x,y,z) appeared in some step of the algorithm. The only such triple is $(t_{R+S}, t_{R+S}, t_{R+S})$ and we need to show that $t_{R+S} \le t_Q$. At the beginning we have $t_R \le t_Q$ and $t_S \le t_Q$. In the step (i), new triples (c,c,c) can appear such that $(a,b,c) \in R$, $(b,a,c) \in S$. The set Q cannot contain both (a,b,c) and (b,a,c). If $(a,b,c) \notin Q$ (the other case is similar) we have $a \le t_Q$ or $c \le t_Q$. If $a \le t_Q$ then $(b,a,c) \notin Q$, because Q is reduced. Consequently, $b \le t_Q$ or $c \le t_Q$. If $b \le t_Q$ then also $c \le a+b \le t_Q$. Hence, in any case, $c \le t_Q$. This means that after the step (ii) we have $t_{T_2} \le t_Q$. Consider now the step (iii). Let $(a,b,c) \in R \cup S$, $b \le t_{T_2}$. Then $b \le t_Q$, hence $(a,b,c) \notin Q$ and therefore $a \le t_Q$ or $c \le t_Q$. The case $c \le t_Q$ gives us the desired inequality $c + t_{T_2} \le t_Q$. If $a \le t_Q$ then $c \le a+b \le t_Q$ with the same conclusion. Hence, after the step (iii) we have $t_{R+S} = t_{T_3} \le t_Q$ and the step (iv) is inessential. $\square$

Lemma 2.2. *The assignment $x \mapsto \{(x,x,x)\}$ defines a 0-preserving semilattice embedding $L \to \mathcal{R}(L)$.*

Proof. Obvious. $\square$

By an *interpolant* on a semilattice L we mean any function $\iota : C(L) \to L$ that satisfies the conditions $\iota(x,y,z) + \iota(y,x,z) = z$ and $\iota(x,y,z) \le x$ for every $(x,y,z) \in C(L)$. It is obvious that an interpolant exists on a semilattice L if and only if L is distributive. Also notice that an interpolant on a distributive semilattice L is not determined uniquely.

Theorem 2.3. *Let $f : L \to M$ be a semilattice homomorphism. Suppose that ι is an interpolant on M. Let us define a map $f_\iota : \mathcal{R}(L) \to M$ by*

$$f_\iota(R) = \sum_{(x,y,z)\in R} \iota(f(x), f(y), f(z)).$$

Then f_ι is a semilattice homomorphism and $f_\iota \restriction L = f$.

Proof. For $(x,y,z) \in C(L)$ we have $(f(x), f(y), f(z)) \in C(M)$, so f_ι is well defined. Since $x = \iota(x,x,x) + \iota(x,x,x) = \iota(x,x,x)$, we have $f_\iota \restriction L = f$. Now we claim that

$$f_\iota(R+S) = \sum_{(x,y,z)\in R\cup S} \iota(f(x), f(y), f(z)) = f_\iota(R) + f_\iota(S).$$

Let $(x,y,z) \in R \cup S$. Since $R, S \le R+S$, we have $(x,y,z) \in R+S$ or $x \le t_{R+S}$ or $z \le t_{R+S}$. If $(x,y,z) \in R+S$ then $\iota(f(x), f(y), f(z)) \le f_\iota(R+S)$. If $z \le t_{R+S}$ then $\iota(f(x), f(y), f(z)) \le f(z) \le f(t_{R+S}) = \iota(f(t_{R+S}), f(t_{R+S}), f(t_{R+S})) \le f_\iota(R+S)$. A similar argument holds in the case $x \le t_{R+S}$. This shows that $f_\iota(R) + f_\iota(S) \le f_\iota(R+S)$.

To prove the inverse inequality, it suffices to show that

$$f(t_{R+S}) = \iota(f(t_{R+S}), f(t_{R+S}), f(t_{R+S})) \leq f_\iota(R) + f_\iota(S).$$

Clearly, $f(t_R) \leq f_\iota(R)$, $f(t_S) \leq f_\iota(S)$. Let us consider $R+S$ computed by the algorithm from 2.1. Suppose that $(a,b,c) \in R$, $(b,a,c) \in S$. Then $f(c) = \iota(f(a), f(b), f(c)) + \iota(f(b), f(a), f(c)) \leq f_\iota(R) + f_\iota(S)$. Since f is a homomorphism, after step (ii) we have $f(t_{T_2}) \leq f_\iota(R) + f_\iota(S)$. Now let $(a,b,c) \in R \cup S$, $b \leq t_{T_2}$. Then $f(c) = \iota(f(a), f(b), f(c)) + \iota(f(b), f(a), f(c)) \leq \iota(f(a), f(b), f(c)) + f(b) \leq f_\iota(R) + f_\iota(S)$, hence $f(c + t_{T_2}) = f(c) + f(t_{T_2}) \leq f_\iota(R) + f_\iota(S)$. This shows that $f(t_{R+S}) \leq f_\iota(R) + f_\iota(S)$ holds at the end of the algorithm. □

Let us set $\mathcal{R}_0(L) = L$ and $\mathcal{R}_{n+1}(L) = \mathcal{R}(\mathcal{R}_n(L))$ for $n = 0, 1, 2, \ldots$. Up to isomorphism we can assume that $\mathcal{R}_n(L)$ is a subsemilattice of $\mathcal{R}_{n+1}(L)$. Let us set $\mathcal{D}(L) = \bigcup_{n=0}^{\infty} \mathcal{R}_n(L)$. Then every semilattice homomorphism $f : L \to M$ (with M distributive having an interpolant ι) can be extended to a homomorphism $f_{[\iota]} : \mathcal{D}(L) \to M$, similarly as in 2.3.

Theorem 2.4. *For every semilattice L, $\mathcal{D}(L)$ is distributive.*

Proof. Let $x, y, z \in \mathcal{D}(L)$, $x + y \geq z$. Let n be the least number such that $x, y, z \in \mathcal{R}_n(L)$. We set $\iota(x,y,z) = \{(x,y,z), (0,0,0)\} \in \mathcal{R}_{n+1}(L)$. (If $x = y = z$ or $0 \in \{x,y,z\}$ then $\iota(x,y,z) = z$.) This defines an interpolant on $\mathcal{D}(L)$. □

Let L be a 0-subsemilattice of M. Then, obviously, $C(L)$ is a subset of $C(M)$ and $\mathcal{R}(L)$ is a 0-subsemilattice of $\mathcal{R}(M)$. Moreover, the diagram

$$\begin{array}{ccc} L & \longrightarrow & M \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{R}(L) & \longrightarrow & \mathcal{R}(M) \end{array}$$

commutes, which means that natural embeddings $L \to \mathcal{R}(L)$ and $M \to \mathcal{R}(M)$ can be simultaneously regarded as inclusions. As a consequence we obtain

Lemma 2.5. *If L is a 0-subsemilattice of M then $\mathcal{D}(L)$ is a 0-subsemilattice of $\mathcal{D}(M)$.*

Further, let M be a directed union of its 0-subsemilattices L_i $(i \in I)$. Then $C(M)$ is a directed union of its subsets $C(L_i)$ and therefore $\mathcal{R}(M)$ is a directed union of its 0-subsemilattices $\mathcal{R}(L_i)$. Consequently we have

Lemma 2.6. *If M is a directed union of its 0-subsemilattices L_i $(i \in I)$ then $\mathcal{D}(M)$ is a directed union of its 0-subsemilattices $\mathcal{D}(L_i)$.*

3. Failure of WURP

Let Ω be a set and for every $\alpha \in \Omega$ let a_0^α, a_1^α be two different elements. Suppose moreover that the sets $\{a_0^\alpha, a_1^\alpha\}$, $\{a_0^\beta, a_1^\beta\}$ are disjoint for any two $\alpha, \beta \in \Omega$. Let $L(\Omega)$ be the family of all $A \subseteq U = \{a_0^\alpha \mid \alpha \in \Omega\} \cup \{a_1^\alpha \mid \alpha \in \Omega\}$ that satisfy

(1) if $\{a_0^\alpha, a_1^\alpha\} \subseteq A$ for some α then $A = U$;
(2) if $A \neq U$ then A is finite.

It is clear that $L(\Omega)$ is a semilattice in which the join of two elements is either set-theoretical union or $1 = U$. (In fact, $L(\Omega)$ is the free semilattice with generators satisfying $a_0^\alpha + a_1^\alpha = 1$ for all α.)

Now let us consider the 0-subsemilattice $L(2)$ of $L(\Omega)$ generated by the set $\{a_0^0, a_1^0, a_0^1, a_1^1\}$ $(0, 1 \in \Omega)$. Let G be the 0-subsemilattice of $\mathcal{R}(L(2))$ generated by the elements

(1) $c^{00} = \{(0,0,0), (a_0^0, a_1^0, a_0^1), (a_0^1, a_1^1, a_0^0)\}$;
(2) $c^{01} = \{(0,0,0), (a_0^0, a_1^0, a_1^1), (a_1^1, a_0^1, a_0^0)\}$;
(3) $c^{10} = \{(0,0,0), (a_1^0, a_0^0, a_0^1), (a_0^1, a_1^1, a_1^0)\}$;
(4) $c^{11} = \{(0,0,0), (a_1^0, a_0^0, a_1^1), (a_1^1, a_0^1, a_1^0)\}$.

Let $\mathbf{2}$ be the 2-element semilattice $\{0, 1\}$. It is well known that, for a natural number n, $\mathbf{2}^n$ is the free 0-semilattice with n generators $(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots$ $(0, 0, 0, \ldots, 1)$.

Lemma 3.1. *G is isomorphic to $\mathbf{2}^4$.*

Proof. Since $\mathbf{2}^4$ is free, we have a homomorphism $\varphi : \mathbf{2}^4 \to G$ with $\varphi(1, 0, 0, 0) = c^{00}$, $\varphi(0, 1, 0, 0) = c^{01}$, $\varphi(0, 0, 1, 0) = c^{10}$, $\varphi(0, 0, 0, 1) = c^{11}$. Further we define the interpolant ι on $\mathbf{2}^4$ by $\iota(x, y, z) = x \wedge z$. ($\mathbf{2}^4$ is a lattice.) Let us consider the embedding $\psi : L(2) \to \mathbf{2}^4$ determined by $\psi(a_0^0) = (1, 1, 0, 0)$, $\psi(a_1^0) = (0, 0, 1, 1)$, $\psi(a_0^1) = (1, 0, 1, 0)$, $\psi(a_1^1) = (0, 1, 0, 1)$. By 2.3, ψ can be extended to $\psi_\iota : \mathcal{R}(L(2)) \to \mathbf{2}^4$. We have $\psi_\iota(c^{00}) = \iota(\psi(a_0^0), \psi(a_1^0), \psi(a_0^1)) + \iota(\psi(a_0^1), \psi(a_1^1), \psi(a_0^0)) = (1, 0, 0, 0) + (1, 0, 0, 0) = (1, 0, 0, 0)$ and similarly, $\psi_\iota(c^{01}) = (0, 1, 0, 0)$, $\psi_\iota(c^{10}) = (0, 0, 1, 0)$, $\psi_\iota(c^{11}) = (0, 0, 0, 1)$. This shows that ψ_ι is inverse to φ, hence it is an isomorphism. $\square$

Theorem 3.2. *If card$(\Omega) \geq \aleph_2$ then $\mathcal{D}(L(\Omega))$ does not have WURP at 1.*

Proof. For every $\alpha \in \Omega$ we have $a_0^\alpha + a_1^\alpha = 1$. For contradiction, suppose that elements $c^{\alpha\beta}$ have required properties.

For any set $X \subseteq \Omega$ let $L(X)$ be the 0-subsemilattice of $L(\Omega)$ generated by $\{a_0^\alpha \mid \alpha \in X\} \cup \{a_1^\alpha \mid \alpha \in X\}$. By 2.5 we have the canonical embedding $\mathcal{D}(L(X)) \to \mathcal{D}(L(\Omega))$, which we regard as inclusion. Obviously, $L(\Omega)$ is a directed union of its 0-subsemilattices $L(X)$, where X runs through finite subsets of Ω. By 2.6, $\mathcal{D}(L(\Omega))$ is a directed union of its 0-subsemilattices $\mathcal{D}(L(X))$, X finite. Hence, for every $\alpha, \beta \in \Omega$ there is a finite set $X_{\{\alpha,\beta\}}$ ($X_{\alpha\beta}$ for short) such that $c^{\alpha\beta}$ and $c^{\beta\alpha}$ belong to $\mathcal{D}(L(X_{\alpha\beta}))$. By Kuratowski's theorem there is a 3-element set $3 = \{0, 1, 2\} \subseteq \Omega$ such that $0 \notin X_{12}$, $1 \notin X_{02}$, $2 \notin X_{01}$.

Let us consider the map $f : L(\Omega) \to L(3) \subseteq \mathcal{D}(L(3))$ defined by

$$f(A) = \begin{cases} 1 & \text{if } a_1^\alpha \in A \text{ for some } \alpha \notin \{0,1,2\} \\ A \cap \{a_0^0, a_0^1, a_0^2, a_1^0, a_1^1, a_1^2\} & \text{otherwise.} \end{cases}$$

Now we define a special interpolant on $\mathcal{D}(L(3))$. Let G_{01} be the 0-subsemilattice of $\mathcal{D}(L(3))$ described in 3.1. Let G_{02} and G_{12} be analogous subsemilattices using the elements a_0^2, a_1^2. Since all G_{ij} are lattices, we can use the meet operation. Thus, for $x, y, z \in G_{ij}$, $z \leq x + y$, we set $\iota(x, y, z) = x \wedge z$. (The intersection is taken in G_{ij}. There is no ambiguity, since the intersection of two different G_{ij} is a sublattice of both. For instance, $G_{01} \cap G_{02} = \{0, 1, a_0^0, a_1^0\}$.) If $\{x, y, z\} \nsubseteq G_{01}, G_{02}, G_{12}$, we define $\iota(x, y, z)$ as in the proof of 2.4.

It is easy to see that f is a homomorphism. By 2.3 it can be extended to a homomorphism $f_{[\iota]}: \mathcal{D}(L(\Omega)) \to \mathcal{D}(L(3))$. We denote $d_0 = f_{[\iota]}(c^{12})$, $d_1 = f_{[\iota]}(c^{02})$, $d_2 = f_{[\iota]}(c^{01})$.

The definition of $f_{[\iota]}$ implies that $d_i \in \mathcal{D}(L(3 \setminus \{i\}))$ for $i = 0, 1, 2$. Indeed, by induction we can prove that $\mathcal{R}_n(L(\Omega \setminus \{i\}))$ is mapped by $f_{[\iota]}$ into $\mathcal{D}(L(3 \setminus \{i\}))$ for every n. In fact, the definition of ι ensures that $d_0 \in G_{12}$, $d_1 \in G_{02}$ and $d_2 \in G_{01}$.

Since $f_{[\iota]}$ is a homomorphism, we have $d_0 \le a_0^1, a_1^2$, $d_1 \le a_0^0, a_1^2$, $d_2 \le a_0^0, a_1^1$. Further, we have the equalities $d_0 + a_1^1 + a_0^2 = d_1 + a_1^0 + a_0^2 = d_2 + a_1^0 + a_0^1 = 1$. In the Boolean algebra G_{12} there exists only one element d_0 satisfying these requirements, namely
$d_0 = \{(0,0,0), (a_0^1, a_1^1, a_1^2), (a_1^2, a_0^2, a_0^1)\}$.

For similar reasons,
$d_1 = \{(0,0,0), (a_0^0, a_1^0, a_1^2), (a_1^2, a_0^2, a_0^0)\}$,
$d_2 = \{(0,0,0), (a_0^0, a_1^0, a_1^1), (a_1^1, a_0^1, a_0^0)\}$.

From the inequality $c^{02} \le c^{01} + c^{12}$ we obtain that $d_1 \le d_0 + d_2$. But we can check directly that this is not true - a contradiction. ☐

4. Free distributive extensions of semilattices - another version

In this section we present an alternative construction of a "free distributive extension" of a given semilattice L.

Let $(L, +)$ be a (join-) semilattice with 0. Let us denote

$$D_0(L) = \{(a_0, a_1, b_0, b_1) \in L^4 \mid a_0 + a_1 = b_0 + b_1\}.$$

Further we denote

$$D(L) = (D_0(L) \times \{0,1\} \times \{0,1\}) \cup L.$$

Instead of $((a_0, a_1, b_0, b_1), i, j)$ we shall write $(a_0, a_1, b_0, b_1)_{ij}$. Moreover, if $i, j \in \{0,1\}$, then we denote by i' the other element of $\{0,1\}$ different from i and similarly j' is the other element of $\{0,1\}$ different from j. A finite set $R \subseteq D(L)$ is called reduced if the following conditions hold:

(1) R contains exactly one element of L; this element x will be denoted by t_R;
(2) if $i \in \{0,1\}$, then $(a_0, a_1, b_0, b_1)_{i0}, (a_0, a_1, b_0, b_1)_{i1}$ do not belong to R simultaneously;
(3) if $j \in \{0,1\}$, then $(a_0, a_1, b_0, b_1)_{0j}, (a_0, a_1, b_0, b_1)_{1j}$ do not belong to R simultaneously;
(4) if $(a_0, a_1, b_0, b_1)_{ij} \in R$, then $a_0 \not\le t_R$, $a_1 \not\le t_R$, $b_0 \not\le t_R$, $b_1 \not\le t_R$.

Let $\mathcal{Q}(L)$ be the family of all reduced sets. We define an order relation on $\mathcal{Q}(L)$ by $R \le S$ if and only if $t_R \le t_S$ and, moreover,

$$\text{for every} \quad (a_0, a_1, b_0, b_1)_{ij} \in R \setminus S, \quad \text{either} \quad a_i \le t_S \quad \text{or} \quad b_j \le t_S.$$

Lemma 4.1. *$(\mathcal{Q}(L), \le)$ is a semilattice. The supremum $R + S$ can be computed by the following algorithm.*

(i) *Set $T_0 = R \cup S$. If $(a_0, a_1, b_0, b_1)_{i0} \in T_0$ and $(a_0, a_1, b_0, b_1)_{i1} \in T_0$ for some $i \in \{0,1\}$, then include also a_i to the set T_0. If $(a_0, a_1, b_0, b_1)_{0j} \in T_0$ and $(a_0, a_1, b_0, b_1)_{1j} \in T_0$ for some $j \in \{0,1\}$, then include b_j to T_0. After all such inclusions we obtain a set T_1.*
(ii) *Let $x_1, \dots, x_n$ be all elements of $L \cap T_1$. We replace all these elements by the single element $\sum x_i$ and denote the resulting set by T_2.*

(iii) *If there is* $(a_0,a_1,b_0,b_1)_{ij} \in T_2$ *and* $a_{i'} \le t_{T_2}$*, then replace the elements* $(a_0,a_1,b_0,b_1)_{ij}$*,* t_{T_2} *by the element* $b_j + t_{T_2}$*. If there is* $(a_0,a_1,b_0,b_1)_{ij} \in T_2$ *and* $b_{j'} \le t_{T_2}$*, then replace* $(a_0,a_1,b_0,b_1)_{ij}$ *and* t_{T_2} *by* $a_i + t_{T_2}$*. (Of course, this changes the value of* t_{T_2}*.) We repeat this procedure until no such situation occurs. Let* T_3 *be the resulting set.*

(iv) $R+S$ *is obtained from* T_3 *by deleting all elements* $(a_0,a_1,b_0,b_1)_{ij}$ *such that either* $a_i \le t_{T_3}$ *or* $b_j \le t_{T_3}$*.*

Proof. I. First we show that $R+S \in \mathcal{Q}(L)$. The condition (1) is ensured on the step (ii) and remains valid after performing steps (iii) and (iv). The conditions (2) and (3) are satisfied because of the step (i). The condition $a_{i'}, b_{j'} \nleq t_{R+S}$ for every $(a_0,a_1,b_0,b_1)_{ij} \in R+S$ holds because of the step (iii) and the conditions $a_i, b_j \nleq t_{R+S}$ hold because of the step (iv) of our algorithm.

II. Now we show that $R \le R+S$. (The proof that $S \le R+S$ is similar.) Because of (ii) we have $t_R \le t_{R+S}$. Suppose that $(a_0,a_1,b_0,b_1)_{ij} \in R \setminus (R+S)$. Since $(a_0,a_1,b_0,b_1)_{ij} \in R \cup S$, the six-tuple $(a_0,a_1,b_0,b_1)_{ij}$ must have been deleted in some step of the algorithm. If this happened in step (iii), then either $a_i + t_{T_2} \le t_{R+S}$ or $b_j + t_{T_2} \le t_{R+S}$. If this happened in step (iv), the case is trivial.

III. Let $Q \in \mathcal{Q}(L)$, $R \le Q$, $S \le Q$. We are going to show that $R+S \le Q$. Let $(a_0,a_1,b_0,b_1)_{ij} \in (R+S) \setminus Q$. Since only elements of L can appear in $R+S$ without being already in $R \cup S$, we get that $(a_0,a_1,b_0,b_1)_{ij} \in R \cup S$. Thus either $(a_0,a_1,b_0,b_1)_{ij} \in R \setminus Q$ or $(a_0,a_1,b_0,b_1)_{ij} \in S \setminus Q$. In both cases, either $a_i \le t_Q$ or $b_j \le t_Q$.

It remains to prove $t_{R+S} \le t_Q$. We have that $t_R + t_S \le t_Q$. First of all we prove that for every element $x \in T_1 \cap L$ we get $x \le t_Q$. So suppose that $(a_0,a_1,b_0,b_1)_{i0}, (a_0,a_1,b_0,b_1)_{i1} \in R \cup S$. We get that the elements $(a_0,a_1,b_0,b_1)_{i0}$ and $(a_0,a_1,b_0,b_1)_{i1}$ cannot be simultaneously in Q since Q is reduced. If none of the elements $(a_0,a_1,b_0,b_1)_{i0}, (a_0,a_1,b_0,b_1)_{i1}$ is contained in Q, then (since $R, S \le Q$) we get that either $a_i \le t_Q$ or $b_0, b_1 \le t_Q$. But in the latter case also $a_i \le b_0+b_1 \le t_Q$. In the remaining case, one of the elements, say $(a_0,a_1,b_0,b_1)_{i0}$ belongs to Q and $(a_0,a_1,b_0,b_1)_{i1} \notin Q$. Again, the assumption $(a_0,a_1,b_0,b_1)_{i1} \notin Q$ implies that either $a_i \le t_Q$ or $b_1 \le t_Q$,which (together with $(a_0,a_1,b_0,b_1)_{i0} \in Q$) contradicts the condition (4) of the definition of reduced sets for Q. Thus in each case $a_i \le t_Q$. The case $(a_0,a_1,b_0,b_1)_{0j}, (a_0,a_1,b_0,b_1)_{1j} \in R \cup S$ is treated similarly. Hence for every $x \in T_1 \cap L$ we get $x \le t_Q$. But then also their sum is less than or equal to t_Q, hence $t_{T_2} \le t_Q$.

The element t_{T_2} can be further increased at step (iii). If this is the case suppose that $(a_0,a_1,b_0,b_1)_{ij} \in T_2$ and $a_{i'} \le t_{T_2} \le t_Q$. Thus $(a_0,a_1,b_0,b_1)_{ij} \notin Q$ because of condition (4). Then either $a_i \le t_Q$ or $b_j \le t_Q$. In the former case $b_j \le a_0 + a_1 = a_i + a_{i'} \le t_Q$. Thus in all cases $b_j + t_{T_2} \le t_Q$. The other case of step (iii) is treated similarly.

Since $t_{R+S} = t_{T_3}$, we have proved that $t_{R+S} \le t_Q$. □

Lemma 4.2. *The semilattice* L *is a* 0*-subsemilattice of* $\mathcal{Q}(L)$*. (We identify* $x \in L$ *with* $\{x\} \in \mathcal{Q}(L)$*.)*

Proof. Obvious. □

By a *refinement operator* on a semilattice L we mean a collection of four functions $\iota_{ij} : D_0(L) \to L$, $i,j = 0,1$, satisfying the conditions $\iota_{i0}(a_0,a_1,b_0,b_1) + \iota_{i1}(a_0,a_1,b_0,b_1)) = a_i$ and $\iota_{0j}(a_0,a_1,b_0,b_1) + \iota_{1j}(a_0,a_1,b_0,b_1) = b_j$ for every

$i, j = 0, 1$. It is obvious that a refinement operator exists on a semilattice L if and only if L is distributive. Also notice that a refinement operator on a distributive semilattice L is not determined uniquely.

Theorem 4.3. *Let $f : L \to M$ be a semilattice homomorphism. Suppose that ι_{ij}, $i, j = 0, 1$, is a refinement operator on M. Let us define a map $f_\iota : \mathcal{Q}(L) \to M$ by*

$$f_\iota(R) = f(t_R) + \sum_{(a_0,a_1,b_0,b_1)_{ij} \in R} \iota_{ij}(f(a_0), f(a_1), f(b_0), f(b_1)).$$

Then f_ι is a semilattice homomorphism and $f_\iota \restriction L = f$.

Proof. For every $(a_0, a_1, b_0, b_1) \in D_0(L)$ we have that $(f(a_0), f(a_1), f(b_0), f(b_1)) \in D_0(M)$, so f_ι is well defined. Moreover, $f_\iota \restriction L = f$.

To show that $f_\iota(R)+f_\iota(S) \leq f_\iota(R+S)$ for any $R, S \in \mathcal{Q}(L)$ we have to consider how the algorithm 4.1. computes $R + S$ from $R \cup S$. First of all we introduce the following notation. If $T \subseteq D(L)$, then we set $V = T \cap L$. Thus if $T \subseteq D(L)$ is reduced, then $|V| = 1$.

The reduced set $R + S$ is computed from $T_0 = R \cup S$ by the algorithm 4.1. We start with $V_0 = \{t_R, t_S\}$. Thus

$$f_\iota(R) + f_\iota(S) = \sum_{x \in V_0} f(x) + \sum_{(a_0,a_1,b_0,b_1)_{ij} \in R \cup S} \iota_{ij}(f(a_0), f(a_1), f(b_0), f(b_1)).$$

In the first step of the algorithm, any pair of elements $(a_0, a_1, b_0, b_1)_{i0}$ and $(a_0, a_1, b_0, b_1)_{i1}$ in $R \cup S$ is replaced in T_1 by the single element $a_i \in L \cap T_1 = V_1$. Since $\iota_{i0}(f(a_0), f(a_1), f(b_0), f(b_1)) + \iota_{i1}(f(a_0), f(a_1), f(b_0), f(b_1)) = f(a_i)$ the replacement does not change the value of the right-hand side of the last displayed equality. Since the same is true for the replacement of $(a_0, a_1, b_0, b_1)_{0j}$ and $(a_0, a_1, b_0, b_1)_{1j}$ by b_j, we get

$$f_\iota(R) + f_\iota(S) = \sum_{x \in V_1} f(x) + \sum_{(a_0,a_1,b_0,b_1)_{ij} \in T_1} \iota_{ij}(f(a_0), f(a_1), f(b_0), f(b_1)).$$

The second step of the algorithm changes only the set V_1, all the elements of V_1 are replaced by their sum denoted by t_{T_2}. Since $f(t_{T_2}) = \sum_{x \in V_1} f(x)$, we get

$$f_\iota(R) + f_\iota(S) = f(t_{T_2}) + \sum_{(a_0,a_1,b_0,b_1)_{ij} \in T_2} \iota_{ij}(f(a_0), f(a_1), f(b_0), f(b_1)).$$

The third step of the algorithm replaces any $(a_0, a_1, b_0, b_1)_{ij} \in T_2$ such that $a_{i'} \leq t_{T_2}$ and the element t_{T_2} by the single element $b_j + t_{T_2}$. Since $f(b_j) + f(t_{T_2}) = \iota_{0j}(f(a_0), f(a_1), f(b_0), f(b_1)) + \iota_{1j}(f(a_0), f(a_1), f(b_0), f(b_1)) + f(t_{T_2}) \leq \iota_{ij}(f(a_0), f(a_1), f(b_0), f(b_1)) + f(a_{i'}) + f(t_{T_2}) = \iota_{ij}(f(a_0), f(a_1), f(b_0), f(b_1)) + f(t_{T_2}) \leq f(b_j) + f(t_{T_2})$, the replacement again does not change the value of the right-hand side of the last displayed equality. Since this is true for any other replacement made in the third step, we get

$$f_\iota(R) + f_\iota(S) = f(t_{T_3}) + \sum_{(a_0,a_1,b_0,b_1)_{ij} \in T_3} \iota_{ij}(f(a_0), f(a_1), f(b_0), f(b_1)).$$

Finally, in the last step of the algorithm we remove from the set T_3 all the elements $(a_0, a_1, b_0, b_1)_{ij}$ such that either $a_i \leq t_{T_3}$ or $b_j \leq t_{T_3}$. In this case either $\iota_{ij}(f(a_0), f(a_1), f(b_0), f(b_1)) \leq f(a_i) \leq f(t_{T_3})$ or $\iota_{ij}(f(a_0), f(a_1), f(b_0), f(b_1)) \leq$

$f(b_j) \leq f(t_{T_3})$. None of the removals changes the value of the right-hand side of the last equality. But after the removals we obtain the reduced set $R+S$, hence

$$f_\iota(R) + f_\iota(S) = f_\iota(R+S).$$

$\square$

Now let us set $\mathcal{Q}_0(L) = L$ and $\mathcal{Q}_{n+1}(L) = \mathcal{Q}(\mathcal{Q}_n(L))$ for a non-negative integer n. Thus every $\mathcal{Q}_n(L)$ is a subsemilattice of $\mathcal{Q}_{n+1}(L)$ by Lemma 4.2. Let us set $\mathcal{C}(L) = \bigcup_n \mathcal{Q}_n(L)$. Thus given a distributive semilattice M, a refinement operator ι_{ij}, $i,j = 0,1$, and a semilattice homomorphism $f : L \to M$, then by repeated application of Theorem 4.3 we get that there exists a special homomorphism $f_{[\iota]} : \mathcal{C}(L) \to M$ extending f.

Theorem 4.4. *For every semilattice L, $\mathcal{C}(L)$ is distributive.*

Proof. Let $a_0, a_1, b_0, b_1 \in \mathcal{C}(L)$ be such that $a_0 + a_1 = b_0 + b_1$. There exists n such that $a_0, a_1, b_0, b_1 \in \mathcal{Q}_n(L)$, hence $(a_0, a_1, b_0, b_1) \in D_0(\mathcal{Q}_n(L))$. If $0 \notin \{a_0, a_1, b_0, b_1\}$, we set $\iota_{ij}(a_0, a_1, b_0, b_1) = \{(a_0, a_1, b_0, b_1)_{ij}, 0\} \in \mathcal{Q}_{n+1}(L)$, $i,j \in 0,1$. Applying the algorithm 4.1 we easily compute that $\{(a_0, a_1, b_0, b_1)_{i0}, 0\} + \{(a_0, a_1, b_0, b_1)_{i1}, 0\} = \{a_i\}$ and $\{(a_0, a_1, b_0, b_1)_{0j}, 0\} + \{(a_0, a_1, b_0, b_1)_{1j}, 0\} = \{b_j\}$ in $\mathcal{Q}_{n+1} \subseteq \mathcal{C}(L)$. If $a_0 = 0$ (the other cases are similar), we set $\iota_{00}(a_0, a_1, b_0, b_1) = \iota_{01}(a_0, a_1, b_0, b_1) = 0$, $\iota_{10}(a_0, a_1, b_0, b_1) = b_0$, $\iota_{11}(a_0, a_1, b_0, b_1) = b_1$. We have a refinement operator on L. $\square$

It is easy to see that assertions analogous to 2.5 and 2.6 hold also for $\mathcal{C}(L)$.

5. Failure of WURP in some $\mathcal{C}(L)$

Let $L(\Omega)$ be the same lattice as in the section 3. We consider the 0-subsemilattice $L(2)$ of $L(\Omega)$ generated by the elements $\{a_0^0, a_1^0, a_0^1, a_1^1\}$ $(0, 1 \in \Omega)$. Let G be the 0-subsemilattice of $\mathcal{Q}(L(2))$ generated by the elements $c^{ij} = \{0, (a_0^0, a_1^0, a_0^1, a_1^1)_{ij}\}$, $i,j = 0,1$.

Lemma 5.1. *G is isomorphic to $\mathbf{2}^4$.*

Proof. Let $\varphi : \mathbf{2}^4 \to G$ and $\psi : L(2) \to \mathbf{2}^4$ be as in the proof of 3.1. Further we define the refinement operator ι_{ij} on $\mathbf{2}^4$ by $\iota_{ij}(x_0, x_1, y_0, y_1) = x_i \wedge y_j$. By 4.3, ψ can be extended to $\psi_\iota : \mathcal{Q}(L(2)) \to \mathbf{2}^4$. We have $\psi_\iota(c^{ij}) = \iota_{ij}(\psi(a_0^0), \psi(a_1^0), \psi(a_0^1), \psi(a_1^1)) + \psi(0) = \psi(a_i^0) \wedge \psi(a_j^1)$. Hence $\psi_\iota(c^{00}) = (1,0,0,0)$, $\psi\iota(c^{01}) = (0,1,0,0)$, $\psi_\iota(c^{10}) = (0,0,1,0)$ and $\psi_\iota(c^{11}) = (0,0,0,1)$. This shows that ψ_ι is inverse to φ, hence it is an isomorphism. $\square$

Theorem 5.2. *If card$(\Omega) \geq \aleph_2$ then $\mathcal{C}(L(\Omega))$ does not have WURP at 1.*

Proof. We proceed similarly as in 3.2. Let us consider the same map $f : L(\Omega) \to L(3) \subseteq \mathcal{C}(L(3))$ as in 3.2. Now we define a special refinement operator on $\mathcal{C}(L(3))$. Let G_{01} be the 0-subsemilattice of $\mathcal{C}(L(3))$ described in 5.1. Let G_{02} and G_{12} be analogous subsemilattices defined by the elements $a_0^0, a_1^0, a_0^2, a_1^2$ and $a_0^1, a_1^1, a_0^2, a_1^2$, resp. Since all G_{kl} and their intersections are lattices, we can use the meet operation. Thus, for $x_0, x_1, y_0, y_1 \in G_{kl}$, $x_0 + x_1 = y_0 + y_1$, we can set $\iota_{ij}(x_0, x_1, y_0, y_1) = x_i \wedge y_j$. (The meet is taken in G_{kl}.) If $\{x_0, x_1, y_0, y_1\} \nsubseteq G_{01}, G_{02}, G_{12}$, we define $\iota_{ij}(x_0, x_1, y_0, y_1) = \{0, (x_0, x_1, y_0, y_1)_{ij}\}$, similarly as in the proof of 4.4.

It is easy to verify that f is a homomorphism. By remarks after 4.3 it can be extended to a homomorphism $f_{[\iota]}: \mathcal{C}(L(\Omega)) \to \mathcal{C}(L(3))$. We denote $d_0 = f_{[\iota]}(c^{12})$, $d_1 = f_{[\iota]}(c^{02})$, $d_2 = f_{[\iota]}(c^{01})$. Again we have $d_0 \in G_{12}$, $d_1 \in G_{02}$ and $d_2 \in G_{01}$.

Since $f_{[\iota]}$ is a homomorphism, we have $d_0 \le a_0^1, a_1^2$, $d_1 \le a_0^0, a_1^2$, $d_2 \le a_0^0, a_1^1$. Further, we have the equalities $d_0 + a_1^1 + a_0^2 = d_1 + a_1^0 + a_0^2 = d_2 + a_1^0 + a_0^1 = 1$. In the Boolean algebra G_{12} (isomorphic to $\mathbf{2}^4$) there exists only one element d_0 satisfying these requirements, namely
$d_0 = \{0, (a_0^1, a_1^1, a_0^2, a_1^2)_{01}\}$.

For similar reasons,
$d_1 = \{0, (a_0^0, a_1^0, a_0^2, a_1^2)_{01}\}$,
$d_2 = \{0, (a_0^0, a_1^0, a_0^1, a_1^1)_{01}\}$.

From the inequality $c^{02} \le c^{01} + c^{12}$ we obtain that $d_1 \le d_0 + d_2$. But we can check directly that $d_0 + d_2 = \{0, (a_0^1, a_1^1, a_0^2, a_1^2)_{01}, (a_0^0, a_1^0, a_0^1, a_1^1)_{01}\}$. Using definition of order on $\mathcal{Q}(L(3)))$ we see that $d_1 = \{0, (a_0^0, a_1^0, a_0^2, a_1^2)_{01}\} \not\le d_0 + d_2$ - a contradiction. $\square$

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