Combinatorics of ordered sets

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An order (more precisely, a partial order) on a set A is a binary relation ρ on A, such that the following conditions hold for every $x, y, z \in A$:

(1) $(x, x) \in \rho$ (reflexivity);

(2) if $(x, y) \in \rho$ and $(y, x) \in \rho$, then $x = y$ (antisymmetry);

(3) if $(x, y) \in \rho$ and $(y, z) \in \rho$, then also $(x, z) \in \rho$ (transitivity).

Some examples

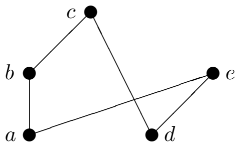
- The relation $\subseteq$ (set inclusion) is a partial order on various systems of sets.
- The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ a $\mathbb{R}$ have their natural order.
- We can define the relation ρ on the set $\mathbb{N}$ by

$$(x, y) \in \rho \text{ if and only if } y \text{ is divisible by } x.$$

An ordered set, in which every two elements are comparable is called *linearly ordered*, or *a chain*. Another extreme is an *antichain*, which is an ordered set in which no two distinct elements are comparable.

Hasse diagrams

Finite ordered sets are determined by the covering relation. We say that y *covers* x if $x < y$ and there is no element between x and y (i.e. $x < z < y$).



We have a 5-element ordered set $\{a, b, c, d, e\}$. In the diagram we can see that $a < b < c$, $d < c$, $a < e$, $d < e$. The relation $a < c$ is also true, but this line is not drawn, as (a, c) is not a covering pair.

Isomorphisms and isotone maps

Partially ordered sets (A, ρ) and (B, σ) are called *isomorphic*, if there exists a bijective function $\varphi\colon A \to B$ such that

$$(x, y) \in \rho \text{ if and only if } (\varphi(x), \varphi(y)) \in \sigma.$$

Such a function φ is called *an isomorphism*.
When comparing different ordered sets, it is often convenient to consider the following weakening of the concept of an isomorphism.
A mapping $\varphi\colon A \to B$ between partially ordered sets (A, ρ) and (B, σ) is called *isotone* if the following condition holds:

$$(x, y) \in \rho \text{ implies that } (\varphi(x), \varphi(y)) \in \sigma.$$

Nech $(P, \leq)$ be a partially ordered set and $A \subseteq P$. We say that $x \in P$ is *a supremum* of the set A, if it is its least upper bound, that is, the least element in the set

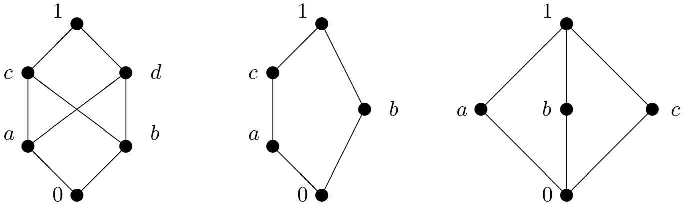
$$U(A) = \{y \in P \mid y \geq a \text{ for every } a \in A\}.$$

Analogically, the infimum of the set A is its greatest lower bound, that is, the greatest element in the set

$$L(A) = \{y \in P \mid y \leq a \text{ for every } a \in A\}.$$

Of course, the supremum or infimum of a set A need not exist. *A lattice* is a partially ordered set, in which every two-element subset has a supremum and a infimum.

Lattice examples



P N_5 M_3

Here, the ordered sets M_3 and N_5 are lattices, while P is not. (The set $\{a, b\}$ does not have a supremum.)

Denotations

In the theory of lattices, the usual denotation for supremum and infimum is

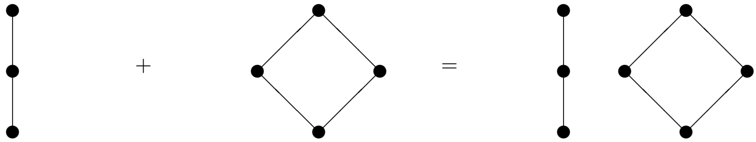
$$a \vee b = \sup\{a, b\},$$

$$a \wedge b = \inf\{a, b\}.$$

(The motivation comes from logic: consider the 2-element ordered set $0 < 1$.)

Addition

We define the sum of two (or more) ordered sets as their disjoin union. Informally, we put the ordered sets together without drawing any lines between them. For example,

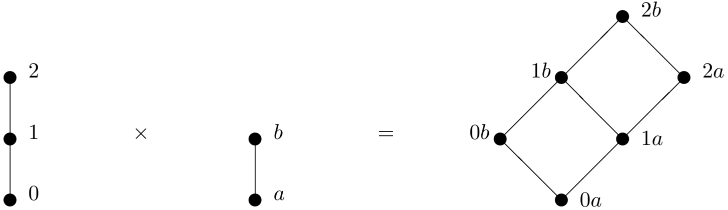


Multiplication

The product of ordered sets P and Q is their Cartesian product $P \times Q$ (or $P \cdot Q$) with the order defined componetwise, that is,

$$(a, b) \leq (c, d) \quad \text{if} \quad a \leq c \quad \text{and} \quad b \leq d.$$

$(a, c \in P,\ b, d \in Q)$



Some arithmetics

The arithmetics of ordered sets can be viewed as an extension of the usual arithmetics of natural numbers, because every natural number n can be represented as a n-element *antichain* (an ordered set in which no two distinct elements are comparable).

Theorem

For every ordered sets A, B, C, the following holds:

1. $A + B = B + A$;
2. $(A + B) + C = A + (B + C)$;
3. $A \cdot B = B \cdot A$;
4. $(A \cdot B) \cdot C = A \cdot (B \cdot C)$;
5. $A \cdot (B + C) = A \cdot B + A \cdot C$;
6. $0 + A = A$ *(here 0 denotes an empty set)*;
7. $1 \cdot A = A$ *(here 1 denotes a 1-element set)*;
8. $0 \cdot A = 0$.

Product decomposition

An ordered set is *connected* if it is not isomorphic to the sum of two nonempty ordered sets. It is clear that every ordered set is a sum of connected ordered sets.
Now we consider the decompositions with respect to the product. We have the following analogue of the Fundamental Theorem of Arithmetics.

Theorem

Every finite ordered set is a product of indecomposable ordered sets. If the ordered set is connected, the decomposition is unique.

If the ordered set is not connected, the product decomposition need not be unique:

$$(1+\mathbf{2})(1+\mathbf{2}^2+\mathbf{2}^4) = (1+\mathbf{2}^3)(1+\mathbf{2}+\mathbf{2}^2) = 1+\mathbf{2}+\mathbf{2}^2+\mathbf{2}^3+\mathbf{2}^4+\mathbf{2}^5.$$

($\mathbf{2}$ denotes the 2-element chain.)

Cancellability

Theorem

For any finite ordered sets A, B and C the following holds:

1. *if $A \cdot B = A \cdot C$ and $A \neq 0$, then $B = C$;*
2. *if $A^n = B^n$ $(n \geq 1)$, then $A = B$.*

Exponentation

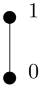
We define the power B^A as the set of all isotone functions $A \to B$. The order on this set is defined by the rule

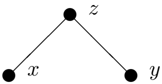
$$f \leq g \quad \text{if} \quad f(x) \leq g(x) \quad \text{for every} \quad x \in A.$$

This operation has properties similar to the power operation on natural numbers, for instance

- $B^0 = 1$;
- $B^1 = B$;
- $1^A = 1$;
- if A is a n-element antichain, then $B^A = B^n = B \cdot B \cdot \cdots \cdot B$.

Exponentation example

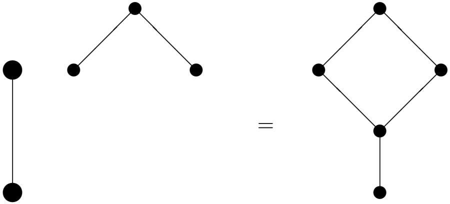
$B =$ 

$A =$ 

There are 8 functions $A \to B$, of which 5 are isotone, namely

*	x	y	z
a	0	0	0
b	0	0	1
c	0	1	1
d	1	0	1
e	1	1	1

Exponentation example 2



Theorem

For every ordered sets A, B, C, the following is true:

1. $A^{B+C} = A^B \cdot A^C$;
2. $(A \cdot B)^C = A^C \cdot B^C$;
3. $A^{B \cdot C} = (A^B)^C$;
4. *if C is connected, then $(A+B)^C = A^C + B^C$.*

If C is not connected, for instance $C = D + E$ with D, E connected, then we can calculate:

$$(A+B)^{D+E} = (A+B)^D \cdot (A+B)^E = (A^D + B^D) \cdot (A^E + B^E)$$

Power cancellation

The following theorem is difficult and was proved relatively recently (1999).

Theorem

Let A, B, C be finite ordered sets.

1. *If $A^B = A^C$ and A is not an antichain, then $B = C$.*
2. *If $A^C = B^C$, then $A = B$.*

This theorem enables to define the logarithms of ordered sets, with the base A, which is not an antichain. The simplest case is when A is a 2-element chain.

Distributive lattices

A distributive lattice is a lattice which satisfies the laws of distributivity:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

For instance, every chain is a distributive lattice. The lattice $\mathcal{P}(X)$ (all subsets of X ordered by the set inclusion) is distributive for any X. The ordered set $(\mathbb{N}, \rho)$ (divisibility) is a distributive lattice. On the other hand, the lattices M_3 and N_5 are not distributive.
An element of a finite lattice is called $\wedge$-*irreducible*, if it has exactly one upper cover (successor). The set of all $\wedge$-irreducible elements of a lattice L will be denoted by $M(L)$. We view $M(L)$ as an ordered set, with the order inverse to the order in L, that is

$$x \leq y \text{ in } M(L) \text{ if } y \leq x \text{ in } L.$$

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Birkhoff's representation

Theorem

1. *For every finite partially ordered set P, the set $\mathbf{2}^P$ is a finite distributive lattice and, up to isomorphism, $M(\mathbf{2}^P) = P$.*
2. *For every finite distributive lattice L, the set $M(L)$ is partially ordered and, up to isomorphism, $L = \mathbf{2}^{M(L)}$.*

This theorem says that $M(L)$ behaves much like the logarithm of L with the base $\mathbf{2}$. In fact, we also have other properties similar to calculation with the logarithms.

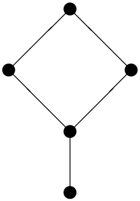
Theorem

For any finite distributive lattices P, Q,

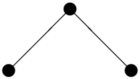
1. $M(P \cdot Q) = M(P) + M(Q)$;
2. P^Q *is a finite distributive lattice and* $M(P^Q) = M(P) \cdot Q$.

Representation example

If L is the lattice



then the ordered set $M(L)$ is (recall that the order is inverse to the order of L)



and we have already found that $\mathbf{2}^{M(L)} = L$.

Fixed point property

A partially ordered set P has the fixed point property (FPP) if for every isotone $f : P \to P$ there exists a fixed point, that is $x \in P$ with $f(x) = x$.
The characterization of partially ordered sets with the fixed point property is an unsolved problem. We can only present some partial results, which are not difficult to prove.

Theorem

Every finite lattice has FPP.

This theorem can be generalized to infinite lattices which are complete, i.e. partially ordered sets in which every subset has a supremum and infimum.

Dismantlable sets

A partially ordered set P is called *dismantlable* if there exists a sequence of ordered sets

$$P = P_0 \supset P_1 \supset P_2 \supset \cdots \supset P_n$$

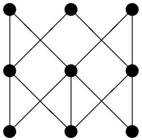
such that P_n is 1-element and every P_i arises from P_{i-1} by omitting one element, which has exactly one upper cover or exactly one lower cover.
Every finite lattice is dismantlable, but there are dismantlable ordered sets that are not lattices.

Theorem

Every dismantlable ordered set has FPP.

Nondismantlable set with FPP

A conjecture that a finite ordered set has FPP if and only if it is dismantlable has been rejected by the following example.



Product theorem for FPP

A recent solution to another difficult problem is contained in the following assertion.

Theorem

The product of two finite ordered sets with FPP has FPP.

Dimension

The definition is based on the following classical result (Dushnik, Miller 1941).

Theorem

Every partial order is an intersection of linear orders.

Thus, if we have a finite partially ordered set $(P, \leq)$, then there are linear orders $\leq_1$, $\leq_2, \ldots, \leq_n$ such that

$$x \leq y \text{ if and only if } x \leq_i y \text{ for every } i.$$

The smallest number of such linear orders is called the *dimension* of $(P, \leq)$ and denoted $\dim(P)$.
Equivalently, P has dimension n if it is isomorphic to a subset of the product of n chains, and it is not isomorphic to a subset of the product of $n-1$ chains.

Easy examples

For some ordered sets, the dimension is easy to determine:

(1) The ordered sets of dimension 1 are exactly the chains;
(2) Every antichain has dimension 2.
(3) The ordered set $\mathbf{2}^n$ has dimension n.

For the dimension of the product we have the following assertion.

Theorem

For every finite ordered sets P and Q,

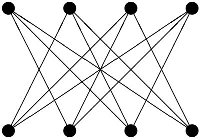
$$\dim(P \times Q) \leq \dim(P) + \dim(Q).$$

If P and Q has both the smallest and the greatest element, then the equality holds.

The equality $\dim(P \times Q) = \dim(P) + \dim(Q)$ does not hold in general (antichains).

One difficult example

Let S_n denote the ordered set whose elements are $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ ordered in such a way that $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are antichains, and $a_i < b_j$ whenever $i \neq j$. (See the picture of S_4 below.)



Theorem

For every $n > 1$, $\dim(S_n) = n$.

Snyder's theorem

Let $G = (V, E)$ be a (unoriented) graph with V as the set of vetrices and E the set of edges. The incidence partially ordered set of G, denoted $P(G)$ has elements $E \cup G$, ordered in such a way that E and G are antichains and $v < e$ whenever the edge e contains the vertex v.

Theorem

A graph G is planar if and only if $\dim(P(G)) \leq 3$.

Bibliography

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