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CONGRUENCE PRESERVING FUNCTIONS ON MEDIAN ALGEBRAS

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Abstract. Affine completeness and local affine completeness for median algebras has been characterized by H. J. Bandelt [1] and M. Ploščica [10]. In this paper we take a more general approach and try to describe the clone of all congruence preserving functions on median algebras which are not necessarily (locally) affine complete. We define several types of congruence preserving functions and conjecture that every congruence preserving function is a composition of functions of these types. We are able to confirm this conjecture in the unary case.

An analogous problem has been recently solved for distributive lattices (see [11]), so we try to apply similar methods.

1. Introduction

A finitary function $f: A^n \to A$ on an algebra A is called *congruence preserving* (or *compatible*) if, for any congruence θ of A, $(a_i, b_i) \in \theta$, $i = 1, \dots, n$, implies that

$$(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \in \theta.$$

A polynomial function (or simply *a polynomial*) of A is any function that can be obtained by composition of the basic operations of A, the projections and the constant functions. *A local polynomial* of A is any function which can be interpolated by polynomials on all finite subsets of its domain.

Obviously, (local) polynomials are compatible functions. An algebra is called *(locally) affine complete* if the converse holds: every compatible function is a (local) polynomial.

The (local) affine completeness has been investigated for various classes of algebras, including median algebras. (See [1] and [10].) An overview of results until the 1990s can be found in the monograph by K. Kaarli and A.F. Pixley [7].

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We believe that the concept of compatibility is worth investigating also in algebras that are not affine complete, because it is so closely connected with the fundamental algebraic notions of a congruence and a polynomial.

The project of describing the compatible functions of an algebra in a given (favourite) variety has been explicitly formulated in [8] and [11]. Since compatible functions form a clone, our wish is to express every compatible function as a composition of functions from some nice and well understood family. (We recall that a *clone* on a set A is a set of finitary functions containing all projections and closed under composition.)

Problem 1.1. *Given an algebra A, find nice generating sets for the clone $C(A)$ of all compatible functions of A and the clone $LP(A)$ of all local polynomial functions of A.*

A description of this kind is obvious for the clone of all polynomial functions, where the generating set consists of the basic operations and constants. Thus the answer is known if the algebra A is affine complete. If A is not affine complete, we seek for an extension of this generating set by some typical compatible non-polynomial functions.

The above problem has been successfully solved for distributive lattices in [11]. In the present paper we obtain a partial solution for median algebras. Since these algebras are very close to distributive lattices, one could expect a similar description of compatible functions.

Now we recall the definition and some terminology for median algebras. For more information see also Bandelt and Hedlíková [3], van de Vel [12] and Isbell [6]. Note that the median algebras we are dealing with in this note are called symmetric media in [6].

On any distributive lattice D we define the median polynomial by

$$m(x,y,z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z).$$

This operation turns D into a median algebra. In general, *a median algebra* is an algebra endowed with a single ternary operation which can be embedded in (D, m) for some distributive lattice D. Median algebras form a variety (equational class). This variety can be defined, for example, by the following identities (see [6]):

$$m(a,a,b) = a, \quad m(a,c,d) = m(a, m(a,c,d), m(b,c,d)),$$
$$m(a,b,c) = m(a,c,b) = m(b,a,c) = m(b,c,a) = m(c,a,b) = m(c,b,a).$$

(See [3] for other systems of axioms.)

Let M be a median algebra. For elements $a,b,c \in M$ we say that c is *between* a and b if $c = m(a,b,c)$. A subset C of M is *convex* if $a,b \in C$ and $x \in M$ imply $m(a,b,x) \in C$. Equivalently, C is convex if, for every $a,b \in C$, C contains all elements of M that are between a and b. It is easy to see that the intersection of any number of convex sets is convex. Hence, for any subset

A of M there is a smallest convex set containing A. We denote it by Conv A. A set of the form Conv$\{a, b\}$ is called *a segment* and will be denoted by $[a, b]$. (Of course, the segments $[a, b]$ and $[b, a]$ coincide.) It is not difficult to show that the segment Conv$\{a, b\}$ consists of all elements that are between a and b. Any segment (or, more generally, any convex set) is a subalgebra of M. On every segment $[a, b]$ we can define lattice operations by

$$x \wedge y = m(a, x, y), \quad x \vee y = m(b, x, y).$$

This turns $[a, b]$ into a bounded distributive lattice (with a the smallest and b the greatest element). Moreover, m on $[a, b]$ coincides with the lattice median operation. So, $[a, b]$ as a median algebra and $[a, b]$ as a lattice are polynomially equivalent, which means that they have the same polynomials, congruences and also compatible functions.

A segment is called *Boolean* if it is isomorphic to (B, m) for some Boolean lattice B.

A nonempty convex set is *prime* if its complement is also convex and nonempty. Any prime convex set C determines a congruence θ of M with the equivalence classes C and $M \setminus C$. Congruences of this form are called *split congruences*. The quotient M/θ is isomorphic to $\mathbf{2}$, the 2-element lattice with its median operation. Using the embedding of M into a distributive lattice D one can show that for every $x, y \in M$, $x \neq y$, there is a prime convex set P with $x \in P$, $y \notin P$. (Indeed, the intersection of M with any prime ideal of D is a prime convex set.) As a consequence, every median algebra is a subdirect power of $\mathbf{2}$. Another consequence is that every congruence on a median algebra is the intersection of split congruences. Hence, to prove the compatibility of a function on a median algebra, it suffices to check the condition for all split congruences.

2. Unary compatible functions

Two types of non-polynomial compatible functions have been introduced in [1] and [10]. Let (M, m) be a median algebra.

Lemma 2.1 (see [1], page 26)**.** *For any Boolean segment $Z = [a, b]$, the function $c^Z : M \to M$ defined by letting $c^Z(x)$ to be the complement of $m(x, a, b)$ in Z is compatible.*

The functions of the above type will be called *Boolean segment complementations.* If $a \neq b$, then such a function is not a local polynomial. The lattice analogue of this type of compatible functions has been discussed in [5].

A subset C of M is called *a Čebyšev set*, if for every $x \in M$ there exists $x_C \in C$ such that $m(x, x_C, y) = x_C$ for every $y \in C$ (i.e. x_C is between x and every element of C).

Every Čebyšev set is convex, but not vice versa. If $C = \text{Conv}\, A$ for a finite set $A \subseteq M$, then C is Čebyšev and the function $x \mapsto x_C$ is a polynomial. (See

[10], Lemmas 2.3 and 2.5.) Such a function will be called *a projection on a Čebyšev set.*

Lemma 2.2. *For any Čebyšev set C, the function $M \to M$ defined by $x \mapsto x_C$ is a local polynomial, and hence compatible.*

Proof. For every finite set $X \subseteq M$ consider the convex hull $B = \mathrm{Conv}\{x_C \mid x \in X\}$. Clearly, $B \subseteq C$, so $x_B = x_C$ for every $x \in X$. Thus, the projection on B is a polynomial, which interpolates the projection on C. $\square$

If a Čebyšev set C is not a convex hull of a finite set, then the projection on C is not a polynomial ([10], Lemma 4.4). As an example, consider the real plane $\mathbb{R} \times \mathbb{R}$ with the median operation derived from the pointwise lattice operations. The set $C = \{(x, y) \in \mathbb{R}^2 \mid x \leq 0\}$ is a Čebyšev set, which is not a convex hull of a finite set. The projection on C is a local polynomial, which is not a polynomial.

Projections on Čebyšev sets also have their analogue in distributive lattices, namely projections on almost principal ideals and filters (see [9]).

In this section we prove that that every unary compatible function on a median algebra is a composition of polynomials, Boolean segment complementations and projections on Čebyšev sets. In the sequel, let $f : M \to M$ be a compatible function.

Lemma 2.3. *f is an endomorphism of M and $f^3 = f$.*

Proof. Let $x, y, z \in M$ and assume that $f(m(x, y, z)) \neq m(f(x), f(y), f(z))$. Then there is a prime convex set C such that $m(f(x), f(y), f(z)) \in C$ and $f(m(x, y, z)) \notin C$. Let θ_C be the split congruence corresponding to C. The convexity of $M \setminus C$ implies that at least two of $f(x)$, $f(y)$, $f(z)$ must belong to C. Without loss of generality, $f(x) \in C$, $f(y) \in C$. The compatibility of f implies that $(x, m(x, y, z)) \notin \theta_C$, $(y, m(x, y, z)) \notin \theta_C$. Since θ_C has only two equivalence classes, we obtain that $(x, y) \in \theta_C$. If $x, y \in C$, then also $m(x, y, z) \in C$. If $x, y \in M \setminus C$, then also $m(x, y, z) \in M \setminus C$. In both cases $(x, m(x, y, z)) \in \theta_C$, a contradiction.

To prove the second statement, assume that $f^3(x) \neq f(x)$ for some $x \in M$. Then $f(x) \in C$ and $f^3(x) \notin C$ for some prime convex set C. If $x \in C$, then $(x, f(x)) \in \theta_C$ and the compatibility of f implies that $(f(x), f^2(x)) \in \theta_C$ and $(f^2(x), f^3(x)) \in \theta_C$, so $(f(x), f^3(x)) \in \theta_C$, a contradiction. If $f^2(x) \in C$, then $(f(x), f^2(x)) \in \theta_C$ and $(f^2(x), f^3(x)) \notin \theta_C$, a contradiction. The remaining case is $x \notin C$, $f^2(x) \notin C$. Then $(x, f^2(x)) \in \theta_C$, but $(f(x), f^3(x)) \notin \theta_C$, another contradiction with the compatibility of f. $\square$

Lemma 2.4. *For every $a \in M$ the segment $Z = [f(a), f^2(a)]$ is Boolean. The complement of $x \in Z$ is $y = m(f(x), f(a), f^2(a))$.*

Proof. Clearly, $y \in Z$. We claim that $f(a) = m(f(a), x, y)$ and $f^2(a) = m(f^2(a), x, y)$. Suppose for contradiction that $f(a) \neq m(f(a), x, y)$. Then there is a prime convex set C with $f(a) \in C$, $m(f(a), x, y) \notin C$. This is only possible if $x, y \notin C$. From $f(a) \in C$ and $y = m(f(x), f(a), f^2(a)) \notin C$ we deduce that $f^2(a) \notin C$ and $f(x) \notin C$. So $(f(a), f^2(a)) \notin \theta_C$ and the compatibility of f implies that $(a, f(a)) \notin \theta_C$, hence $a \notin C$. Now we have, $(x, a) \in \theta_C$ and $(f(x), f(a)) \notin \theta_C$, which contradicts the compatibility of f. The proof of $f^2(a) = m(f^2(a), x, y)$ is the same, with the roles of $f(a)$ and $f^2(a)$ interchanged (using the fact that $f^3(a) = f(a)$). □

Let us fix $a \in M$ and define

$$I = \{f(x) \mid x \in M,\ m(f(x), f(a), f^2(a)) = f(a)\},$$
$$J = \{f(x) \mid x \in M,\ m(f(x), f(a), f^2(a)) = f^2(a)\}.$$

Lemma 2.5. *I and J are Čebyšev sets. For every $x \in M$,*

$$x_I = m(f(x), f^2(x), f(a)), \qquad x_J = m(f(x), f^2(x), f^2(a)).$$

Proof. Let $x \in M$. Denote $y = m(f(x), f^2(x), f(a)) = f(m(x, f(x), a))$. We claim that $y \in I$. For contradiction, let $m(y, f(a), f^2(a)) \neq f(a)$. Then $f(a) \in C$ and $m(y, f(a), f^2(a)) \notin C$ for some prime convex set C. Since C is convex, we have $y, f^2(a) \notin C$. For the same reason, $y \notin C$ implies $f(x), f^2(x) \notin C$. Since $f(a) \in C$ and $f^2(a) \notin C$, the compatibility of f yields that $a \notin C$. Now we have $(a, f(x)) \in \theta_C$, while $(f(a), f^2(x)) \notin \theta_C$, which contradicts the compatibility of f.

Thus, $y \in I$. Now we need to show that $m(x, f(t), y) = y$ for every $f(t) \in I$. Suppose that this equality is not true and choose a prime convex set C with $y \in C$, $m(x, f(t), y) \notin C$. Then $x, f(t) \notin C$. We consider the following two cases.

I. If $f(x) \notin C$, then $(x, f(x)) \in \theta_C$ and the compatibility of f yields that $(f(x), f^2(x)) \in \theta_C$, hence $f^2(x) \notin C$. But then $y = m(f(x), f^2(x), f(a)) \notin C$, a contradiction.

II. Suppose that $f(x) \in C$. Then $(f(t), f(x)) \notin \theta_C$ implies $(x, t) \notin \theta_C$. Since also $(x, f(x)) \notin \theta_C$ and θ_C has only two equivalence classes, we have $(f(x), t) \in \theta_C$ and $(f^2(x), f(t)) \in \theta_C$, hence $f^2(x) \notin C$. From $m(f(x), f^2(x), f(a)) = y \in C$ we deduce that $f(a) \in C$. So, $(f(a), f(x)) \in \theta_C$, hence $(f^2(a), f^2(x)) \in \theta_C$, and therefore $f^2(a) \notin C$. But then $f(t) \in I$ implies that $m(f(t), f(a), f^2(a)) = f(a) \notin C$, a contradiction.

The proof for J is the same, with $f(a)$ playing the role of a. (Notice that $f^3(a) = f(a)$.) □

We can illustrate the above Lemma by the following example. Let L be the direct product $\mathbb{Z} \times \mathbf{2}$ of the chain of integers and the 2-element chain $\{0, 1\}$ (the "infinite ladder"). Consider the median operation on L derived from the pointwise lattice operations. Let $f : L \to L$ be given by $f(x, i) = (x, 1 - i)$. Let a be any element of the form $(x, 0)$. Then $I = \{(x, i) \in L \mid i = 1\}$ and $J = \{(x, i) \in L \mid i = 0\}$ are the Čebyšev sets defined in Lemma 2.5. (The choice of $a = (x, 1)$ would interchange I and J.) Also, notice that the Boolean segments discussed in Lemma 2.4 have the form $[(x, 0), (x, 1)]$.

Theorem 2.6. *For every $x \in M$,*

$$f(x) = m(c^Z(x), x_I, x_J),$$

where Z denotes the Boolean segment $[f(a), f^2(a)]$.

Proof. Let $x \in M$ and assume that $f(x) \neq m(c^Z(x), x_I, x_J)$. Let C be a prime convex set such that $f(x) \in C$ and $m(c^Z(x), x_I, x_J) \notin C$. We distinguish the following two cases.

I. Let $x \in C$. Then $(x, f(x)) \in \theta_C$ and the compatibility of f implies that $(f(x), f^2(x)) \in \theta_C$, so $f^2(x) \in C$. By 2.5, $x_I, x_J \in C$, which is a contradiction with $m(c^Z(x), x_I, x_J) \notin C$.

II. Let $x \notin C$. If $f^2(x) \in C$, then we get the same contradiction as above. So, let $f^2(x) \notin C$. We consider the following two cases for a.

If $a \in C$, then $(a, f(x)) \in \theta_C$, so $(f(a), f^2(x)) \in \theta_C$, $(f^2(a), f(x)) \in \theta_C$. Hence, $f(a) \notin C$, $f^2(a) \in C$. Consequently, $x_I \notin C$, $x_J \in C$. Further, $y = m(x, f(a), f^2(a)) \notin C$, so $(x, y) \in \theta_C$, $(f(x), f(y)) \in \theta_C$ and therefore $f(y) \in C$. By 2.4, $c^Z(x) = m(f(y), f(a), f^2(a)) \in C$. Then $m(c^Z(x), x_I, x_J) \in C$, a contradiction.

If $a \notin C$, then $(a, x) \in \theta_C$ and we obtain that $(f(a), f(x)) \in \theta_C$ and also $(f^2(a), f^2(x)) \in \theta_C$, so $f(a) \in C$, $f^2(a) \notin C$. The argument leading to a contradiction is the same as above. $\square$

As a consequence, we obtain our main result.

Theorem 2.7. *Every unary compatible function on a median algebra is a composition of polynomials, Boolean segment complementations and projections on Čebyšev sets.*

3. Some binary compatible functions

In order to describe two new types of congruence preserving functions, it is convenient to consider a median algebra M embedded in a distributive lattice D with a smallest element 0, such that

(i) M is a lower subset of D ($x \leq y \in M$ implies $x \in M$);
(ii) M is a $\wedge$-subsemilattice of D;
(iii) every element of D is a join of finitely many elements from M.

Such an embedding is always possible (see [2]), and moreover, the element $0 \in M$ can be chosen arbitrarily.

The embedding of M into D allows us to use the lattice operations in D in order to describe the compatible functions in M.

A Čebyšev ideal in M is a Čebyšev set, which is also an ideal in the lattice D. (Notice that this concept depends on the choice of D.)

Lemma 3.1. *For every Čebyšev ideal P, the function $f: M^2 \to M$ defined by $f(x,y) = m(x,y,x_P \vee y_P)$ is a local polynomial.*

Proof. Let $X \subseteq M$ be finite. Since P is an ideal, there exists $t \in P$ such that $x_P \leq t$ for every $x \in X$. The function $p(x,y) = m(x,y,t)$ is a polynomial and for $x, y \in X$ we have $x_P \vee y_P = p(x_P, y_P)$. By Lemma 2.2, the assignment $x \mapsto x_P$ is a local polynomial. It follows that f is a composition of polynomials and local polynomials, hence f itself is a local polynomial. □

The above lemma describes a new type of compatible functions. Since the restriction of this function to P is the lattice-theoretical join, we refer to functions of this type as *local joins.*

As an example, consider the chain $\mathbb{R}$ of real numbers with the natural median operation. Let 0 be the usual real zero. (Any other element would serve our purpose equally well.) The lattice D in this case is the product of the real intervals $[0,\infty)$ and $[0,-\infty)$, with the inverse ordering on $[0,-\infty)$, so that 0 is the smallest element there. The natural embedding $\mathbb{R} \to D$ maps $x \in \mathbb{R}$ into $(x,0)$ if x is positive, and into $(0,x)$ if x is negative. The set $P = \{(x,0) \mid x \geq 0\}$ is a Čebyšev ideal and it is easy to check that $m(x,y,x_P \vee y_P) = x \vee y$ for every $x, y \in \mathbb{R}$. (Here $\vee$ is the usual join in the natural ordering of $\mathbb{R}$, which on P coincides with the join in D.)

Now we introduce our last type of compatible functions.

A generalized Boolean segment is a pair (P,Q) of Čebyšev sets such that the segment $[x_P, x_Q]$ is Boolean for every $x \in M$.

Lemma 3.2. *Let $Z = [a,b]$ be a Boolean segment in M. Let $x \in Z$ and let x' denote the complement of x in Z. Let C be a prime convex set such that $a \in C$, $b \notin C$. Then $x \in C$ if and only if $x' \notin C$.*

Proof. Let $x \in C$. Since $m(x,x',b) = b \notin C$, we have $x' \notin C$. Similarly, if $x \notin C$, then $m(x,x',a) = a \in C$ implies that $x' \in C$. □

Lemma 3.3. *For every generalized Boolean segment (P,Q), the function $f: M^2 \to M$ defined by*

$$f(x,y) = c^{[x_P,x_Q]}(y),$$

where c^Z is the function defined in Lemma 2.1, is compatible.

Proof. Let θ be a split congruence on M with the congruence classes C and $M \setminus C$. Let $(a,b),(c,d) \in \theta$. Since the projections on P and Q are compatible, we have $(a_P,b_P),(a_Q,b_Q) \in \theta$. Then also $(m(c,a_P,a_Q), m(d,b_P,b_Q)) \in \theta$. Now, if $a_P, a_Q \in C$, then also $b_P, b_Q \in C$ and consequently $f(a,c), f(b,d) \in C$, so $(f(a,c), f(b,d)) \in \theta$. If $a_P, a_Q \notin C$, the same argument applies (using $M \setminus C$ instead of C). Finally, let $a_P \in C$, $a_Q \notin C$. (The case $a_P \notin C$, $a_Q \in C$ is similar.) Then 3.2 implies that $f(a,c) \in C$ iff $m(c,a_P,a_Q) \notin C$ and $f(b,d) \in C$ iff $m(d,b_P,b_Q) \notin C$. Since $(m(c,a_P,a_Q), m(d,b_P,b_Q)) \in \theta$, we obtain that $f(a,c) \in C$ iff $f(b,d) \in C$, so $(f(a,c), f(b,d)) \in \theta$. □

The compatible functions described in the above lemma will be called generalized Boolean segment complementations. Notice, that this type of functions is a common generalization of projections on Čebyšev sets and Boolean segment complementations. Indeed, if $P = Q$ then $f(x, y) = x_P$, and if $P = \{a\}$, $Q = \{b\}$, then $f(x, y) = c^{[a,b]}(y)$.

As an example, let M be the lattice of all finite subsets of an infinite set S, ordered by set inclusion, and regarded as a median algebra. It is easy to see that every segment of M is Boolean. Consider the Čebyšev sets $P = \{\emptyset\}$, $Q = M$. Then (P, Q) is a generalized Boolean segment and the induced function is $f(X, Y) = X \setminus Y$ (the set-theoretical difference). Indeed, $X_P = \emptyset$, $X_Q = X$, $m(X_P, X_Q, Y) = X \cap Y$, so $f(X, Y)$ is the complement of $X \cap Y$ in the segment (interval) $[\emptyset, X]$.

Let us remark that the lattice-theoretical analogue of generalized Boolean complementations has been studied in [11]. The results on compatible functions on distributive lattices suggest the following conjectures.

Conjecture 3.4. *Every compatible function on a median algebra is a composition of polynomials, local joins, and generalized Boolean complementations.*

Conjecture 3.5. *Every local polynomial on a median algebra is a composition of polynomials, projections on Čebyšev sets and local joins.*

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