A NATURAL REPRESENTATION OF BOUNDED LATTICES

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Abstract. We establish a new representation theory for bounded lattices, which generalizes the well known Priestley duality. Our basic tool is the concept of a maximal partial homomorphism.

There exist several representation theories for lattices. Well known is the Priestley duality for bounded distributive lattices (cf. [4]). This duality is natural in the sense of B. A. Davey and H. Werner ([1]). For obvious reasons, it is impossible to build a natural duality for the variety of all bounded lattices. In our representation we try to preserve as much from the natural dualities theory as possible.

Our theory has common points with other representation theories for lattices. It is especially close to A. Urquhart's theory developed in [5]. In fact, in both cases the dual spaces have the same elements and the same topology, but they differ in relational structure. Our choice of relational structure seems to be the most natural from the point of view of natural dualities.

1. The representation theorem

We work with bounded lattices as algebras of the signature (2,2,0,0). A partial map $f: L_1 \longrightarrow L_2$ between bounded lattices is called a partial homomorphism if its domain $\mathrm{dom}(f)$ is a 0,1-sublattice of L_1 and the restriction $f \restriction \mathrm{dom}(f)$ is a bounded lattice homomorphism. A partial homomorphism is called maximal (MPH, for short), if there is no partial homomorphism properly extending it. Let us notice that by the Zorn's lemma, every homomorphism can be extended to a MPH.

Let $\overline{2}$ denote the 2-element lattice with elements 0, 1. For any bounded lattice L, let $D(L)$ be the set of all MPH's $L \longrightarrow \overline{2}$ equipped with the binary relation E defined by the rule

$$(f,g) \in E \quad \text{iff } f(x) \leq g(x) \text{ for every } x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$$

and with the topology τ whose subbasis of closed sets consists of all sets of the form $A_x = \{f \mid f(x) = 0\}$ and $B_x = \{f \mid f(x) = 1\}$ $(x \in L)$.

It is easy to prove that if the lattice L is distributive, then any MPH $L \longrightarrow \overline{2}$ is a total map (its domain is the whole L), E is a partial ordering and $D(L)$ is the usual dual space corresponding to L in the Priestley duality.

There is another way how to regard MPH's. It is easy to see that MPH's $L \longrightarrow \overline{2}$ correspond to so called maximal filter-ideal pairs in L. If F is a filter and I is an ideal in L such that $F \cap I = \emptyset$, then (F, I) is called a filter-ideal pair. Such a pair is said to be maximal, if neither F nor I can be enlarged without breaking the

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2000 *Mathematics Subject Classification.* 06B15, 06B10.
Key words and phrases. lattice, homomorphism, natural duality.
Supported by GA SAV Grant 362/93 .

disjointness. For any MPH $f:\ L \longrightarrow \overline{2}$ the pair $(f^{-1}(1), f^{-1}(0))$ is maximal; for any maximal pair (F, I) the partial map $f:\ L \longrightarrow \overline{2}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in I, \\ 1 & \text{if } x \in F, \\ \text{undefined otherwise} \end{cases}$$

is a MPH. If we regard $D(L)$ as the set of all maximal filter-ideal pairs, then

$$((F, I), (G, J)) \in E \quad \text{iff} \quad F \cap J = \emptyset.$$

Further, $A_x = \{(F, I) \,|\, x \in I\}$, $B_x = \{(F, I) \,|\, x \in F\}$.

Hence, for every bounded lattice L, its dual space $D(L)$ is a set equipped with a reflexive binary relation and a topology. Any object of this type will be called a topological graph. We keep the reflexivity assumption throughout the paper.

Now we decribe how to reconstruct a lattice from its dual space. A map $\varphi:\ (X_1, E_1, \tau_1) \longrightarrow (X_2, E_2, \tau_2)$ between topological graphs is called morphism if it preserves the binary relation (that is $(x, y) \in E_1$ implies $(\varphi(x), \varphi(y)) \in E_2$) and if it is continuous with respect to τ_1 and τ_2. A partial map $\varphi:\ (X_1, E_1, \tau_1) \longrightarrow (X_2, E_2, \tau_2)$ is called a partial morphism if its domain is a τ_1-closed subset of X_1 and the restriction of φ to its domain is a morphism. (We assume that $\mathrm{dom}(\varphi)$ inherits the binary relation and the topology from X_1.) A partial morphism is called maximal (MPM, for short), if there is no partial morphism properly extending it.

Let $\widetilde{2}$ denote the set $\{0, 1\}$ equipped with the discrete topology and the binary relation $\leq$ $(0 < 1)$. Hence, $\widetilde{2}$ is a topological graph.

Lemma 1.1. *Let L be a bounded lattice and $h:\ L \longrightarrow \overline{2}$ a MPH. If $x \in L$ is such that $x \notin h^{-1}(0)$, then there is a MPH $k:\ L \longrightarrow \overline{2}$ such that $h^{-1}(0) \subseteq k^{-1}(0)$ and $k(x) = 1$.*

Proof. The partial map $k':\ L \longrightarrow \overline{2}$ defined by

$$k'(y) = \begin{cases} 0 & \text{if } y \in h^{-1}(0), \\ 1 & \text{if } y \geq x, \\ \text{undefined otherwise} \end{cases}$$

is a partial homomorphism and can be extended to a MPH k, which obviously has the required properties. □

Lemma 1.2. *For every bounded lattice L and any $x \in L$, the evaluation function $e_x:\ D(L) \longrightarrow \widetilde{2}$ defined by*

$$e_x(f) = \begin{cases} f(x) & \textit{if } x \in \mathrm{dom}(f), \\ \textit{undefined otherwise} \end{cases}$$

is a MPM.

Proof. Clearly, $f \in \mathrm{dom}(e_x)$ iff $x \in \mathrm{dom}(f)$, hence $\mathrm{dom}(e_x) = A_x \cup B_x$, which is a closed set. Since the sets $A_x = e_x^{-1}(0)$ and $B_x = e_x^{-1}(1)$ are closed, the restriction of e_x to its domain is continuous.

Further, Let $f, g \in \mathrm{dom}(e_x)$, $(f, g) \in E$. We have to show that $e_x(f) \leq e_x(g)$, or $f(x) \leq g(x)$. But this follows directly from the definition of E.

It remains to prove the maximality of e_x. Let $\varphi:\ D(L) \longrightarrow \widetilde{2}$ be a partial morphism, $e_x \subseteq \varphi$. Let $h:\ L \longrightarrow \overline{2}$ be a MPH such that $h \notin \mathrm{dom}(e_x)$, that is

$x \notin \mathrm{dom}(h)$. We have to show that $h \notin \mathrm{dom}(\varphi)$. By 1.1, there is a MPH h_0 such that $h_0^{-1}(0) \supseteq h^{-1}(0)$, $h_0(x) = 1$. Similarly we can find a MPH h_1 with $h_1^{-1}(1) \supseteq h^{-1}(1)$, $h_1(x) = 0$. Now clearly $(h_0, h) \in E$, $(h, h_1) \in E$ and $h_0, h_1 \in \mathrm{dom}(e_x) \subseteq \mathrm{dom}(\varphi)$. Hence, $h \in \mathrm{dom}(\varphi)$ would imply $\varphi(h_0) \leq \varphi(h) \leq \varphi(h_1)$, which is impossible because $\varphi(h_0) = e_x(h_0) = h_0(x) = 1$ and $\varphi(h_1) = e_x(h_1) = h_1(x) = 0$. □

It is easy to see that the topology of $D(L)$ is always T_1. Moreover, this topology is always compact. (See [5, Lemma 6] for the proof.)

Lemma 1.3. *Suppose that $\widetilde{X} = (X, E, \tau)$ is a topological graph equipped with a T_1-topology τ. Let φ be a MPM $\widetilde{X} \longrightarrow \widetilde{2}$. Then*

(i) $\varphi^{-1}(0) = \{x \in X \mid$ *there is no* $y \in \varphi^{-1}(1)$ *with* $(y, x) \in E\}$*;*
(ii) $\varphi^{-1}(1) = \{x \in X \mid$ *there is no* $y \in \varphi^{-1}(0)$ *with* $(x, y) \in E\}$*.*

Proof. We prove (i). If $x \in \varphi^{-1}(0)$, then for every $y \in \varphi^{-1}(1)$ we have $1 = \varphi(y) \not\leq \varphi(x) = 0$. Since φ preserves E, we obtain $(y, x) \notin E$.

To prove the other inclusion, suppose that $x \in X$ is such that there is no $y \in \varphi^{-1}(1)$ with $(y, x) \in E$. Then $\varphi \cup \{(x, 0)\}$ is a partial morphism. (Its continuity follows from the fact that the topology τ is T_1 and therefore the set $\{x\}$ is closed.) The maximality of φ yields that $\varphi(x) = 0$. □

Lemma 1.4. *Let L be a bounded lattice, $\varphi : D(L) \longrightarrow \widetilde{2}$ a maximal partial morphism and $x \in L$. Then*

(i) $\varphi^{-1}(0) \cap B_x = \emptyset$ *implies* $\varphi^{-1}(0) \subseteq A_x$*;*
(ii) $\varphi^{-1}(1) \cap A_x = \emptyset$ *implies* $\varphi^{-1}(1) \subseteq B_x$*.*

Proof. It is enough to prove (i). Suppose that there is a MPH r with $r \notin A_x$ (that is $r(x) = 1$ or $r(x)$ is undefined) and $\varphi(r) = 0$. By 1.1 there exists a MPH p such that $r^{-1}(0) \subseteq p^{-1}(0)$ and $p(x) = 1$, hence $p \in B_x$. Further, for every $s \in \varphi^{-1}(1)$ we have $(s, r) \notin E$, because $\varphi(s) \not\leq \varphi(r)$. Hence $s^{-1}(1) \cap r^{-1}(0) \neq \emptyset$ and then also $s^{-1}(1) \cap p^{-1}(0) \neq \emptyset$, which means that $(s, p) \notin E$. By 1.3, $p \in \varphi^{-1}(0) \cap B_x \neq \emptyset$. □

Lemma 1.5. *Let L be a bounded lattice. Then every MPM $\varphi : D(L) \longrightarrow \widetilde{2}$ is of the form e_x for some $x \in L$.*

Proof. Let φ be a MPM. For every $p \in \varphi^{-1}(0)$, $q \in \varphi^{-1}(1)$ we have $(q, p) \notin E$, hence there is $x_p^q \in L$ with $q(x_p^q) = 1$, $p(x_p^q) = 0$. The set $X_p^q = \{r \in D(L) \mid r(x_p^q) \neq 0\}$ is open and the family $\mathcal{A}_p = \{X_p^q \mid q \in \varphi^{-1}(1)\}$ covers the set $\varphi^{-1}(1)$ because $q \in X_p^q$. Since the set $\varphi^{-1}(1)$ is compact (it is a closed subset of the compact space $D(L)$), there are $x_1, \ldots, x_n \in p^{-1}(0)$ such that $\varphi^{-1}(1) \subseteq \bigcup_{i=1}^n \{r \in D(L) \mid r(x_i) \neq 0\}$. Let us set $x_p = \bigvee_{i=1}^n x_i$. Then $p(x_p) = 0$ and $\varphi^{-1}(1) \subseteq \{r \in D(L) \mid r(x_p) \neq 0\}$. By 1.4, $\varphi^{-1}(1) \subseteq \{r \in D(L) \mid r(x_p) = 1\}$. The compact set $\varphi^{-1}(0)$ is covered by the open sets of the form $\{r \in D(L) \mid r(x_p) \neq 1\}$, where $p \in \varphi^{-1}(0)$. Hence, there are $x^1, \ldots, x^m \in L$ with $\varphi^{-1}(0) \subseteq \bigcup_{i=1}^m \{r \in D(L) \mid r(x^i) \neq 1\}$ and $\varphi^{-1}(1) \subseteq \bigcap_{i=1}^m \{r \in D(L) \mid r(x^i) = 1\}$. Let us set $x = \bigwedge_{i=1}^m x^i$. Then $\varphi^{-1}(1) \subseteq \{r \in D(L) \mid r(x) = 1\}$ and $\varphi^{-1}(0) \subseteq \{r \in D(L) \mid r(x) \neq 1\}$. From 1.4 we obtain that $\varphi^{-1}(0) \subseteq \{r \in D(L) \mid r(x) = 0\}$. Hence, $\varphi \subseteq e_x$ and the maximality of φ yields $\varphi = e_x$. □

The lemmas 1.2 and 1.5 establish a correspondence between elements of L and MPM's $D(L) \longrightarrow \widetilde{2}$. In order to reconstruct the lattice L from its dual space we need to introduce the lattice operations on the set of all MPM's. This is not a problem in the Priestley duality for distributive lattices (or in any other natural duality). If L is distributive, then all MPM's $D(L) \longrightarrow \widetilde{2}$ are total maps and the lattice operations can be defined pointwise. In general however, if $\widetilde{X}$ is any topological graph and φ, $\psi :\ \widetilde{X} \longrightarrow \widetilde{2}$ are MPM's, then the pointwise meet $\varphi \wedge \psi$ (defined on $\mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi)$) is a partial morphism, but not necessarily maximal. And, even if $\widetilde{X}$ is a dual space of some lattice, the maximal extension of $\varphi \wedge \psi$ need not be unique. But, fortunately, we are able to recover the order relation of L.

Lemma 1.6. *Let L be a bounded lattice. For elements $x, y \in L$ the following conditions are equivalent:*

(i) $x \leq y$;
(ii) $e_x^{-1}(0) \supseteq e_y^{-1}(0)$;
(iii) $e_x^{-1}(1) \subseteq e_y^{-1}(1)$.

Proof. If $x \leq y$ then, for every MPH $f :\ L \longrightarrow \overline{2}$, $f(x) = 1$ implies $f(y) = 1$ and $f(y) = 0$ implies $f(x) = 0$. Hence, (i) implies (ii) and (iii). Conversely, if $x \nleq y$, then there exists a MPH f with $f(x) = 1$ and $f(y) = 0$, hence $e_x^{-1}(0) \not\supseteq e_y^{-1}(0)$ and $e_x^{-1}(1) \not\subseteq e_y^{-1}(1)$. □

As a consequence we obtain our representation theorem.

Theorem 1.7. *Any bounded lattice L is isomorphic to the set of all MPM's $D(L) \longrightarrow \widetilde{2}$ ordered by the rule $\varphi \leq \psi$ iff $\varphi^{-1}(1) \subseteq \psi^{-1}(1)$.*

Our way of reconstructing a lattice from its dual space is similar to the methods used in the formal concept analysis of R. Wille ([7]). If L is a finite lattice, then we can consider the triple $(D(L), D(L), -E)$ as a context (using the terminology from [7]), where $-E$ is the complement of the relation E. It is easy to see that the concepts connected with this context are precisely the pairs of the form $(\varphi^{-1}(1), \varphi^{-1}(0))$, where φ is any MPM $D(L) \longrightarrow \widetilde{2}$. By 1.7, any finite lattice is isomorphic to the concept lattice of the context $(D(L), D(L), -E)$. The generalization to the infinite case cannot be straightforward, since concept lattices are always complete. There is a representation theory of G. Hartung ([2]), where the notion of a topological context was introduced in order to represent all bounded lattices. The relationship between Hartung's theory and our representation is essentially explained in [2], since this paper discusses Urquhart's representation, which is very close to our theory.

We close this section with one example, which shows how our representation works. Let L be the lattice depicted in Figure 1. Its dual space is drawn in Figure 2. The arrows indicate the relation E, the topology is discrete. The transition between L and $D(L)$ is described by the table in Figure 3. In this table, the rows are MPH's $L \longrightarrow \overline{2}$ and the columns are MPM's $D(L) \longrightarrow \widetilde{2}$ (iėėvaluation maps corresponding to the elements of L). This example also illustrates the remark preceding 1.6. The pointwise meet $e_x \wedge e_z$ defined on $\mathrm{dom}(e_x) \cap \mathrm{dom}(e_z) = \{f, g\}$ is a partial morphism whose extension to a MPM is not unique.

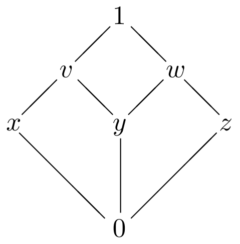

Figure 1

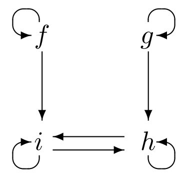

Figure 2

	0	x	y	z	v	w	1
f	0	1	0	0	1	0	1
g	0	0	0	1	0	1	1
h	0	0	1	−	1	1	1
i	0	−	1	0	1	1	1

Figure 3

2. Urquhart's representation

A. Urquhart in [5] introduced two binary (quasiorder) relations $\leq_1$, $\leq_2$ on the set of all MPH's $L \longrightarrow \overline{2}$ as follows:
$f \leq_1 g$ iff $f^{-1}(1) \subseteq g^{-1}(1)$;
$f \leq_2 g$ iff $f^{-1}(0) \subseteq g^{-1}(0)$.
So his dual of a lattice L is the set of all MPH's $L \longrightarrow \overline{2}$ equipped with $\leq_1$, $\leq_2$ and the same topology τ as in our representation.

Theorem 2.1. *Let L be a bounded lattice and let f, g be MPH's $L \longrightarrow \overline{2}$. Then*

(i) *$(f, g) \in E$ iff there is a MPH h with $f \leq_1 h$ and $g \leq_2 h$;*
(ii) *$f \leq_2 g$ iff there is no MPH h with $(h, g) \in E$ and $(h, f) \notin E$;*
(iii) *$f \leq_1 g$ iff there is no MPH h with $(g, h) \in E$ and $(f, h) \notin E$.*

Proof. (i) If $(f, g) \in E$ then $f^{-1}(1) \cap g^{-1}(0) = \emptyset$. The filter-ideal pair $(f^{-1}(1), g^{-1}(0))$ can be extended to a MPH h for which we have $f^{-1}(1) \subseteq h^{-1}(1)$ and $g^{-1}(0) \subseteq h^{-1}(0)$. Conversely, if $f \leq_1 h$, $g \leq_2 h$, then $f^{-1}(1) \cap g^{-1}(0) \subseteq h^{-1}(1) \cap h^{-1}(0) = \emptyset$, hence $(f, g) \in E$.

(ii) Let $f \leq_2 g$. Then for any MPH h with $(h, g) \in E$ we have $h^{-1}(1) \cap g^{-1}(0) = \emptyset$, which implies that $h^{-1}(1) \cap f^{-1}(0) = \emptyset$, hence $(h, f) \in E$. Conversely, if $f^{-1}(0) \nsubseteq g^{-1}(0)$ then we have $x \in f^{-1}(0) \setminus g^{-1}(0)$ and by 1.1 there exists a MPH h with $g^{-1}(0) \subseteq h^{-1}(0)$ and $h(x) = 1$. Clearly, $(h, g) \in E$, $(h, f) \notin E$.

The proof of (iii) is similar to (ii). □

In order to recover a lattice from its dual space, Urquhart introduces the following concepts.

Let L be a bounded lattice. For a set $Y \subseteq D(L)$ we define
$l(Y) = \{f \in D(L) \,|\, \text{there is no } g \in Y \text{ with } f \leq_1 g\}$;
$r(Y) = \{f \in D(L) \,|\, \text{there is no } g \in Y \text{ with } f \leq_2 g\}$.
The set Y is called stable if $Y = lr(Y)$. The set Y is called doubly closed if both Y and $r(Y)$ are (topologically) closed. Notice that the operators r and l are inclusion-reversing, that is $Y \subseteq Z$ implies $r(Y) \supseteq r(Z)$, $l(Y) \supseteq l(Z)$.

Theorem 2.2. *Let L be a bounded lattice. For a set $Y \subseteq D(L)$ the following are equivalent:*

(i) *Y is a doubly closed stable set;*
(ii) *$Y = \varphi^{-1}(1)$ for some MPM $\varphi : D(L) \longrightarrow \widetilde{2}$.*

Proof. Let Y be a doubly closed stable set. Let us define a partial map $\varphi : D(L) \longrightarrow \widetilde{2}$ by

$$\varphi(p) = \begin{cases} 1 & \text{iff } p \in Y, \\ 0 & \text{iff } p \in r(Y). \end{cases}$$

Since $Y \cap r(Y) = \emptyset$, φ is well defined. Further, its domain is closed and it is continuous. We claim that φ preserves E. By way of contradiction, suppose that $p, q \in \mathrm{dom}(\varphi)$ are such that $(p,q) \in E$ and $\varphi(p) \not\leq \varphi(q)$. Necessarily, $\varphi(p) = 1$, $\varphi(q) = 0$, hence $p \in Y = lr(Y)$, $q \in r(Y)$. By 2.1(i) there is $h \in D(L)$ with $p \leq_1 h$, $q \leq_2 h$. By the definition of the operator l, $p \in l(r(Y))$ implies that $h \notin r(Y)$. By the definition of r, there is $k \in Y$ with $h \leq_2 k$. Since the relation $\leq_2$ is transitive, we obtain that $q \leq_2 k$, hence $q \notin r(Y)$, a contradiction. Thus, φ preserves E. To show maximality, let $\psi \supseteq \varphi$ be a partial morphism. Then $Y \subseteq \psi^{-1}(1)$ and $r(Y) \subseteq \psi^{-1}(0)$. Since ψ preserves E, we have $(p,q) \notin E$ whenever $p \in \psi^{-1}(1)$, $q \in \psi^{-1}(0)$. Thus, neither $p \leq_1 q$ nor $q \leq_2 p$ can hold for such p, q. We obtain that $\psi^{-1}(1) \subseteq l(\psi^{-1}(0))$, $\psi^{-1}(0) \subseteq r(\psi^{-1}(1))$. Consequently, $Y = lr(Y) \supseteq l(\psi^{-1}(0)) \supseteq \psi^{-1}(1)$ and therefore $Y = \psi^{-1}(1)$ and $r(Y) = \psi^{-1}(0)$. Hence, $\varphi = \psi$. We have proved that φ is a MPM with $Y = \varphi^{-1}(1)$.

Conversely, let $\varphi : D(L) \longrightarrow \widetilde{2}$ be a MPM. By 1.5 we can assume that $\varphi = e_x$ for some $x \in L$. Let $Y = \varphi^{-1}(1)$. Then $Y = B_x$ and we claim that $r(Y) = A_x$. Indeed, if $p \in A_x$, $q \in B_x$, then $p \not\leq_2 q$, hence $A_x \subseteq r(B_x)$. On the other hand, if $p \notin A_x$ then 1.1 yields $q \in D(L)$ such that $p^{-1}(0) \subseteq q^{-1}(0)$ and $q(x) = 1$, hence $p \notin r(B_x)$. Symetrically we can show that $B_x = l(A_x)$. Thus, $lr(Y) = Y$, which means that Y is a stable set. The continuity of φ implies that both Y and $r(Y)$ are closed. □

Putting together 1.7 and 2.2 we obtain the representation theorem from [5]: any bounded lattice is isomorphic to the set of all doubly closed stable sets ordered by the set inclusion.

The theorems 2.1 and 2.2 provide transition between our and Urquhart's duals of bounded lattices. They allow to translate Urquhart's characterization of dual spaces and his resuts about representation of complete lattices, surjective homomorphisms, congruences, etc. We will not carry out these tasks here.

3. More about dual spaces

Our aim in this section is to obtain some information about topological graphs that arise as duals of bounded lattices. It turns out however, that a wider class of topological graphs can be interesting for representing bounded lattices.

First, let us notice that for any topological graph $\widetilde{X} = (X, E, \tau)$ with τ a T_1-topology, the set of all MPM's $\widetilde{X} \longrightarrow \widetilde{2}$ is partially ordered by the rule

$$\varphi \leq \psi \quad \text{iff } \varphi^{-1}(1) \subseteq \psi^{-1}(1).$$

Indeed, only the antisymmetry is not quite trivial. But, by 1.3, $\varphi^{-1}(1) = \psi^{-1}(1)$ implies $\varphi^{-1}(0) = \psi^{-1}(0)$ and hence $\varphi = \psi$. The partially ordered set of all MPM's need not be a lattice in general. But it will be a lattice under some condition.

If φ and ψ are MPM's $\widetilde{X} \longrightarrow \widetilde{2}$ then we define the partial maps $k^{\wedge}_{\varphi,\psi}$, $k^{\vee}_{\varphi,\psi}$: $X \longrightarrow \{0,1\}$ by

$$k^{\wedge}_{\varphi,\psi}(x) = \begin{cases} 1 & \text{iff } \varphi(x) = 1 \text{ and } \psi(x) = 1, \\ 0 & \text{iff } \varphi(x) = 0 \text{ or } \psi(x) = 0, \end{cases}$$

$$k^{\vee}_{\varphi,\psi}(x) = \begin{cases} 1 & \text{iff } \varphi(x) = 1 \text{ or } \psi(x) = 1, \\ 0 & \text{iff } \varphi(x) = 0 \text{ and } \psi(x) = 0. \end{cases}$$

We claim that these maps are partial morphisms. The partial map $k = k^{\wedge}_{\varphi,\psi}$ preserves E. Indeed, if $(k(x), k(y)) \notin E$, then $k(x) = 1$ and $k(y) = 0$ and without loss of generality $\varphi(x) = 1$, $\varphi(y) = 0$. Since φ preserves E, we obtain $(x, y) \notin E$. Further, k is continuous and its domain is closed, since $k^{-1}(0) = \varphi^{-1}(0) \cup \psi^{-1}(0)$ and $k^{-1}(1) = \varphi^{-1}(1) \cap \psi^{-1}(1)$ are closed sets. Hence k is a partial morphism. The proof for $k^{\vee}_{\varphi,\psi}$ is similar.

We say that $\widetilde{X}$ satisfies the condition (EXT) if, for every MPM's φ and ψ, the partial maps $k^{\wedge}_{\varphi,\psi}$, $k^{\vee}_{\varphi,\psi}$ can be extended to MPM's.

Lemma 3.1. *Let $\widetilde{X} = (X, E, \tau)$ be a topological graph with a T_1-topology τ satisfying (EXT). Then the ordered set of all MPM's $\widetilde{X} \longrightarrow \widetilde{2}$ is a bounded lattice.*

Proof. Let φ, ψ be MPM's. To prove the existence of $\varphi \wedge \psi$ it is sufficient to find a MPM η with $\eta^{-1}(1) = \varphi^{-1}(1) \cap \psi^{-1}(1)$. Let η be the MPM extending $k = k^{\wedge}_{\varphi,\psi}$. Then $k^{-1}(1) \subseteq \eta^{-1}(1)$, it remains to show that $k^{-1}(1) \supseteq \eta^{-1}(1)$. Let $\eta(x) = 1$. Then $(x, y) \in E$ holds for no $y \in \eta^{-1}(0) \supseteq \varphi^{-1}(0) \cup \psi^{-1}(0)$. By 1.3(ii) we obtain that $\varphi(x) = 1$, $\psi(x) = 1$ and hence $k(x) = 1$.

We have proved that $\varphi \wedge \psi$ exists for every φ, ψ. The proof for $\varphi \vee \psi$ is analogous. The universal bounds are the constant maps. □

Example 3.2. The condition (EXT) cannot be omitted in 3.1. We define $\widetilde{X} = (X, E, \tau)$ as follows. (See Figure 4.) We set $X = \{a, b_0, b_1\} \cup \{c_i \mid i \in \omega\} \cup \{d_i \mid i \in \omega\}$ (all the elements are distinct). Further, $(x, y) \in E$ iff:
$x = y$ or
$x \in \{b_0, b_1\}$ and $y \in \{c_i \mid i \in \omega\}$ or
$x \in \{c_i \mid i \in \omega\}$ and $y \in \{b_0, b_1\}$.
Finally, a set $Y \subseteq X$ will be closed in τ if it is finite or contains the element a. It is easy to see that τ is a T_1-topology. Now we define MPM's φ and ψ by $\varphi^{-1}(0) = \{b_0\}$, $\varphi^{-1}(1) = \{a, b_1, d_0, d_1, d_2, \dots\}$, $\psi^{-1}(0) = \{b_1\}$, $\psi^{-1}(1) = \{a, b_0, d_0, d_1, d_2, \dots\}$. It is not difficult to check that we indeed have MPM's. We prove that $\varphi \wedge \psi$ does not exists. Suppose that η is a MPM, $\eta \leq \varphi$, $\eta \leq \psi$. By the definition of ordering, $\eta^{-1}(1) \subseteq \varphi^{-1}(1) \cap \psi^{-1}(1) = \{a, d_0, d_1, d_2, \dots\}$. By 1.3(i), $\eta^{-1}(0) \supseteq \{b_0, b_1, c_0, c_1, c_2, \dots\}$. The set $\eta^{-1}(0)$ is closed and infinite, hence $a \in \eta^{-1}(0)$. Then $a \notin \eta^{-1}(1)$ and therefore $\eta^{-1}(1)$ must be a finite subset of $\{d_0, d_1, d_2, \dots\}$, which implies that $\eta^{-1}(0) = X \setminus \eta^{-1}(1)$.

We have shown that the lower bounds of the set $\{\varphi, \psi\}$ are exactly such MPM's η, for which $\eta^{-1}(1)$ is a finite subset of $\{d_0, d_1, d_2, \dots\}$ and $\eta^{-1}(0) = X \setminus \eta^{-1}(1)$. Among them, there is no largest element, and that is why $\varphi \wedge \psi$ does not exist.

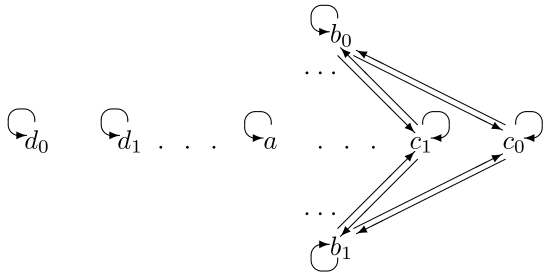

Figure 4

Notice that (EXT) is satisfied whenever $\widetilde{X}$ is isomorphic to dual of some bounded lattice. (Indeed, if $\varphi = e_x$, $\psi = e_y$, then $e_{x \wedge y}$ extends $k^{\wedge}_{\varphi,\psi}$.) Further, it is obviously satisfied, whenever $\widetilde{X}$ is finite (or, more generally, when the topology is discrete). In the finite case, even a stronger condition is satisfied: every partial morphism can be extended to a MPM. A question now arises, whether this (simpler) condition could replace (EXT). However, the next example shows that duals of bounded lattices need not satisfy it.

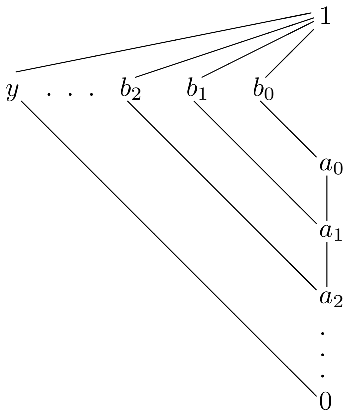

Figure 5

Example 3.3. Let L be the lattice depicted on Figure 5. We shall exhibit a partial morphism $\varphi : \ D(L) \longrightarrow \widetilde{2}$ that cannot be extended to a MPM.

Let $A = \{f \in D(L) \mid f(y) = 0\}$, $B = \{f \in D(L) \mid f(y) = 1\}$. It is not difficult to verify that A consists of a single $h \in D(L)$ defined by

$$h(x) = \begin{cases} 0 & \text{if } x \le y, \\ 1 & \text{otherwise,} \end{cases}$$

while $B = \{f_0, f_1, f_2, \dots\}$, where

$$f_i(x) = \begin{cases} 0 & \text{if } x \le b_i, \\ 1 & \text{if } x \ge y, \\ \text{undefined otherwise.} \end{cases}$$

By the definition of the topology in $D(L)$, the sets A and B are closed. Further, $(h, f_i) \notin E$ for any $i \in \omega$. Hence the partial mapping $\varphi : \ D(L) \longrightarrow \widetilde{2}$ defined by $\varphi^{-1}(0) = B$, $\varphi^{-1}(1) = A$, is a partial morphism. For the contradiction, suppose that φ can be extended to a MPM ψ. By 1.5, $\psi = e_z$ for some $z \in L$. For any $i \in \omega$ we have $f_i(z) = e_z(f_i) = \varphi(f_i) = 0$. The only $z \in L$ satisfying this condition is $z = 0$. But this is impossible, since $e_0(h) = 0 \neq \varphi(h)$.

Hence, any topological graph with T_1-topology satisfying (EXT) represents some bounded lattice. However, not all such topological graphs are equally advantageous for representing bounded lattices. Some of them contain redundancies and are unnecessarily large. That is why we introduce another condition, which ensures that any two points can be properly distinguished by a MPM.

Let $\widetilde{X} = (X, E, \tau)$ be a topological graph. For any $x \in X$ we denote $x^+ = \{y \in X \mid (x, y) \in E\}$, $x^- = \{y \in X \mid (y, x) \in E\}$. We say that $\widetilde{X}$ satisfies (SEP) if, for any $x, y \in X$ such that $x \neq y$, $(x, y) \in E$ and $(y, x) \in E$, the following conditions are satisfied:

(1) $x^+ \neq y^+$ or $x^- \neq y^-$;
(2) if $x^- \not\subseteq y^-$ then $y^- \not\subseteq x^-$.
(3) if $x^+ \not\subseteq y^+$ then $y^+ \not\subseteq x^+$;

Lemma 3.4. *If $\widetilde{X} = D(L)$ for some bounded lattice L, then $\widetilde{X}$ satisfies (SEP).*

Proof. Let f, g be MPH's $L \longrightarrow \underline{2}$, $f \neq g$, $(f, g) \in E$, $(g, f) \in E$.

(1) Without loss of generality, $f^{-1}(0) \not\subseteq g^{-1}(0)$, hence there is $x \in L$ with $f(x) = 0$ and $g(x)$ undefined. By 1.1 there is a MPH h with $g^{-1}(0) \subseteq h^{-1}(0)$ and $h(x) = 1$. Obviously, $(h, g) \in E$, $(h, f) \notin E$, hence $f^- \neq g^-$.

(2) Suppose that $f^- \not\subseteq g^-$. Then we have a MPH h with $(h, f) \in E$, $(h, g) \notin E$. There must be $x \in L$ with $h(x) = 1$, $g(x) = 0$ and $f(x) \neq 0$, hence $g^{-1}(0) \not\subseteq f^{-1}(0)$. We claim that also $f^{-1}(0) \not\subseteq g^{-1}(0)$ Indeed, if $f^{-1}(0) \subsetneq g^{-1}(0)$, then the maximality of f implies that $g^{-1}(0) \cap f^{-1}(1) \neq \emptyset$, hence $(f, g) \notin E$, a contradiction. Thus, $f^{-1}(0) \not\subseteq g^{-1}(0)$ and by the same argument as in (1), there exists a MPH h' with $(h', g) \in E$, $(h', f) \notin E$.

(3) can be proved similarly. □

We have found out that if a topological graph $\widetilde{X}$ has a T_1-topology and satisfies (EXT) then the set of all MPM's $\widetilde{X} \longrightarrow \widetilde{2}$ forms a lattice. We denote this lattice by $C(\widetilde{X})$. Thus, 1.7 says that $L \cong C(D(L))$ holds for any bounded lattice L. Now we are going to prove that if $\widetilde{X}$ in addition satisfies (SEP) and is finite, then $\widetilde{X}$ is canonically embedded in $D(C(\widetilde{X}))$. Since the following assertion concerns finite graphs only, the condition (EXT) is automatically satisfied. We do not know if a similar statement (with (EXT) added) holds for the infinite case.

Let $\widetilde{X}$ be a topological graph with a T_1-topology and (EXT). For every $x \in X$ we define the evaluation function $\varepsilon_x : C(\widetilde{X}) \longrightarrow \{0, 1\}$ by the rule

$$\varepsilon_x(\varphi) = \begin{cases} \varphi(x) & \text{if } x \in \mathrm{dom}(\varphi), \\ \text{undefined otherwise.} \end{cases}$$

Lemma 3.5. *Let $\widetilde{X} = (X, E, \tau)$ be a finite topological graph with a T_1-topology (necessarily discrete), satisfying (SEP). Then*

(i) *for every $x \in X$, ε_x is a MPH $C(\widetilde{X}) \longrightarrow \overline{2}$;*
(ii) *for any $x, y \in X$, $(x, y) \in E$ iff $(\varepsilon_x, \varepsilon_y) \in E$;*
(iii) *for any $x, y \in X$, if $x \neq y$ then $\varepsilon_x \neq \varepsilon_y$.*

Proof. (i) Let $x \in X$. If $\varphi, \psi \in C(\widetilde{X})$, $\varphi \geq \psi$ and $\varphi \in \varepsilon_x^{-1}(0)$, then $\varphi(x) = 0$ and by the definition of the ordering in $C(\widetilde{X})$ we obtain that $\psi(x) = 0$ and hence $\psi \in \varepsilon_x^{-1}(0)$. Further, if $\varphi, \psi \in \varepsilon_x^{-1}(0)$, then $\varphi(x) = \psi(x) = 0$. From the proof of 3.1 one can see that $(\varphi \vee \psi)^{-1}(0) = \varphi^{-1}(0) \cap \psi^{-1}(0)$, hence $(\varphi \vee \psi)(x) = 0$ and $(\varphi \vee \psi) \in \varepsilon_x^{-1}(0)$. We have proved that $\varepsilon_x^{-1}(0)$ is an ideal in $C(\widetilde{X})$. Similarly we can prove that $\varepsilon_x^{-1}(1)$ is a filter. Hence, ε_x is a partial homomorphism. It remains to prove the maximality of ε_x. Let us define partial maps $\psi_0, \psi_1 : \widetilde{X} \longrightarrow \widetilde{2}$ by

$$\psi_0(y) = \begin{cases} 0 & \text{if } y = x, \\ 1 & \text{if } (y,x) \notin E, \end{cases} \qquad \psi_1(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } (x,y) \notin E. \end{cases}$$

Since E is reflexive, the maps are well defined and it is easy to see that they are partial morphisms. Let φ_0 and φ_1 be their extensions to MPM's.

Suppose that φ is a MPM $\widetilde{X} \longrightarrow \widetilde{2}$ with $\varphi \notin \mathrm{dom}(\varepsilon_x)$, that is $x \notin \mathrm{dom}(\varphi)$. We claim that $\varphi \wedge \varphi_1 \leq \varphi_0$ holds in $C(\widetilde{X})$ and we will prove it by showing the inclusion $\varphi^{-1}(1) \cap \varphi_1^{-1}(1) \subseteq \varphi_0^{-1}(1)$. Let $y \in \varphi^{-1}(1) \cap \varphi_1^{-1}(1)$, hence $\varphi(y) = 1$, $\varphi_1(y) = 1$. We need to show that $\varphi_0(y) = 1$. This is clear if $(y,x) \notin E$, because then $\psi_0(y) = 1$. Let us suppose that $(y,x) \in E$. We will show that this assumption leads to a contradiction. We have $(x,y) \in E$ because otherwise $\varphi_1(y) = \psi_1(y) = 0$. Further, $x \neq y$ because $x \notin \mathrm{dom}(\varphi)$, $y \in \mathrm{dom}(\varphi)$. Since $x \notin \varphi^{-1}(1)$, by 1.3 there is $z \in \varphi^{-1}(0)$ with $(x,z) \in E$. Clearly $(y,z) \notin E$, hence $x^+ \not\subseteq y^+$. By (SEP) there is $u \in X$ such that $(x,u) \notin E$, $(y,u) \in E$. Then $\varphi_1(u) = \psi_1(u) = 0$, which implies that $\varphi_1(y) \neq 1$, a contradiction.

Hence, $\varphi \wedge \varphi_1 \leq \varphi_0$ and similarly one can prove that $\varphi \vee \varphi_0 \geq \varphi_1$. This means, that if $f \supseteq \varepsilon_x$ is a partial homomorphism, then $\varphi \notin \mathrm{dom}(f)$. Indeed, if $f(\varphi) = 1$ (the case $f(\varphi) = 0$ is similar), then $f(\varphi \wedge \varphi_1) = f(\varphi) \wedge f(\varphi_1) = 1 \wedge \varepsilon_x(\varphi_1) = 1$, while $f(\varphi_0) = \varepsilon_x(\varphi_0) = 0$. Hence, ε_x is a MPH.

(ii) If $(x,y) \in E$, then $\varphi(x) \leq \varphi(y)$ holds for every MPM φ with $x, y \in \mathrm{dom}(\varphi)$. In other words, $\varepsilon_x(\varphi) \leq \varepsilon_y(\varphi)$ holds for every $\varphi \in \mathrm{dom}(\varepsilon_x) \cap \mathrm{dom}(\varepsilon_y)$, which by the definition of E means that $(\varepsilon_x, \varepsilon_y) \in E$.

Conversely, if $(x,y) \notin E$, then we can find a MPM φ with $\varphi(x) = 1$, $\varphi(y) = 0$, which shows that $(\varepsilon_x, \varepsilon_y) \notin E$.

(iii) Let $x \neq y$. If $(x,y) \notin E$ then the map $k: \{x,y\} \longrightarrow \{0,1\}$ with $k(x) = 1$, $k(y) = 0$ is a partial morphism and can be extended to a MPM $\varphi: \widetilde{X} \longrightarrow \widetilde{2}$. Clearly $\varepsilon_x(\varphi) = 1 \neq 0 = \varepsilon_y(\varphi)$. If $(y,x) \notin E$, we use a similar argument. Suppose now that $(x,y) \in E$, $(y,x) \in E$. By (SEP) we can assume (without loss of generality) that $x^+ \not\subseteq y^+$. Hence, there is $z \in X$ with $(x,z) \in E$, $(y,z) \notin E$. The map $k: \{y,z\} \longrightarrow \{0,1\}$ defined by $k(y) = 1$, $k(z) = 0$ can be extended to a MPM φ. Then $\varphi(y) = 1$, $\varphi(z) = 0$ and, since φ preserves E, $\varphi(x) \neq 1$. Hence, $\varepsilon_y(\varphi) = 1 \neq \varepsilon_x(\varphi)$. □

Hence, under the conditions of 3.5, the assignment $x \mapsto \varepsilon_x$ is an embedding $\widetilde{X} \longrightarrow D(C(\widetilde{X}))$. Of course, this embedding is an isomorphism if and only if $\widetilde{X}$ is a dual space of some bounded lattice. In general however, this embedding is proper. For example, consider $\widetilde{X}$ as a 3-element set $\{x,y,z\}$ equipped with the binary relation $E = \{(x,x),(y,y),(z,z),(x,y),(y,z),(z,x)\}$ and the discrete topology. It is not hard to verify that $C(\widetilde{X})$ is the 5-element modular nondistributive lattice M_3 and that $D(M_3)$ has 6 elements. This shows that in some cases there exist topological graphs smaller than $D(L)$ which represent L (more effectively, one could say). However, $D(L)$ reflects the properties of L better. For example, L and $D(L)$ have the same automorphism group, which need not be true for other topological graphs representing L.

In some cases (for instance, if $L = N_5$, the 5-element nonmodular lattice) $D(L)$ is minimal in the sense that there is no proper subgraph of $D(L)$ representing L. It is not clear which lattices have this property.

Finally, let us mention that some (rather complicated) characterization of topological graphs that are the duals of bounded lattices can be obtained by translating the characterization in [5].

4. Concluding remarks

Several questions of a general nature arise in connection with our representation. The notions of MPH and MPM are applicable also for other algebras and relational structures. One can try to build similar representations for various classes of algebras. This effort is especially promising when one starts from some natural duality (as the Priestley duality in our case). In an implicit form, representations of this kind can be found in the literature. For example, Isbell's proof in [3] that every median algebra is embeddable in a lattice provides a representation that is (at least in the finite case) a generalization of Werner's duality for symmetric median algebras ([6, appendix]) via MPH's.

There are unanswered questions in our representation itself. Some of them have been mentioned in the previous text. In fact, it is not quite clear that the variety of bounded lattices is the "right" class of algebras for the representation based on $\overline{2}$ and $\widetilde{2}$. There might be an algebra A which is not a lattice (but of the same signature), for which elements of A are in a one-to-one correspondence with MPM's $D(A) \longrightarrow \widetilde{2}$ and whose operations can be in some way recovered from $D(A)$. The Priestley duality works with the fact that the variety of bounded distributive lattices is generated by the algebra $\overline{2}$. The relationship between the variety of all bounded lattices and $\overline{2}$ is not that tight. What might be a key property of bounded lattices is that their elements can be effectively separated by MPH's into $\overline{2}$. However, we are unable to specify the word "effectively".

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