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Relative separation in distributive congruence lattices

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Abstract. In [5] we defined separable sets in algebraic lattices and showed a close connection between the types of non-separable sets in congruence lattices of algebras in a finitely generated congruence distributive variety $\mathcal{V}$ and the structure of subdirectly irreducible algebras in $\mathcal{V}$. Now we generalize these results using the concept of relatively separable sets (with respect to subsets) and apply them to some lattice varieties.

1. Introduction

Let $\mathcal{V}$ be a finitely generated and congruence distributive variety. Let $\mathrm{Con}(\mathcal{V})$ denote the class of all lattices isomorphic to $\mathrm{Con}\,A$ (the congruence lattice of an algebra A), for some $A \in \mathcal{V}$. Further, let $\mathrm{SI}(\mathcal{V})$ denote the class of all subdirectly irreducible members of $\mathcal{V}$. The aim of this paper (and its predecessor [5]) is to describe the class $\mathrm{Con}(\mathcal{V})$, using the knowledge of $\mathrm{SI}(\mathcal{V})$. One connection is obvious: for any completely meet-irreducible element $x \in L \in \mathrm{Con}(\mathcal{V})$, the interval $\uparrow x = \{y \in L \mid y \geq x\}$ must be isomorphic to $\mathrm{Con}\,A$ for some $A \in \mathrm{SI}(\mathcal{V})$. In [5], we introduced a new condition satisfied by all $L \in \mathrm{Con}(\mathcal{V})$. It turns out that the congruence lattices of subalgebras of subdirectly irreducible algebras play an important role. In this paper we develop further the ideas from [5] and provide even deeper insight into $\mathrm{Con}(\mathcal{V})$. However, a complete description of $\mathrm{Con}(\mathcal{V})$ remains a much more difficult problem.

Our basic reference books are [1] and [3]. All the unexplained concepts and unreferenced facts used in this paper can be found there.

If B is a subalgebra of an algebra A and $\alpha \in \mathrm{Con}\,A$, then $\alpha\restriction B = \alpha \cap B^2$ denotes the restriction of α to B. If $f\colon X \to Y$ is a mapping and $Z \subseteq X$, then $f\restriction Z$ denotes the restriction of f to Z. Furthermore, $\mathrm{Ker}(f)$ (the kernel of f) is the binary relation on X defined by $(x, y) \in \mathrm{Ker}(f)$ iff $f(x) = f(y)$.

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2. The relative separation

Let L be an algebraic lattice. An element $a \in L$ is called *strictly meet-irreducible* (or completely meet-irreducible) iff $a = \bigwedge X$ implies that $a \in X$, for every subset X of L. Note that the greatest element of L is not strictly meet-irreducible. Let $\mathrm{M}(L)$ denote the set of all strictly meet-irreducible elements of L. Recall that $x = \bigwedge\{a \in \mathrm{M}(L) \mid x \leq a\}$, for every $x \in L$. Thus, every L contains many strictly meet-irreducible elements. Every $x \in \mathrm{M}(L)$ has a unique upper cover, which we denote by x^+. Hence, $x^+ = \min\{y \in L \mid y > x\}$.

If L is distributive and $x_1 \wedge \cdots \wedge x_n \leq x \in \mathrm{M}(L)$ then $x_i \leq x$ for some i. If L is distributive and finite, then $\mathrm{M}(L)$ characterizes L up to isomorphism. If $L = \mathrm{Con}\, A$ then $\alpha \in \mathrm{M}(L)$ iff the quotient algebra A/α is subdirectly irreducible.

The following definition introduces the main concept of this paper.

Definition 2.1. *Let L be an algebraic lattice and $P \subseteq Q \subseteq \mathrm{M}(L)$. We say that Q is separable over P, if there exists a family $\{x_q \mid q \in Q\} \subseteq L$ such that*

(1) *$x_q \not\leq q$ for every $q \in Q$;*
(2) *for every $y \in \mathrm{M}(L)$ with $y \not\geq \bigwedge\{x_q \mid q \in Q\}$, there exists $q \in P$ such that $y^+ \geq x_q$.*

A set $Q \subseteq \mathrm{M}(L)$ is separable in the sense of [5], if it is separable over the empty set. In this case the condition (2) takes the form $\bigwedge\{x_q \mid q \in Q\} = 0$. Thus, we have generalized the concept of separability.

Our definition is easier to understand when we consider the following topological representation. Let L be a distributive algebraic lattice. A set $X \subseteq \mathrm{M}(L)$ is defined to be closed if $X = \mathrm{M}(L) \cap \uparrow x$, for some $x \in L$. It is easy to see (cf. [4]) that this defines a topology on $\mathrm{M}(L)$ and L is isomorphic to $\mathcal{O}(\mathrm{M}(L))$ (the lattice of open subsets of $\mathrm{M}(L)$).

If Z is a topological space and $Y \subseteq Z$, then $\overline{Y}$ denotes the closure of Y. For every open set $A \subseteq Z$, we define $A^\circ = \{x \in A \mid A \cap (\overline{\{x\}} \setminus \{x\}) \neq \emptyset\}$. It is not difficult to check that the following holds.

Lemma 2.2. *Let L be a distributive algebraic lattice, $P \subseteq Q \subseteq \mathrm{M}(L)$. The following conditions are equivalent.*

(1) *Q is separable over P;*
(2) *there are open sets $A_q \subseteq \mathrm{M}(L)$ ($q \in Q$) such that $q \in A_q$, for every q, and*

$$\bigcap_{q \in Q \setminus P} A_q \cap \bigcap_{q \in P} A_q^\circ = \emptyset.$$

It is easy to see that if $Q \subseteq \mathrm{M}(L)$ is separable over P and $P \subseteq P_1 \subseteq Q \subseteq Q_1$, then Q_1 is separable over P_1.

Now we will prove some general results for finitely generated congruence distributive varieties. Recall that for any such variety $\mathcal{V}$, the class $\mathrm{SI}(\mathcal{V})$ contains only finite algebras.

Lemma 2.3. *Suppose that $\mathcal{V}$ is a finitely generated congruence distributive variety, $A \in \mathcal{V}$, $\alpha_1, \dots, \alpha_n \in \mathrm{M}(\mathrm{Con}\, A)$, $n \in \omega$. Denote $\alpha = \bigcap\{\alpha_i \mid i = 1, \dots, n\}$. Then there exists a finite subalgebra B of A such that*

(1) *for every $a \in A$, there is $b \in B$ with $(a, b) \in \alpha$;*
(2) *for every $i = 1, \dots, n$, the algebra $B/(\alpha_i \restriction B)$ is isomorphic to A/α_i;*
(3) *for every $i = 1, \dots, n$, there is $\beta_i \in \mathrm{Con}\, A$ such that $\beta_i \restriction B \nsubseteq \alpha_i \restriction B$ and, for every $\beta \in \mathrm{Con}\, A$, either $\beta_i \subseteq \beta$ or $\beta \restriction B \subseteq \alpha_i$.*

Proof. For every $i = 1, \dots, n$, the algebra A/α_i is subdirectly irreducible, and hence finite. Therefore, all α_i have finitely many congruence classes. Consequently, α has finitely many congruence classes, so it is possible to choose a finite set $B_0 \subseteq A$ such that for every $a \in A$ there is $b \in B_0$ with $(a, b) \in \alpha$. Let B be the subalgebra of A generated by B_0. Obviously, B is finite and satisfies (1).

Clearly, (2) is a direct consequence of (1).

To prove (3), let $i \in \{1, \dots, n\}$. By (2), the algebra $B/(\alpha_i \restriction B)$ is subdirectly irreducible, hence $\alpha_i \restriction B \in \mathrm{M}(\mathrm{Con}\, B)$. Since $\mathrm{Con}\, B$ is a finite distributive lattice, there is the smallest $\gamma_i \in \mathrm{Con}\, B$ with $\gamma_i \nsubseteq (\alpha_i \restriction B)$. Let β_i be the congruence on A generated by γ_i. It is easy to see that (3) is satisfied. □

If $\alpha \in \mathrm{Con}\, A$, $x \in A$, then x_α denotes the congruence class of α containing x. It is well known that $\mathrm{Con}\, A/\alpha = \{\gamma/\alpha \mid \gamma \in \mathrm{Con}\, A,\ \gamma \geq \alpha\}$, where $(x_\alpha, y_\alpha) \in \gamma/\alpha$ iff $(x, y) \in \gamma$. For a subdirectly irreducible algebra S, let μ_S denote the smallest nonzero congruence on S.

Theorem 2.4. *Let the algebra A belong to a finitely generated congruence distributive variety $\mathcal{V}$. Let $\alpha_1, \dots, \alpha_n \in \mathrm{M}(\mathrm{Con}\, A)$, $\alpha = \alpha_1 \cap \dots \cap \alpha_n$. Suppose that $K \subseteq \{1, \dots, n\}$ is such that the set $\{\alpha_1, \dots, \alpha_n\}$ is not separable over $\{\alpha_i \mid i \in K\}$. Then there exist $S \in \mathrm{SI}(\mathcal{V})$, a subalgebra $T \leq S$ and a surjective homomorphism $t\colon T \to A/\alpha$ such that*

($*$) *for every $i \in K$ and every $(c, d) \in \mu_S \restriction T$, the pair $(t(c), t(d))$ belongs to α_i/α.*

Proof. Let $B \leq A$ and $\beta_1, \dots, \beta_n \in \mathrm{Con}\, A$ be the subalgebra and the congruences constructed in 2.3. Clearly, $\beta_i \nsubseteq \alpha_i$ for every $i = 1, 2, \dots, n$. By our non-separability assumption, there exists $\delta \in \mathrm{M}(\mathrm{Con}\, A)$ such that $\bigcap_{i=1}^{n} \beta_i \nsubseteq \delta$ and $\beta_i \nsubseteq \delta^+$, for every $i \in K$. We set $S = A/\delta$, $T = B/(\delta \restriction B)$. By 2.3, $\delta \restriction B \subseteq \alpha_i \restriction B$, for every i, hence $\delta \restriction B \subseteq \alpha \restriction B$. Thus, there is a natural homomorphism $t\colon T \to A/\alpha$ defined by $t(x_\delta) = x_\alpha$, for every $x \in B$. By (1) of 2.3, t is surjective. It remains to verify ($*$).

Clearly, $\mu_S = \delta^+/\delta$. If $(c,d) \in \mu_S \restriction T$, then $(c,d) = (x_\delta, y_\delta)$, for some $x, y \in B$ with $(x,y) \in \delta^+$. For every $i \in K$, we have $\beta_i \nsubseteq \delta^+$, which by (3) of 2.3 implies that $\delta^+ \restriction B \subseteq \alpha_i \restriction B$, hence $(t(c), t(d)) = (x_\alpha, y_\alpha) \in \alpha_i/\alpha$. □

For $K = \emptyset$ we obtain the following consequence.

Theorem 2.5. *Let A belong to a finitely generated congruence distributive variety $\mathcal{V}$. Let $\alpha_1, \dots, \alpha_n \in \mathrm{M}(\mathrm{Con}\, A)$, $\alpha = \alpha_1 \cap \dots \cap \alpha_n$. If the set $\{\alpha_1, \dots, \alpha_n\}$ is not separable, then A/α is a homomorphic image of a subalgebra of some $S \in \mathrm{SI}(\mathcal{V})$. Consequently, S has a subalgebra with at least n meet-irreducible congruences.*

The converse to 2.4 is true for infinite free algebras. Let $F_{\mathcal{V}}(X)$ denote the free algebra in $\mathcal{V}$ with X as the set of free generators.

Theorem 2.6. *Let $\mathcal{V}$ be a finitely generated congruence distributive variety. Let $F = F_{\mathcal{V}}(X)$, $|X| \geq \aleph_0$, $\alpha_1, \dots, \alpha_n \in \mathrm{M}(\mathrm{Con}\, F)$, $\alpha = \alpha_1 \cap \dots \cap \alpha_n$. Let $S \in \mathrm{SI}(\mathcal{V})$, $T \leq S$ and let $t\colon T \to F/\alpha$ be a surjective homomorphism. Let*

$$K = \{i \mid (t(c), t(d)) \in \alpha_i/\alpha, \text{ for every } (c,d) \in \mu_S \restriction T\}.$$

Then the set $\{\alpha_1, \dots, \alpha_n\}$ is not separable over $\{\alpha_i \mid i \in K\}$.

Proof. Let $\beta_1, \dots, \beta_n \in \mathrm{Con}\, F$, $\beta_i \nsubseteq \alpha_i$ for every i. We need to find $\delta \in \mathrm{M}(\mathrm{Con}\, A)$ such that $\beta_i \nsubseteq \delta$, for every $i = 1, \dots, n$, and $\beta_i \nsubseteq \delta^+$, for $i \in K$.

For every i we have $(x_i, y_i) \in \beta_i \setminus \alpha_i$. There is a finite set $Y \subseteq X$ such that all x_i and y_i belong to $\langle Y \rangle$ (the subalgebra of F generated by Y). Since S is finite, it is possible to choose a surjective map $h_0\colon X \to S$ such that $h_0(y) \in t^{-1}(y_\alpha)$, for every $y \in Y$. Since F is free, this map can be extended to a (surjective) homomorphism $h\colon F \to S$. Especially, $h(Y) \subseteq T$. We set $\delta = \mathrm{Ker}(h)$. Then F/δ is isomorphic to $S \in \mathrm{SI}(\mathcal{V})$, so we have $\delta \in \mathrm{M}(\mathrm{Con}\, F)$.

The restriction map $h_1 = h \restriction \langle Y \rangle$ is a homomorphism $\langle Y \rangle \to T$. The composition th_1 is a homomorphism $\langle Y \rangle \to F/\alpha$ which coincides with the natural projection $p\colon \langle Y \rangle \to F/\alpha$ ($p(x) = x_\alpha$). Indeed, the homomorphisms th_1 and p coincide on Y, and this set generates $\langle Y \rangle$. For every $i = 1, \dots, n$, we have $x_i, y_i \in \langle Y \rangle$ and $(x_i, y_i) \notin \alpha_i \supseteq \alpha$, hence $th_1(x_i) = (x_i)_\alpha \neq (y_i)_\alpha = th_1(y_i)$, which implies $h(x_i) = h_1(x_i) \neq h_1(y_i) = h(y_i)$, and therefore $(x_i, y_i) \notin \mathrm{Ker}(h) = \delta$. Since $(x_i, y_i) \in \beta_i$, we have obtained that $\beta_i \nsubseteq \delta$.

Finally, let $i \in K$. Since $(x_i, y_i) \notin \alpha_i$, we have $(th(x_i), th(y_i)) = ((x_i)_\alpha, (y_i)_\alpha) \notin \alpha_i/\alpha$. By our assumption this means that $(h(x_i), h(y_i)) \notin \mu_S \restriction T$. Since $h(x_i), h(y_i)$ belong to T, we have $(h(x_i), h(y_i)) \notin \mu_S$. Clearly, $(x,y) \in \delta^+$ iff $(h(x), h(y)) \in \mu_S$, hence $(x_i, y_i) \notin \delta^+$, which shows that $\beta_i \nsubseteq \delta^+$. □

In the next sections we will demonstrate how to apply the general theorems to concrete varieties.

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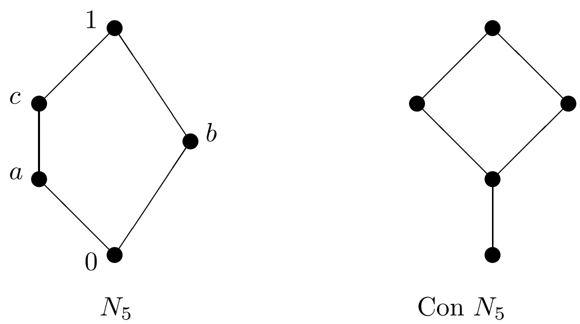

FIGURE 1

3. The variety $\mathcal{N}_5$

The results from the previous section can be used to obtain some information about congruence lattices of algebras in various congruence distributive varieties. In this paper we will deal with some finitely generated varieties of lattices. Our source of information for these varieties is [2] and [1] (App. F).

The variety $\mathcal{N}_5$ is the variety generated by the lattice $N_5 = \{0, 1, a, b, c\}$ depicted in Figure 1. The lattice $\operatorname{Con} N_5$ has 3 meet-irreducible elements, namely $\gamma_1 = (0ac)(b1)$ (that is, the congruence classes of γ_1 are $\{0, a, c\}$, $\{b, 1\}$), $\gamma_2 = (0b)(ac1)$ and the zero congruence Δ_{N_5}. The smallest nonzero congruence is $\mu_{N_5} = (ac)(0)(b)(1)$.

There are 2 subdirectly irreducible algebras in $\mathcal{N}_5$: the algebra N_5 and the 2-element chain C_2. It is not difficult to check that the complete list of subalgebras of subdirectly irreducible algebras (up to isomorphism) is $\mathcal{S} = \{\emptyset, C_1, C_2, C_3, C_4, C_2 \times C_2, N_5\}$. Here C_k denotes the k-element chain. It is also easy to see that the set $\mathcal{S}$ is closed under the formation of homomorphic images (up to isomorphism).

If $A \in \mathcal{N}_5$ and $\alpha \in \mathrm{M}(\operatorname{Con} A)$, then A/α is isomorphic to C_2 or to N_5. In the first case, α is a maximal element of $\mathrm{M}(\operatorname{Con} A)$. In the second case, there are exactly two elements of $\mathrm{M}(\operatorname{Con} A)$ above α. Thus, $L = \operatorname{Con} A$ satisfies the following condition:

(N5) $\mathrm{M}(L)$ is a union of two antichains M_1 and M_2 such that, for every $x \in M_1$, there are exactly two $y \in M_2$ with $x < y$.

In fact, this condition characterizes the finite members of $\operatorname{Con}(\mathcal{N}_5)$.

Theorem 3.1. (See [6].) *A finite distributive lattice L belongs to* $\operatorname{Con}(\mathcal{N}_5)$ *iff it satisfies* (N5).

On the infinite level, the condition (N5) is not sufficient. The separability properties come into play.

Theorem 3.2. *Let $A \in \mathcal{N}_5$, $\alpha_1, \dots, \alpha_n \in \mathrm{M}(\mathrm{Con}\, A)$. Suppose that all α_i are distinct. Then the following holds.*

(1) *If $n > 3$, then the set $\{\alpha_1, \dots, \alpha_n\}$ is separable.*

(2) *If $n = 3$, then there is $i \in \{1, 2, 3\}$ such that $\{\alpha_1, \alpha_2, \alpha_3\}$ is separable over $\{\alpha_i\}$.*

Proof. Let $\alpha = \alpha_1 \cap \dots \cap \alpha_n$. It is easy to check that all algebras in $\mathcal{S}$ have at most 3 meet-irreducible congruences. If $n > 3$, then A/α has more then 3 meet-irreducible congruences, namely all α_i/α. Thus, (1) follows from 2.5.

Now let $n = 3$. The only algebras in $\mathcal{S}$ with 3 meet-irreducible congruences are N_5 and C_4. If A/α is not isomorphic to N_5 or C_4 then $\{\alpha_1, \alpha_2, \alpha_3\}$ is separable by 2.5 and hence it is separable over any subset.

Assume that A/α is isomorphic to N_5. Then, for some $i \in \{1, 2, 3\}$, α_i/α is the zero congruence on A/α, which means $\alpha_i = \alpha$. We claim that $\{\alpha_1, \alpha_2, \alpha_3\}$ is separable over $\{\alpha_i\}$. For contradiction, suppose it is not the case. By 2.4, there are $T \leq S \in \mathrm{SI}(\mathcal{N}_5)$ and a surjective homomorphism $t \colon T \to A/\alpha$ such that $(t(x), t(y)) \in \alpha_i/\alpha$ for every $(x, y) \in \mu_S \restriction T$. The only possibility is $T = S = N_5$ and then t is an isomorphism. We have $(a, c) \in \mu_S \restriction T$ and $t(a) \neq t(c)$, because t is injective. Since α_i/α is the zero congruence, we obtain that $(t(a), t(c)) \notin \alpha_i/\alpha$, a contradiction.

Finally, assume that A/α is equal to $C_4 = \{x_1, x_2, x_3, x_4\}$, $x_1 < x_2 < x_3 < x_4$. Then, for some $i \in \{1, 2, 3\}$, the congruence α_i/α is equal to $(x_1x_2)(x_3x_4)$. We claim that $\{\alpha_1, \alpha_2, \alpha_3\}$ is separable over $\{\alpha_i\}$. Similarly as above, we assume it is not the case. Now the only possibility is $S = N_5$, $T = \{0, a, c, 1\}$. Again, $t \colon T \to C_4$ is an isomorphism. Thus, $(a, c) \in \mu_S \restriction T$ and $(t(a), t(c)) = (x_2, x_3) \notin \alpha_i/\alpha$, a contradiction. □

On the other hand, it is possible to find $A \in \mathcal{N}_5$ and distinct $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{M}(\mathrm{Con}\, A)$, such that $\{\alpha_1, \alpha_2, \alpha_3\}$ is not separable over $\{\alpha_1, \alpha_2\}$. One can map $A = F_{\mathcal{N}_5}(\aleph_0)$ homomorphically onto C_4 or N_5 and use 2.6.

4. The varieties $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_1 \vee \mathcal{L}_2$

$\mathcal{L}_1$ and $\mathcal{L}_2$ are the varieties generated by the lattices L_1 and L_2 respectively. (See Figure 2.)

Since both L_1 and L_2 contain a subalgebra isomorphic to N_5, we have $\mathcal{N}_5 \subseteq \mathcal{L}_1$, $\mathcal{N}_5 \subseteq \mathcal{L}_2$. (In fact, both $\mathcal{L}_1$ and $\mathcal{L}_2$ cover $\mathcal{N}_5$ in the lattice of lattice varieties.) The varieties $\mathcal{L}_1$, $\mathcal{L}_2$ contain mutually dual lattices. Since dual lattices have the same congruences, we have $\mathrm{Con}(\mathcal{L}_1) = \mathrm{Con}(\mathcal{L}_2)$.

The congruence lattice of L_1 is isomorphic to $\mathrm{Con}\, N_5$. The three meet-irreducible members of $\mathrm{Con}\, L_1$ are $\delta_1 = (0ac)(bed1)$, $\delta_2 = (0be)(acd1)$ and the zero congruence

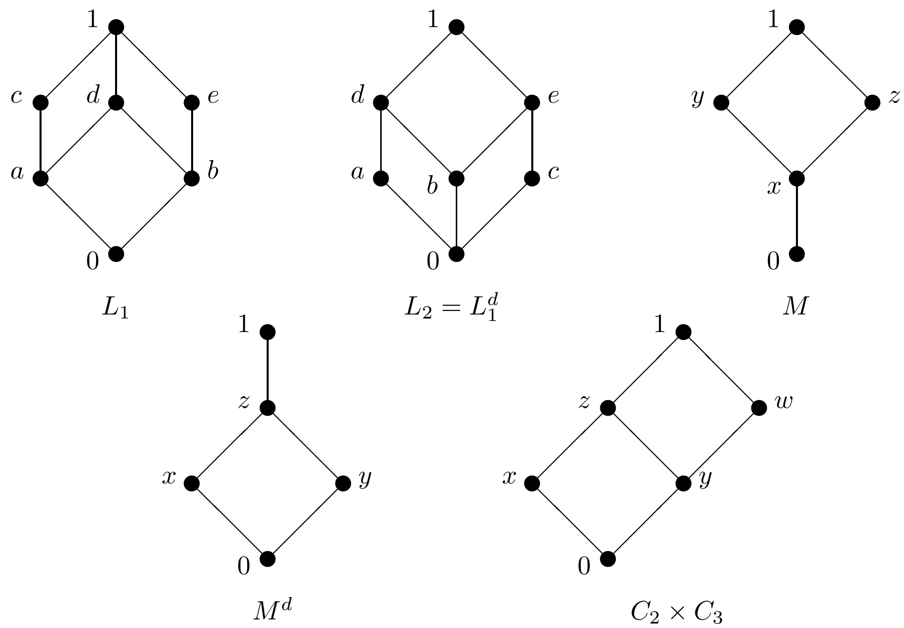

FIGURE 2

Δ_{L_1}. The smallest nonzero congruence is $\mu_{L_1} = (ac)(d1)(be)(0)$. The only subdirectly irreducible algebras in $\mathcal{L}_1$ (up to isomorphism) are C_2, N_5 and L_1. Every $L \in \operatorname{Con}(\mathcal{L}_1)$ satisfies the condition (N5) for the same reason as in the case of the variety $\mathcal{N}_5$. And the same holds for the varieties $\mathcal{L}_2$ and $\mathcal{L}_1 \vee \mathcal{L}_2$. (The subdirectly irreducible algebras in $\mathcal{L}_1 \vee \mathcal{L}_2$ are C_2, N_5, L_1 and L_2.) Hence, we have the following consequence of 3.1.

Theorem 4.1. *The classes* $\operatorname{Con}(\mathcal{N}_5)$, $\operatorname{Con}(\mathcal{L}_1)$, $\operatorname{Con}(\mathcal{L}_2)$ *and* $\operatorname{Con}(\mathcal{L}_1 \vee \mathcal{L}_2)$ *contain the same finite lattices, characterized by (N5).*

However, we will show that these classes contain different infinite lattices. It turns out that the relative separability plays a crucial role.

It is not difficult to check that every homomorphic image of a subalgebra of a subdirectly irreducible algebra in $\mathcal{L}_1$ is isomorphic to some algebra in $\mathcal{T} = \{\emptyset, C_1, C_2, C_3, C_4, C_2 \times C_2, C_2 \times C_3, M, M^d, N_5, L_1\}$. The lattice M and its dual M^d are depicted in Figure 2. These lattices will be especially important in our considerations. It is easy to see that M has exactly 3 meet-irreducible congruences, namely $\varepsilon_1 = (0xy)(z1)$, $\varepsilon_2 = (0)(xyz1)$, $\varepsilon_3 = (0xz)(y1)$.

None of the algebras in $\mathcal{T}$ have more than 3 meet-irreducible congruences. Thus, every 4-element subset of $\mathrm{M}(\mathrm{Con}\, A)$, for every $A \in \mathcal{L}_1$, is separable.

Lemma 4.2. *Let $\mathcal{V}$ be a finitely generated variety of lattices containing L_1. Let $F = F_{\mathcal{V}}(X)$ for some infinite set X. Let $f \colon F \to M$ be a surjective homomorphism. For $i = 1, 2, 3$, denote $\alpha_i = f^{-1}(\varepsilon_i) = \{(u, v) \in F^2 \mid (f(u), f(v)) \in \varepsilon_i\}$. Then $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{M}(\mathrm{Con}\, F)$ and the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is separable neither over $\{\alpha_1, \alpha_2\}$ nor over $\{\alpha_2, \alpha_3\}$.*

Proof. It is easy to see that $\alpha_i \in \mathrm{M}(\mathrm{Con}\, F)$. Denote $\alpha = \mathrm{Ker}(f)$. Since $\varepsilon_1 \cap \varepsilon_2 \cap \varepsilon_3 = \Delta_M$, we have $\alpha_1 \cap \alpha_2 \cap \alpha_3 = \alpha$. For simplicity assume that $F/\alpha = M$, so $\varepsilon_i = \alpha_i/\alpha$.

Let $S = L_1$ and $T = \{0, a, c, d, 1\} \subseteq L_1$. The only nontrivial pairs in $\mu_S \restriction T$ are (a, c) and $(d, 1)$. If we define $t_1 \colon T \to M$ by $t_1(0) = 0$, $t_1(a) = x$, $t_1(c) = y$, $t_1(d) = z$, $t_1(1) = 1$, then we have $(t_1(a), t_1(c)) = (x, y) \in \varepsilon_1 \cap \varepsilon_2$, $(t_1(d), t_1(1)) = (z, 1) \in \varepsilon_1 \cap \varepsilon_2$. By 2.6 we obtain that $\{\alpha_1, \alpha_2, \alpha_3\}$ is not separable over $\{\alpha_1, \alpha_2\}$. The proof for the set $\{\alpha_2, \alpha_3\}$ is analogous, using the homomorphism $t_2 \colon T \to M$ defined by $t_2(0) = 0$, $t_2(a) = x$, $t_2(c) = z$, $t_2(d) = y$, $t_2(1) = 1$. □

Consequence 4.3. *For any infinite set X, $\mathrm{M}(\mathrm{Con}\, F_{\mathcal{L}_1}(X))$ contains distinct α_1, α_2, α_3 such that the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is separable neither over $\{\alpha_1, \alpha_2\}$ nor over $\{\alpha_2, \alpha_3\}$.*

If $\{\alpha_1, \alpha_2, \alpha_3\}$ is separable neither over $\{\alpha_1, \alpha_2\}$ nor over $\{\alpha_2, \alpha_3\}$, then it is not separable over $\{\alpha_i\}$, for any $i = 1, 2, 3$. By 3.2, this is impossible in $\mathrm{Con}(\mathcal{N}_5)$ and we have the following result.

Theorem 4.4. $\mathrm{Con}\, F_{\mathcal{L}_1}(\aleph_0) \notin \mathrm{Con}(\mathcal{N}_5)$*, hence* $\mathrm{Con}(\mathcal{N}_5) \neq \mathrm{Con}(\mathcal{L}_1)$.

Now we will show that $\mathrm{Con}(\mathcal{L}_1) \neq \mathrm{Con}(\mathcal{L}_1 \vee \mathcal{L}_2)$. This is somewhat surprising, since $\mathrm{Con}(\mathcal{L}_1) = \mathrm{Con}(\mathcal{L}_2)$, as we have mentioned above. To show this, we need to investigate the relatively separable sets for $A \in \mathcal{L}_1$ in more detail.

Let $\alpha_1, \alpha_2 \in \mathrm{Con}\, A$ for a lattice A be such that both A/α_i are isomorphic to $C_2 = \{0, 1\}$, $0 < 1$. Then α_1, α_2 are the kernels of some surjective homomorphisms p_1, $p_2 \colon A \to C_2$. We write $\alpha_1 \sqsubseteq \alpha_2$ iff $p_1(x) \leq p_2(x)$ for every $x \in A$.

Theorem 4.5. *Let $\alpha_1, \alpha_2, \alpha_3$ be distinct congruences in $\mathrm{M}(\mathrm{Con}\, A)$, $A \in \mathcal{L}_1$, $\alpha = \alpha_1 \cap \alpha_2 \cap \alpha_3$. Suppose that $\{\alpha_1, \alpha_2, \alpha_3\}$ is not separable over $\{\alpha_1, \alpha_2\}$ nor over $\{\alpha_2, \alpha_3\}$. Then A/α is isomorphic to M or C_4. Furthermore, $\alpha_1 \sqsubseteq \alpha_2$, $\alpha_3 \sqsubseteq \alpha_2$.*

Proof. The algebra A/α has at least three meet-irreducible congruences, namely α_1/α, α_2/α, α_3/α. By 2.5, A/α must belong to $\mathcal{T}$ (up to isomorphism), hence $A/\alpha \in \{C_4, M, M^d, C_2 \times C_3, N_5, L_1\}$. We will examine all the cases. By 2.4 there are $T \leq S \in \mathrm{SI}(\mathcal{L}_1)$ and $t \colon T \to A/\alpha$ such that 2.4($*$) holds for $K = \{1, 2\}$, and another (perhaps different) S, T and t such that 2.4($*$) holds for $K = \{2, 3\}$.

(a) Let A/α be isomorphic to L_1. For some $i \in \{1, 2, 3\}$, the congruence α_i/α is the zero congruence. The only choice for S and T is $S = T = L_1$ so t is an isomorphism. Then $(a, c) \in \mu_S \restriction T$ and $(t(a), t(c)) \notin \alpha_i/\alpha$, which shows that $\{\alpha_1, \alpha_2, \alpha_3\}$ is separable over any set containing α_i, a contradiction.

(b) Let A/α be isomorphic to N_5. Again, α_i/α is the zero congruence for some i. We have 3 possibilities for S and T, namely $S = T = N_5$, $T = \{0, 1, a, c, e\} \leq S = L_1$ and $T = \{0, 1, b, c, e\} \leq S = L_1$. In every case, T is isomorphic to N_5, t is an isomorphism and there is $(u, v) \in \mu_S \restriction T$ with $u \neq v$. Hence, $(t(u), t(v)) \notin \alpha_i/\alpha$, so $\{\alpha_1, \alpha_2, \alpha_3\}$ is separable over any set containing α_i, a contradiction.

(c) Let $A/\alpha = M^d$. Then $\alpha_i/\alpha = (0xyz)(1) \in \operatorname{Con} M^d$ for some i. We claim that $K \subseteq \{1, 2, 3\}$ containing i cannot satisfy 2.4($*$). Indeed, the only possibility is $T = \{0, a, b, d, 1\} \leq S = L_1$, so $t\colon T \to M^d$ is an isomorphism. Hence, $(d, 1) \in \mu_S \restriction T$ and $(t(d), t(1)) = (z, 1) \notin \alpha_i/\alpha$.

(d) Let $A/\alpha = C_2 \times C_3$. Then $\alpha_i/\alpha = (0xyz)(w1) \in \operatorname{Con} C_2 \times C_3$, for some i. We claim that any K containing i cannot satisfy 2.4($*$). We have two possibilities: $T = \{0, a, b, c, d, 1\} \leq S = L_1$ or $T = \{0, a, b, d, e, 1\} \leq S = L_1$. In both cases $t\colon T \to A/\alpha$ is an isomorphism, $(d, 1) \in \mu_S \restriction T$ and $(t(d), t(1)) = (z, 1) \notin \alpha_i/\alpha$, which proves our claim.

(e) Let $A/\alpha = C_4 = \{0, 1, x, y\}$, $0 < x < y < 1$. The 3 meet-irreducible congruences of this algebra are $\gamma_1 = (0x)(y1)$, $\gamma_2 = (0)(xy1)$, $\gamma_3 = (0xy)(1)$. There are 5 possibilities for S and T. In every case t must be an isomorphism.

If $T = \{0, a, c, 1\} \leq N_5 = S$ or $T = \{0, a, c, 1\} \leq L_1 = S$, then the only nontrivial pair in $\mu_S \restriction T$ is (a, c) and we have $(t(a), t(c)) = (x, y) \in \gamma_i$ iff $i \in \{2, 3\}$. If $T = \{0, b, e, 1\} \leq L_1 = S$, then $(b, e) \in \mu_S \restriction T$ and $(t(b), t(e)) = (x, y) \in \gamma_i$ iff $i \in \{2, 3\}$.

If $T = \{0, a, d, 1\} \leq L_1 = S$ or $T = \{0, b, d, 1\} \leq L_1 = S$ then the only nontrivial pair in $\mu_S \restriction T$ is $(d, 1)$. We have $(t(d), t(1)) = (y, 1)$, which belongs to γ_i iff $i \in \{1, 2\}$.

We have $\{\alpha_1/\alpha, \alpha_2/\alpha, \alpha_3/\alpha\} = \{\gamma_1, \gamma_2, \gamma_3\}$. Our assumption of non-separability implies that $\gamma_2 = \alpha_2/\alpha$. All algebras C_4/γ_i are isomorphic to C_2, hence all algebras A/α_i are isomorphic to C_2. Moreover, $\gamma_1 \sqsubseteq \gamma_2$, $\gamma_3 \sqsubseteq \gamma_2$, which implies that $\alpha_1 \sqsubseteq \alpha_2$ and $\alpha_3 \sqsubseteq \alpha_2$.

(f) Let $A/\alpha = M$. We have two isomorphic possibilities for T, namely $T = \{0, a, c, d, 1\} \leq L_1 = S$ and $T = \{0, b, d, e, 1\} \leq L_1 = S$. We examine the first case. There are two different surjective homomorphisms $T \to M$, described in the proof of 4.2. If $t = t_1$ then 2.4($*$) holds for $K = \{i \mid \alpha_i/\alpha \in \{\varepsilon_1, \varepsilon_2\}\}$. If $t = t_2$ then 2.4($*$) holds for $K = \{i \mid \alpha_i/\alpha \in \{\varepsilon_2, \varepsilon_3\}\}$. Thus, α_2/α must be equal to ε_2. Similarly as in (e) we have $\varepsilon_1 \sqsubseteq \varepsilon_2$, $\varepsilon_3 \sqsubseteq \varepsilon_2$, which implies that $\alpha_1 \sqsubseteq \alpha_2$ and $\alpha_3 \sqsubseteq \alpha_2$. □

The relation $\sqsubseteq$ is antisymmetric, that is $\alpha \sqsubseteq \beta \sqsubseteq \alpha$ implies $\alpha = \beta$. Hence, 4.5 has the following consequence.

Theorem 4.6. *Let α_1, α_2, α_3, α_4 be distinct members of* $\mathrm{M}(\mathrm{Con}\, A)$*, $A \in \mathcal{L}_1$. Then at least one of the following conditions holds:*

(1) *$\{\alpha_1, \alpha_2, \alpha_3\}$ is separable over $\{\alpha_1, \alpha_2\}$;*
(2) *$\{\alpha_1, \alpha_2, \alpha_3\}$ is separable over $\{\alpha_2, \alpha_3\}$;*
(3) *$\{\alpha_2, \alpha_3, \alpha_4\}$ is separable over $\{\alpha_2, \alpha_3\}$;*
(4) *$\{\alpha_2, \alpha_3, \alpha_4\}$ is separable over $\{\alpha_3, \alpha_4\}$.*

Proof. Suppose that neither of (1)-(4) is satisfied. By 4.5, $\alpha_2 \sqsubseteq \alpha_3 \sqsubseteq \alpha_2$, hence $\alpha_2 = \alpha_3$, a contradiction. □

The relation $\sqsubseteq$ is not determined by the lattice structure of $\mathrm{Con}\, A$ alone; it depends on the concrete form of the congruence relations. The algebras in $\mathcal{L}_2$ have the same congruence lattices as the algebras in $\mathcal{L}_1$. However, instead of 4.5 they satisfy the dual condition: if $\{\alpha_1, \alpha_2, \alpha_3\}$ is not separable over $\{\alpha_1, \alpha_2\}$ nor over $\{\alpha_2, \alpha_3\}$, then $\alpha_2 \sqsubseteq \alpha_1$ and $\alpha_2 \sqsubseteq \alpha_3$. In the variety $\mathcal{L}_1 \vee \mathcal{L}_2$ we have both L_1 and L_2 at our disposal, the both situations can occur.

Theorem 4.7. *Let $F = F_{\mathcal{L}_1 \vee \mathcal{L}_2}(X)$ for some infinite set X. There are distinct congruences $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in$* $\mathrm{M}(\mathrm{Con}\, F)$ *such that neither of the conditions 4.6(1)-(4) is satisfied.*

Proof. Pick 3 different elements of X, say $0, 1, 2 \in X$. There are uniquely determined homomorphisms $f_i \colon F \to C_2$ satisfying for all $u \in X$

(1) $f_1(u) = 1$ iff $u = 2$;
(2) $f_2(u) = 1$ iff $u \in \{1, 2\}$;
(3) $f_3(u) = 1$ iff $u = 1$;
(4) $f_4(u) = 1$ iff $u \in \{0, 1\}$.

We claim that the congruences $\alpha_i = \mathrm{Ker}(f_i)$ have the required properties.

Let $f \colon F \to M$ be the homomorphism defined by $f(1) = y$, $f(2) = z$ and $f(u) = 0$, for all $u \in X \setminus \{1, 2\}$. Let $p_i \colon M \to C_2$ be the homomorphism with $\mathrm{Ker}(p_i) = \varepsilon_i$ ($i = 1, 2, 3$). It is easy to check that $f_i = p_i f$, which implies that $\alpha_i = f^{-1}(\varepsilon_i)$. By 4.2, $\{\alpha_1, \alpha_2, \alpha_3\}$ is neither separable over $\{\alpha_1, \alpha_2\}$ nor over $\{\alpha_2, \alpha_3\}$.

The proof for $\{\alpha_2, \alpha_3, \alpha_4\}$ is similar. We define a homomorphism $f \colon F \to M^d$ by $f(0) = x$, $f(1) = 1$, $f(2) = y$ and $f(u) = 0$, for all $u \in X \setminus \{0, 1, 2\}$. The meet-irreducible congruences of M^d are $\delta_2 = (0x)(yz1)$, $\delta_3 = (0xyz)(1)$, $\delta_4 = (0y)(xz1)$. We have $\alpha_i = f^{-1}(\delta_i)$, for $i = 2, 3, 4$. Using the statement dual to 4.2 (with L_2 and M^d instead of L_1 and M) we obtain that $\{\alpha_2, \alpha_3, \alpha_4\}$ is neither separable over $\{\alpha_3, \alpha_4\}$ nor over $\{\alpha_2, \alpha_3\}$. □

The inequalities $\mathrm{Con}(\mathcal{N}_5) \neq \mathrm{Con}(\mathcal{L}_1) \neq \mathrm{Con}(\mathcal{L}_1 \vee \mathcal{L}_2)$ solve problem 5.5 of [5].

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