Balanced $d$-lattices are complemented*

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According to Chajda and Eigenthaler ([1]), a $d$-lattice is a bounded lattice $L$ satisfying for all $a, c \in L$ the implications

(i) $(a, 1) \in \theta(0, c) \rightarrow a \lor c = 1$;

(ii) $(a, 0) \in \theta(1, c) \rightarrow a \land c = 0$;

where $\theta(x, y)$ denotes the least congruence on $L$ containing the pair $(x, y)$. Every bounded distributive lattice is a $d$-lattice. The 5-element nonmodular lattice $N_5$ is a $d$-lattice.

Theorem 1 A bounded lattice is a $d$-lattice if and only if all maximal ideals and maximal filters are prime.

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Proof. Let $I$ be a maximal ideal in a $d$-lattice $L$. Let $x, y \in L \setminus I$. We need to show that $x \land y \in L \setminus I$. Since $I$ is maximal, there are $c_1, c_2 \in I$ such that $c_1 \lor x = c_2 \lor y = 1$. For $c = c_1 \lor c_2 \in I$ we have $c \lor x = c \lor y = 1$. Then $(x, 1) = (0 \lor x, c \lor x) \in \theta(0, c)$ and similarly $(y, 1) \in \theta(0, c)$, hence $(x \land y, 1) \in \theta(0, c)$. By (i) we have $(x \land y) \lor c = 1$, hence $x \land y \notin I$. The primality of maximal filters can be proved similarly.

Conversely, assume that all maximal ideals and filters in $L$ are prime. To show (i), assume that $a, c \in L$, $a \lor c \neq 1$. By the Zorn lemma, there exists a maximal ideal $I$ containing $a \lor c$. By our assumption, $I$ is prime. Then $\alpha = I^2 \cup (L \setminus I)^2$ is a congruence on $L$. Since $c \in I$, we have $(0, c) \in \alpha$, which implies that $\theta(0, c) \subseteq \alpha$. Since $a \in I$, we have $(a, 1) \notin \alpha$, hence $(a, 1) \notin \theta(0, c)$. This shows (i). The proof of (ii) is similar.

By [1], a bounded lattice is called “balanced”, if the 0-class of any congruence determines the 1-class, and conversely. They showed that complemented lattices are balanced, and they asked:

(⋆) Is there a $d$-lattice which is balanced but not complemented?

We use the above characterization of $d$-lattices to answer this question.

If $A$ is a subset of an algebra, write $\theta_A$ for the smallest congruence that identifies all elements of $A$; if $\phi$ is a congruence, $x$ an element, write $x/\phi$ for the $\phi$-congruence class of $x$.

Further, a congruence $\phi$ (on an algebra with constants 0 and 1) is called balanced if $0/\phi = 0/\theta_{(1/\phi)}$ and $1/\phi = 1/\theta_{(0/\phi)}$; an algebra is called balanced iff all its congruence relations are balanced, or equivalently if: for any congruence relations $\phi, \phi'$ we have:

$$0/\phi = 0/\phi' \iff 1/\phi = 1/\phi'.$$

Fix a $d$-lattice $(L, \lor, \land, 0, 1)$. For $a \in L$ we denote $F_a := \{x : x \lor a = 1\}$, and $I_a := \{x : x \land a = 0\}$.

**Fact 2** $F_a$ is a filter, $I_a$ is an ideal.

Proof. Let $x, y \in F_a$. Similarly as in the proof of Theorem 1, $(x, 1) \in \theta(0, a)$, $(y, 1) \in \theta(0, a)$, hence $(x \land y, 1) \in \theta(0, a)$, which by the definition of a $d$-lattice implies $x \land y \in F_a$. The proof for $I_a$ is similar.
**Fact 3** If $I$ is an ideal disjoint to $F_a$, and $a \notin I$, then also the ideal generated by $I \cup \{a\}$ is disjoint to $F_a$.

**Proof.** If $x \leq i \lor a$ for some $i \in I$, and $x \in F_a$, then also $i \lor a \in F_a$, hence $i \lor a = (i \lor a) \lor a = 1$. Thus, $i \in F_a$, so $F_a \cap I \neq \emptyset$. ■

**Fact 4** If $f : L_1 \to L_2$ is a homomorphism from $L_1$ onto $L_2$, and $L_1$ is balanced, then $L_2$ is balanced.

**Proof.** In fact, this holds “level-by-level”: If $\phi$ is an unbalanced congruence on $L_2$, then the preimage of $\phi$ is unbalanced on $L_1$. ■

**Theorem 5** The following are equivalent (for a $d$-lattice $L$):

1. There is a maximal (hence prime) filter whose complement is not a maximal ideal.
2. There is a maximal (hence prime) ideal whose complement is not a maximal filter.
3. There are two prime ideals in $L$, one properly containing the other.
4. There are two prime filters in $L$, one properly containing the other.
5. There is a homomorphism from $L$ onto the 3-element lattice $\{0,d,1\}$.
6. $L$ is not balanced.
7. $L$ is not complemented.

In particular a $d$-lattice is balanced iff it is complemented.

**Proof.**

\[
\begin{array}{c}
(1) \rightarrow (3) \\
\uparrow & \uparrow & \downarrow \\
(7) & (5) \rightarrow (6) \rightarrow (7) & (2) \rightarrow (4)
\end{array}
\]
(1) → (3): By 1, the complement of a maximal filter is a (necessarily prime) ideal. If this ideal is not maximal, it can be properly extended to a maximal (hence prime) ideal. The proof of (2) → (4) is similar (dual).

(3) → (5): Let $I_1 \subset I_2 \subset L$ be prime ideals. Map $I_1$ to 0, $I_2 \setminus I_1$ to $d$, and $L \setminus I_2$ to 1. Check that this is a lattice homomorphism. The proof of (4) → (5) is dual.

(5) → (6) follows from fact 4, since the three-element lattice is not balanced.

(6) → (7) is from [1].

Now we show (7) → (1). (Again, (7) → (2) is dual.) Assume that $L$ is not complemented, so there is some $a$ such that $F_a \cap I_a = \emptyset$. Let $F_1$ be the filter generated by $F_a \cup \{a\}$. We have $F_1 \cap I_a = \emptyset$ by the dual of Fact 3, so $F_1$ is proper. By the Zorn lemma, $F_1$ can be extended to a maximal filter $F$. Let $I_1 = L \setminus F$. It is enough to see that $I_1$ is not maximal. Let $I$ be the ideal generated by $I_1 \cup \{a\}$. By Fact 3, $I \cap F_a = \emptyset$, so $I$ is a proper ideal properly extending $I_1$. ■

References