AFFINE COMPLETE DISTRIBUTIVE LATTICES

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Abstract. We prove a characterization theorem for affine complete distributive lattices. To do so we introduce the notions of relatively complete ideal and relatively complete filter.

1. Introduction

A k-ary function f on a lattice L is called compatible if for any congruence θ on L and $(a_i, b_i) \in \theta$, $i = 1, \dots, k$, $(f(a_1, \dots, a_k), f(b_1, \dots, b_k)) \in \theta$ holds. It is clear that any polynomial of a lattice L is compatible. Following Schweigert [3] and Werner [4], a lattice L is called affine complete if every compatible function on L is a polynomial.

No internal characterization of affine complete lattices is known. However, in the case of bounded distributive lattices we have the following result of G. Grätzer. An interval in a lattice is called proper if it contains more than one element.

1.1. Theorem ([2]). *A bounded distributive lattice is affine complete if and only if it does not contain a proper interval which is a Boolean lattice.* □

The aim of this paper is to prove a characterization theorem for (in general unbounded) distributive lattices. In the proof we will use the following results due to D. Dorninger and G. Eigenthaler.

1.2. Lemma ([1, p. 102]). *Suppose that every unary compatible function on a distributive lattice L is a polynomial. Then L is affine complete.* □

1.3. Lemma ([1, p. 100]). *Let L be an arbitrary lattice. If L contains a proper Boolean interval, then there is a compatible function on L which is not order-preserving (and hence which cannot be a lattice polynomial).* □

2. Main results

For an element x of a lattice L, let us denote $\uparrow x = \{y \in L \mid x \le y\}$, $\downarrow x = \{y \in L \mid x \ge y\}$.

1991 *Mathematics Subject Classification.* 06D99, 08A40.

Key words and phrases. Distributive lattice, affine completeness.

This research was supported by GA SAV Grant 362/93

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

2.1. Definition. *An ideal I of a lattice L is called relatively complete if for every $x \in L$ there exists* $\max(I \cap \downarrow x)$. *Dually, a filter F of a lattice L is called relatively complete if for every $x \in L$ there exists* $\min(F \cap \uparrow x)$.

It is clear that if an ideal has a largest element, then it is relatively complete. Indeed, if $I = \downarrow b$ then $\max(I \cap \downarrow x) = x \wedge b$. However, there exist relatively complete ideals without a largest element.

An ideal I of a lattice L is proper if $I \neq L$.

2.2. Theorem. *Let I be a proper relatively complete ideal of a distributive lattice L. Suppose that I does not posess a largest element. Then the lattice L is not affine complete.*

Proof. Let us define a function $f : L \longrightarrow L$ by the rule $f(x) = \max(I \cap \downarrow x)$. We will prove that f is compatible and not polynomial.

Let θ be a congruence on L and $(x, y) \in \theta$. We claim that $f(x \wedge y) = f(x) \wedge x \wedge y$. It is clear that $f(x \wedge y) \leq f(x)$ and $f(x \wedge y) \leq x \wedge y$, hence $f(x \wedge y) \leq f(x) \wedge x \wedge y$. On the other hand, the element $f(x) \wedge x \wedge y$ belongs to $I \cap \downarrow (x \wedge y)$, hence $f(x) \wedge x \wedge y \leq f(x \wedge y)$. Now $(x, y) \in \theta$ implies that $(x, x \wedge y) \in \theta$ and also $(x \wedge f(x), x \wedge y \wedge f(x)) \in \theta$, hence $(f(x), f(x \wedge y)) \in \theta$. Similarly one can show that $(f(y), f(x \wedge y)) \in \theta$, thus $(f(x), f(y)) \in \theta$.

It remains to show that f is not a polynomial. Clearly, any unary polynomial g on a distributive lattice L must be either identity or of the form $g(x) = a \vee x$ or $g(x) = b \wedge x$ or $g(x) = (a \vee x) \wedge b$ for suitable $a, b \in L$, $a \leq b$.

Since the ideal I is proper, f is not an identity. It is easy to see that I is the set of all fixed points of the function f. The function f cannot be of the form $b \wedge x$ or $(a \vee x) \wedge b$, because these functions have the largest fixed points, while f has not. Finally, f cannot be of the form $a \vee x$, because the set of all fixed points of this function is $\uparrow a$, which is an ideal only in the case $\uparrow a = L$, hence $\uparrow a \neq I$. □

2.3. Corollary. *If a distributive lattice contains a proper relatively complete filter without a smallest element, then it is not affine complete.* □

2.4. Lemma. *Let $f : L \longrightarrow L$ be a compatible function on a distributive lattice L. Let $x \in L$. If there exists $y \in L$ such that $x \leq y$ and $x \leq f(y)$, then the set $\uparrow x$ is closed under f. Dually, if $x \geq y$ and $x \geq f(y)$ for some $y \in L$, then the set $\downarrow x$ is closed under f.*

Proof. Let $y \in L$ be such that $x \leq y$ and $x \leq f(y)$. For a contradiction, suppose that $z \in \uparrow x$ and $f(z) \notin \uparrow x$. Then there is a prime ideal I such that $x \notin I$ and $f(z) \in I$. Let θ be the congruence on L whose equivalence classes are I and $L \setminus I$. Then $(y, z) \in \theta$ and $(f(y), f(z)) \notin \theta$, which contradicts the compatibility of f. □

2.5. Corollary. *If the set $\downarrow x$ or $\uparrow x$ contains a fixed point of f, then it is closed under f.* □

2.6. Lemma. *Let $f : L \longrightarrow L$ be a compatible function on a distributive lattice L. Suppose that L does not contain a proper Boolean interval. Then*

(i) $f \circ f = f$;
(ii) *the set of all fixed points of f is convex;*
(iii) *the set $\downarrow f(L) = \bigcup_{x \in L} \downarrow f(x)$ is a relatively complete ideal in L.*

Proof. (i) Let $x \in L$. The interval $M = [x \wedge f(x), x \vee f(x)]$ is closed under f because it is an intersection of the sets $\uparrow(x \wedge f(x))$ and $\downarrow(x \vee f(x))$, which are closed under f by 2.4. The restriction $g = f \restriction M$ is a compatible function on the lattice M. Indeed, any congruence on M can be extended to a congruence on L, so f must preserve it. By 1.1, the lattice M is affine complete, hence $g(y) = (a \vee y) \wedge b$ for suitable $a, b \in M$, $a \leq b$. It is easy to verify that $g(g(y)) = g(y)$ holds for every $y \in M$. Since $x \in M$ and $f(x) \in M$, we obtain that $f(f(x)) = g(g(x)) = g(x) = f(x)$.

(ii) Let a and b be fixed points of f, $a < b$. We have to prove that the whole interval $[a, b]$ consists of fixed points. By 2.5, the sets $\uparrow a$ and $\downarrow b$ are closed under f, therefore also $[a, b] = \uparrow a \cap \downarrow b$ is closed under f. Similarly as in (i), the restriction $g = f \restriction [a, b]$ must be a polynomial. Hence, $g(y) = (c \vee y) \wedge d$ for suitable $c, d \in [a, b]$, $c \leq d$. Since $g(a) = a$, $g(b) = b$, we obtain that $a = c$ and $b = d$, which means that g is an identity. Thus, $f(x) = x$ for any $x \in [a, b]$.

(iii) First we show that $\max(\downarrow f(L) \cap \downarrow x) = x \wedge f(x)$ holds for every $x \in L$. Clearly, $x \wedge f(x) \in \downarrow f(L) \cap \downarrow x$. Let y be an arbitrary element of $\downarrow f(L) \cap \downarrow x$. We need to show that $y \leq x \wedge f(x)$. Since $y \in \downarrow f(L)$, the set $\uparrow y$ contains an element of $f(L)$, i. e. fixed point of f. By 2.5, the set $\uparrow y$ is closed under f, hence $y \leq x$ implies that $y \leq f(x)$ and therefore $y \leq x \wedge f(x)$.

It remains to prove that the set $\downarrow f(L)$ is an ideal, i. e. that it is closed under joins. But it is easy to see that if $a, b \in \downarrow f(L)$, then $a, b \leq \max(\downarrow f(L) \cap \downarrow (a \vee b)) \leq a \vee b$, hence $\max(\downarrow f(L) \cap \downarrow (a \vee b)) = a \vee b$. This implies that $a \vee b \in \downarrow f(L)$. □

2.7. Theorem. *A distributive lattice L is affine complete if and only if the following conditions are satisfied:*

(i) *L does not contain a proper Boolean interval;*
(ii) *L does not contain a proper relatively complete ideal without a largest element;*
(iii) *L does not contain a proper relatively complete filter without a smallest element.*

Proof. If some of the above conditions is not fulfilled, then L is not affine complete by 1.3, 2.2 or 2.3. Suppose now that L satisfies (i), (ii) and (iii). We have to prove that any compatible function is a polynomial. In view of 1.2, it suffices to consider unary functions.

Let $f : L \longrightarrow L$ be a compatible function. If the set $f(L)$ does not have a largest element, then $\downarrow f(L)$ is a relatively complete ideal without a largest element and therefore $\downarrow f(L) = L$. Similarly, if $f(L)$ does not have a smallest element, then $\uparrow f(L) = L$. We distinguish four cases.

Suppose that $f(L)$ has neither a largest nor a smallest element. Then $\uparrow f(L) = L = \downarrow f(L)$. For every $x \in L$ there are $a, b \in f(L)$ with $a \leq x \leq b$. By 2.6, $f(L)$ is the set of all fixed points of f, which is convex. That is why $x \in f(L)$, hence x is also a fixed point. We have shown that f is an identity, which is a polynomial.

Suppose that $f(L)$ has a smallest element u and does not have a largest element. Then $\downarrow f(L) = L$ and the convexity of $f(L)$ implies that $f(L) = \uparrow u$. Let $x \in L$. By 2.5 the sets $\uparrow x$ and $\downarrow (x \vee u)$ are closed under f. (They contain $x \vee u \in f(L)$.) Thus, $f(x) \in \uparrow x \cap \downarrow (x \vee u)$. Further, $f(x) \in \uparrow u = f(L)$, hence $f(x) \in \uparrow x \cap \uparrow u \cap \downarrow (x \vee u) = \{x \vee u\}$. We infer that for every $x \in L$, $f(x) = x \vee u$ and therefore f is a polynomial.

Analogously, if $f(L)$ has a largest element v and no smallest element, then $f(x) = x \wedge v$ holds for every $x \in L$.

The remaining case is that $f(L)$ has a smallest element u and a largest element v. From the convexity of $f(L)$ we infer that $f(L)$ is the interval $[u, v]$. For any $x \in L$ the sets $\downarrow(x \vee u)$ and $\uparrow(x \wedge v)$ are closed under f. (They contain the fixed points u and v, respectively.) Thus, $f(x) \in \downarrow(x \vee u)$ and $f(x) \in \uparrow(x \wedge v)$. Further, $f(x) \in \uparrow u$ and $f(x) \in \downarrow v$. We obtain that $f(x) \in \downarrow((x \vee u) \wedge v)$, $f(x) \in \uparrow((x \wedge v) \vee u)$ and therefore $f(x) = (x \wedge v) \vee u$. This completes the proof. $\square$

Now we present some examples. First, the direct product $R \times R$ of the real line with itself is not affine complete. It contains the proper relatively complete ideal

$$I = \{(x, y) \in R \times R \mid x \leq 0\}$$

without a largest element. The theorem 2.2 shows how to construct a compatible function which is not a polynomial.

On the other hand, the sublattice L of $R \times R$ given by the formula

$$L = \{(x, y) \in R \times R \mid x - 1 \leq y \leq x + 1\}$$

is affine complete. Indeed, it is not hard to see that any proper ideal of L has an upper bound in L. And, if b is an upper bound of an ideal I, then $\max(I \cap \downarrow b) = \max I$.

The above example suggests a question if the condition 2.7(ii) could be replaced by a stronger condition

(ii') Every proper ideal of L is bounded.

The negative answer to this question is demonstrated by the following example. Let

$$L = \{(x, y) \in R \times R \mid 0 \geq x \geq y \geq -1\} \setminus \{(0, 0)\}.$$

The lattice L contains the unbounded proper ideal

$$I = \{(x, y) \in L \mid x < 0\}.$$

Nevertheless, the lattice L is affine complete. In fact, I is the only unbounded proper ideal and it is not relatively complete.

Our final remark concerns nondistributive affine complete lattices. There seems to be no example of such a lattice. There are only a few negative results. By [1, p. 100], if a lattice contains a proper subdirectly irreducible interval, then it is not affine complete. Thus, natural questions arises, whether there exist affine complete nondistributive lattices.

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