ON A CHARACTERIZATION OF DISTRIBUTIVE LATTICES BY THE BETWEENNESS RELATION

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Abstract. We construct an example of a ternary structure satisfying certain conditions due to M. Kolibiar, which is not a betweenness relation of any lattice. This answers a question posed by J. Hedlíkova and T. Katriňák.

1. Introduction.

Having a lattice L, one can define a ternary relation R on L as follows:

$$(a, b, c) \in R \quad \text{iff} \quad (a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (b \vee c).$$

We refer to this relation as to the (ternary) betweenness relation on L.

G. Birkhoff and S. A. Kiss in their pioneering paper [2] characterized distributive lattices with universal bounds in terms of a median operator m. In the early fifties, several papers of Sholander were devoted to medians and betweenness relations. The connection between the two subjects is

$$(a, b, c) \in R \quad \text{iff} \quad m(a, b, c) = b.$$

This has lead to the introduction of so-called median algebras; in combination with the Birkhoff-Kiss results it has lead to a characterization of distributive lattices with universal bounds in terms of betweenness.

What makes things more difficult is the lack of universal bounds. M. Kolibiar [5] characterized abstractly the betweenness relation on (in general unbounded) lattices. He proved that a ternary relation R on a set L is a betweenness relation of some distributive lattice defined on L if and only if the relation R satisfies conditions (A), (B), (C), (D) and (F) given below. The condition (F) was especially designed to handle the case of unbounded lattices.

J. Hedlíková and T. Katriňák [3] proved that the conditions (A), (B), (C), (D) and (F) are independent. In the same paper they posed a question of whether the condition (F) can be replaced by an another condition (F_1) (see below). The condition (F_1) has

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the advantage that it is a first-order property, while (F) is not. (The conditions (A), (B), (C) and (D) are first-order properties.) This question is thus connected with the problem of whether the ternary betweeness relation on (distributive) lattices is first-order axiomatizable.

Lattice betweenness is closely connected with median algebras. A recent paper [1] contains several "axiom systems" of betweenness induced by a median operator. By [1, Prop.1.5+Thm.4.6], median betweenness is characterized by the conditions (B), (C), (D) and

(A′) Every element a belongs to the segment $\langle a, a\rangle$.

(See below for the definition of a segment.) It is not difficult to see that (A′) can be replaced by the condition

(A″) Every element a belongs to some segment.

Replacing (A″) by (A) means requiring that every three points of a median algebra are included in a segment. In fact, (A) and (D) imply that each finite subset of a median algebra is included in a segment. Each segment of a median algebra is a distributive lattice (with universal bounds). An example in [3] shows however, that the conditions (A), (B), (C), (D) are not strong enough to characterize lattice betweenness.

In this paper we construct an example of a structure that satisfies (A), (B), (C), (D) and (F_1) but not (F). Hence, even the axiom system (A), (B), (C), (D), (F_1) is not sufficient to characterize lattice betweenness.

Let R be a ternary relation on a set M. For any elements $a, c \in M$ we denote $\langle a, c\rangle = \{b \in M \mid (a, b, c) \in R\}$. Any set of the form $\langle a, c\rangle$ is called a segment on M. If G is a segment, then we define $\mathrm{Fund}(G) = \{(a, c) \in M \times M \mid \langle a, c\rangle = G\}$. Now we can formulate the conditions mentioned above. Because of the presence of the condition (D), we use (A), (B), (C), (F) and (F_1) in a slightly simpler form than in [3]:

(A) For any $a, b, c \in M$ there are $d, e \in M$ such that $\{a, b, c\} \subseteq \langle d, e\rangle$.
(B) For any $a, b, c \in M$, $\langle a, b\rangle \cap \langle a, c\rangle \cap \langle b, c\rangle \neq \emptyset$
(C) If $a, b, c \in M$, then $(a, b, c) \in R$ iff $\langle a, b\rangle \cap \langle c, b\rangle = \{b\}$.
(D) For any $a, b, c, d \in M$, if $\{a, b\} \subseteq \langle c, d\rangle$, then $\langle a, b\rangle \subseteq \langle c, d\rangle$.
(F) There exists a map assigning to every segment J a pair $(a_J, b_J) \in \mathrm{Fund}(J)$ such that for all segments G, H the following holds:

$$\text{if} \quad G \subseteq H, \quad \text{then} \quad (a_H, a_G, b_G) \in R.$$

(F_1) For every segment G there exists $(a, b) \in \mathrm{Fund}(G)$ such that for every segment H satisfying $G \subseteq H$ there exists $(c, d) \in \mathrm{Fund}(H)$ with $(c, a, b) \in R$.

We will need some elementary facts about betweenness relations on lattices. (See [5].) If L^{op} is the dual lattice of a lattice L (i. e. L and L^{op} have the same elements and $x \leq y$ holds in L^{op} iff $y \leq x$ holds in L), then the betweenness relations on L and L^{op} coincide. If R is the betweenness relation of a direct product $L_1 \times L_2$ of lattices L_1 and

L_2 (having R_1 and R_2 as the betweenness relations), then $((x,y),(z,t),(u,v)) \in R$ iff $(x,z,u) \in R_1$ and $(y,t,v) \in R_2$. If R is the betweenness relation of a distributive lattice, then $(a,b,c) \in R$ iff $a \wedge c \leq b \leq a \vee c$.

By a ternary structure we mean a set endowed with a ternary relation.

1.1. Lemma. *Let M_i $(i = 0, 1, 2, \dots)$ be ternary structures satisfying the conditions (A), (B), (C), (D). Suppose that M_i is a substructure of M_j whenever $i \leq j$. Then the ternary structure $M = \bigcup_{i=0}^{\infty} M_i$ satisfies (A), (B), (C), (D), too.*

Proof. Each of the conditions (A), (B), (C) and (D) concerns only a finite number of elements. Their validity in M can be proved by considering M_i containing all the elements involved. □

2. Construction.

Let C_n denote the n-element chain $0 < 1 < \cdots < n-1$ viewed as a lattice. Let

$$K_n = \{(x_0, \dots, x_n\} \in C_2 \times (C_3)^n \mid x_0 = 0 \text{ implies } \{x_1, \dots, x_n\} \subseteq \{0,1\}\}.$$

It is easy to see that K_n is a sublattice of the (distributive) lattice $C_2 \times (C_3)^n$. Further, for any $n \geq 0$ we denote

$$L_n = C_{n+2} \times (K_n)^n.$$

Hence, L_n too is a distributive lattice. We consider elements of L_n in the form (a, A), where $a \in C_{n+2}$ and A is a matrix with n rows and $n+1$ columns. (Each row represents an element of K_n.) We adopt the convention that entries of a matrix A are denoted by a_{ij}, entries of a matrix B by b_{ij}, etc. $(i \in \{1, \dots n\},\ j \in \{0, \dots, n\})$.

For every $n \geq 0$ we define a mapping $f_n : L_n \longrightarrow L_{n+1}$ as follows. For every $x = (a, A) \in L_n$ we set $f_n(x) = (a, B)$, where the matrix B (an extension of A) is given by the following rules:
$b_{ij} = a_{ij}$ whenever $i \leq n$, $j \leq n$;
$b_{i,n+1} = 0$ if $a \leq i$;
$b_{i,n+1} = 1$ if $a > i$;
$b_{n+1,j} = 0$ for every $j > 0$;
$b_{n+1,0} = 0$ if $a \neq 0$;
$b_{n+1,0} = 1$ if $a = 0$.

2.1. Lemma. *For any natural numbers n and k with $n < k$ there exists a lattice $L_{k,n}$ such that the following conditions are satisfied:*

(1) *the lattices L_k and $L_{k,n}$ have the same elements;*
(2) *the betweenness relations of L_k and $L_{k,n}$ coincide;*
(3) *the mapping $f = f_{k-1} \circ f_{k-2} \circ \cdots \circ f_n$ is a lattice embedding $L_n \longrightarrow L_{k,n}$.*

Proof. Let us set $L_{k,n} = C_{k+2} \times (K_k)^n \times (K_k^{op})^{k-n}$. Then (1) and (2) are evident, it remains to prove (3). The injectivity of f follows from the injectivity of $f_n, \dots, f_{k-1}$.

Let $p_0 : L_{k,n} \longrightarrow C_{k+2}$, $p_1 : L_{k,n} \longrightarrow K_k$, ... , $p_n : L_{k,n} \longrightarrow K_k$, ... ,$p_k : L_{k,n} \longrightarrow K_k^{op}$ be the natural projections. Since the operations in $L_{k,n}$ are pointwise, it suffices to show that $p_j \circ f$ is a lattice homomorphism for every $j = 0, 1, \ldots, k$. This is clear for $j = 0$.

Suppose now that $0 < j \leq n$. Let $x, y \in L_n$, $x = (a, A)$, $y = (b, B)$. Then
$p_j(f(x)) = (a_{j0}, a_{j1}, \ldots, a_{jn}, c, c, \ldots, c)$,
$p_j(f(y)) = (b_{j0}, b_{j1}, \ldots, b_{jn}, d, d, \ldots, d)$,
where $c, d \in \{0, 1\}$ are such that $c = 0$ iff $a \leq j$ and $d = 0$ iff $b \leq j$. Clearly, $c \vee d = 0$ iff $\max\{a, b\} \leq j$ and $c \wedge d = 0$ iff $\min\{a, b\} \leq j$. Hence, we have
$p_j(f(x)) \vee p_j(f(y)) = (a_{j0} \vee b_{j0}, \ldots, a_{jn} \vee b_{jn}, c \vee d, \ldots, c \vee d) = p_j(f(x \vee y))$,
$p_j(f(x)) \wedge p_j(f(y)) = (a_{j0} \wedge b_{j0}, \ldots, a_{jn} \wedge b_{jn}, c \wedge d, \ldots, c \wedge d) = p_j(f(x \wedge y))$.

Finally, let $n < j \leq k$ and let $x, y \in L_n$ be as above. Then $p_j(f(x)) = (p, 0, 0, \ldots, 0)$, $p_j(f(y)) = (q, 0, 0, \ldots, 0)$, $p_j(f(x \vee y)) = (r, 0, 0, \ldots, 0)$ and $p_j(f(x \wedge y)) = (s, 0, 0, \ldots, 0)$, where $p, q, r, s \in \{0, 1\}$ are such that $p = 0$ iff $a \neq 0$, $q = 0$ iff $b \neq 0$, $r = 0$ iff $\max\{a, b\} \neq 0$ and $s = 0$ iff $\min\{a, b\} \neq 0$. Since K_k^{op} is dual to K_k, we have $p_j(f(x)) \wedge p_j(f(y)) = (\max\{p, q\}, 0, \ldots, 0) = (s, 0, \ldots, 0) = p_j(f(x \wedge y))$ and $p_j(f(x)) \vee p_j(f(y)) = (\min\{p, q\}, 0, \ldots, 0) = (r, 0, \ldots, 0) = p_j(f(x \vee y))$. $\square$

Let M_n be the ternary (betweenness) structure associated with the lattice L_n. (The ternary betweenness relation itself is denoted by R_n.) As a consequence of 2.1 we obtain that any $f_{kn} = f_{k-1} \circ \cdots \circ f_n$ is an embedding $M_n \longrightarrow M_k$. Let M be the ternary structure that is a limit of the directed system

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \ldots$$

Hence, elements of M are the equivalence classes of the equivalence relation $\sim$ on $\bigcup_{i=0}^{\infty} M_i$ given by the following rule: $x \sim y$ holds for $x \in M_i$, $y \in M_j$ if and only if $f_{ki}(x) = f_{kj}(y)$ for some $k > i, j$. Let $[x]$ denote the equivalence class containing an element x. The ternary relation R on the set M is defined in a natural way: $([x], [y], [z]) \in R$ holds for $x \in M_i$, $y \in M_j$, $z \in M_k$ if and only if $(f_{ni}(x), f_{nj}(y), f_{nk}(z)) \in R_n$ holds for some (and hence for all) $n > i, j, k$. Up to isomorphism, we can assume that M is the union of the increasing chain

$$M_0 \subset M_1 \subset \ldots$$

of its substructures.

2.2. Lemma. *The structure M satisfies (A), (B), (C), (D) and (F_1).*

Proof. (A), (B), (C) and (D) are satisfied by 1.1. To prove (F_1), let $G = \langle [x], [y] \rangle$ be an arbitrary segment of M. We can suppose that $x, y \in M_n$ for some n. Let us set $a = x \wedge y$, $b = x \vee y$, where $\wedge$ and $\vee$ refer to the lattice L_n. It is clear that $([a], [b]) \in \mathrm{Fund}(G)$. Let $H = \langle [z], [t] \rangle$ be any segment with $G \subseteq H$. We can suppose that $z, t \in M_k$ for some $k > n$. Let us set $c = z \wedge t$, $d = z \vee t$, where $\wedge$ and $\vee$ refer to the lattice $L_{k,n}$. Then $([c],[d]) \in \mathrm{Fund}(H)$ and we have to show that $([c],[a],[b]) \in R$, or equivalently, that $(c, f_{kn}(a), f_{kn}(b)) \in R_k$. Since $G \subseteq H$, we have $[a],[b] \in H$, which means that $c \leq f_{kn}(a) \leq d$ and $c \leq f_{kn}(b) \leq d$ are valid in $L_{k,n}$. Since f_{kn} is a lattice homomorphism $L_n \longrightarrow L_{k,n}$, we obtain that $c \leq f_{kn}(a) \wedge f_{kn}(b) = f_{kn}(a \wedge b) = f_{kn}(a) \leq f_{kn}(b)$ and therefore $(c, f_{kn}(a), f_{kn}(b)) \in R_k$. $\square$

Instead of proving that M does not satisfy (F) we will show that it does not satisfy even the following weaker condition:

(F_2) for every segment G there exists $(a,b) \in \mathrm{Fund}(G)$ such that for every segment $H \supseteq G$ there exists $(c,d) \in \mathrm{Fund}(H)$ such that for every segment $J \supseteq H$ there exists $(u,v) \in \mathrm{Fund}(J)$ with $(c,a,b) \in R$ and $(u,c,d) \in R$.

By [5] (assertion 4.3.6), if a ternary structure satisfies (A), (B) and (C) then $a \in \langle c,b \rangle$ and $b \in \langle c,d \rangle$ imply $b \in \langle a,d \rangle$. Hence, in the presence of the conditions (A), (B), (C) we can add the relations $(a,b,d) \in R$ and $(c,d,v) \in R$ to the condition (F_2), making it symmetric.

2.3. Lemma. *The structure M does not satisfy (F_2).*

Proof. By way of contradiction, suppose that (F_2) is fulfilled. Let us set $G = \langle [0],[1] \rangle$, where $0 = (0,\emptyset)$ and $1 = (1,\emptyset)$ are the only two elements of M_0. Let $[a],[b] \in M$ with the property according to (F_2). We can assume that $a, b \in M_n$ for some n, $a = (x,A)$, $b = (y,B)$. Since $([a],[b]) \in \mathrm{Fund}(G)$, we obtain that $f_{n0}(0) \vee f_{n0}(1) = a \vee b$ and $f_{n0}(0) \wedge f_{n0}(1) = a \wedge b$ hold in L_n, which implies that $x \vee y = 1$ and $x \wedge y = 0$. Without loss of generality, $x = 0$ and $y = 1$. Let $z = (0,Z)$ and $t = (n+2,T)$ be the least and the greatest element of L_{n+1}, respectively. Let us set $H = \langle [z],[t] \rangle$. Since $[a],[b] \in H$, we have $G \subseteq H$. Let $([c],[d]) \in \mathrm{Fund}(H)$ be according to (F_2). We can assume that $c, d \in M_k$ for some $k > n$, $c = (x',C)$, $d = (y',D)$. Since $([c],[d]) \in \mathrm{Fund}(H)$, we obtain that $x' \vee y' = n+2$ and $x' \wedge y' = 0$. Let $f_{kn}(a) = (0,A')$, $f_{kn}(b) = (1,B')$ and denote by a'_{ij}, b'_{ij} the entries of the matrices A' and B' respectively. It is easy to see that $a'_{n+1,0} = 1$ and $b'_{n+1,0} = 0$. From $(c, f_{kn}(a), f_{kn}(b)) \in R_k$ we obtain that $x' = 0$ (hence $y' = n+2$, because $x' \vee y' = n+2$) and $c_{n+1,0} = 1$. From $(f_{kn}(a), f_{kn}(b), d) \in R_k$ we infer that $d_{n+1,0} = 0$. Let us set $J = \langle [s],[w] \rangle$, where s and w are the least and the greatest element of L_{k+1} respectively. Then $H \subseteq J$ and according to (F_2) we have $([u],[v]) \in \mathrm{Fund}(J)$ such that $([u],[c],[d]) \in R$ and $([c],[d],[v]) \in R$. We can suppose that $u, v \in M_p$ for some $p > k$, $u = (x'',U)$, $v = (y'',V)$. Let $f_{pk}(c) = (0,C')$, $f_{pk}(d) = (0,D')$, where the entries of matrices C' and D' are denoted by c'_{ij} and d'_{ij}, respectively. It is easy to see that $c'_{n+1,0} = c_{n+1,0} = 1$, $c'_{n+1,k+1} = 0$, $d'_{n+1,0} = d_{n+1,0} = 0$ and $d'_{n+1,k+1} = 1$. From $(u, f_{pk}(c), f_{pk}(d)) \in R_p$ and $(f_{pk}(c), f_{pk}(d), v) \in R_p$ we obtain that $u_{n+1,0} = 1$, $u_{n+1,k+1} = 0$, $v_{n+1,0} = 0$, $v_{n+1,k+1} \in \{1,2\}$. According to the definition of the lattice K_n, this implies that $v_{n+1,k+1} = 1$. Further, we have $f_{p,k+1}(w) = (k+2,W)$, the matrix W having entries denoted by w_{ij}. (If $k+1 = p$, then $f_{p,k+1}$ is the identity mapping.) It is easy to see that $w_{n+1,k+1} = 2$. Since $u_{n+1,k+1} = 0$ and $v_{n+1,k+1} = 1$, we obtain that $(u, f_{p,k+1}(w), v) \notin R_p$, a contradiction with $[w] \in J = \langle [u],[v] \rangle$. $\square$

The main result now follows from 2.2 and 2.3.

2.4. Theorem. *The conditions (A), (B), (C), (D) and (F_1) do not axiomatize the betweenness relation of distributive lattices.*

The question of whether the betweenness relation of (distributive) lattices is first-order axiomatizable remains unsolved. Let us remark that (F_2) is a first-order condition. One can also consider the whole sequence (F_3), (F_4), ... of first-order conditions that arise by adding more quantifiers (in an obvious way) to (F_2). Their strength remains unsettled.

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